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Lyapunov matrices for the stability analysis of a multiple distributed time-delay system with repeated piecewise function kernel ⋆

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Abstract: In this publication, we construct the Lyapunov matrices of a multiple distributed time-delay system in the particular case where the kernel functions are repeated. The matrix construction is reduced to the computation of the solutions of a delay free system of matrix equations. We find that the number of auxiliary matrices which allow us to find the solutions can be significantly reduced. We also carry out a stability analysis with the help of the obtained Lyapunov matrix. Two illustrative examples are presented.

Keywords: Time-delay, Linear systems, Stability analysis, Lyapunov stability.

1. INTRODUCTION

In the last years the stability of delay linear systems have been studied in the framework of the Lyapunov theory. Delay linear systems of retarded type, neutral type and with distributed delays have been addressed with the help of the so-called delay Lyapunov matrix. This matrix generalizes the one obtained in the classical theory of Lyapunov for linear systems, see Kharitonov (2012). When the Lyapunov condition holds, the construction of the Lyapunov matrix is reduced to a two point boundary value problem, which has been studied by means of approximation methods Jarlebring et al. (2009), Huesca et al. (2009) or by the semi-analytic approach. The delay Lyapunov matrix makes possible to address some problems such as H_2 norm computation, robust stability analysis, optimal control design, exponential estimates of the solution and the verification of the stability criterion introduced in Egorov et al. (2017), to name a few. In recent times, the cases of multiple pointwise delays Garcia-Lozano and Kharitonov (2004), single distributed delay Kharitonov (2006), distributed delay with γ-distribution kernel Cuvas et al. (2015), and multiple constant distributed delays Aliseyko (2017), have been subject of active research.

Here, we are interested in the construction of the Lyapunov matrix for systems with multiple concentrated delays and multiple distributed delays with piecewise continuous function kernels, in particular those involving repeated kernels. It is worthy of mention that this Lyapunov matrix is a crucial element of our current studies on predictor control for systems with input and state delays.

The paper is organized as follows. The system definition, the properties of its Lyapunov matrix, as well as the stability conditions are given in Section 2. Next, we present the Lyapunov matrix construction in Section 3. An illustrative example is presented in section 4. Finally, we conclude our work in Section 5.

Notation: The time derivative of a function is expressed by $\dot{g}(t)$, while $g'(\tau)$ denotes the derivative with respect to τ. For a given real matrix $A > 0$, $A \ngeq 0$ means that the matrix is positive definite and not positive semidefinite, respectively. $PC([-H,0],\mathbb{R}^n)$ denotes the space of piecewise continuous and bounded functions defined on $[-H,0]$.

2. PRELIMINARY RESULTS

Consider the linear system with multiple constant delays and a piecewise function kernel,

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-jh) + \sum_{j=0}^{m-1} \int_{-(j+1)h}^{-jh} G_j(\theta) x(t+\theta) d\theta,$$

$$= \sum_{j=0}^{m} A_j x(t-jh) + \sum_{j=0}^{m-1} \int_{-h}^{0} G_j(\theta - jh) x(t+\theta-jh) d\theta, \quad (1)$$

where h is the basic delay, m is a natural number, $mh = H$ is the maximum delay and $A_0, \ldots, A_m$ are $n \times n$ constant real matrices. For an initial function $\varphi \in PC([-H,0],\mathbb{R}^n)$ the restriction of the solution $x(t,\varphi)$ to the interval $[t-H,t]$ is denoted $x_t(\varphi)$. We consider kernels $G_j(\theta - jh)$ of the form

$$G_j(\theta - jh) = \sum_{i=0}^{\tilde{k}} g_{0,i}(\theta) C_{j,i}, \quad (2)$$

where $g_{0,i}(\theta)$ are scalar functions and $C_{j,i}$ are $n \times n$ constant matrices. It is worth mentioning that all kernels depend on the same scalar functions and on different constant matrices. The scalar functions g also satisfy

$$g'_{0,i}(\theta) = \sum_{k=0}^{\tilde{k}} \alpha_{0,i}^{k} g_{0,k}. \quad (3)$$

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In Kharitonov (2012) was shown, that the Lyapunov matrix $U(\tau)$ of system (1) associated to a constant positive definite matrix W, satisfies:
1.- the dynamic property for $\tau > 0$

$$U'(\tau) = \sum_{j=0}^{m} U(\tau - jh)A_j + \sum_{j=0}^{m-1} \int_{-h}^{0} U(\tau + \theta - jh)G_j(\theta - jh)d\theta, \tag{4}$$

2.- the symmetric property, for $\tau \geq 0$

$$U(-\tau) = U^T(\tau), \tag{5}$$

3.- the algebraic property,

$$U'(+0) - U'(-0) = -W. \tag{6}$$

For $\tau < 0$, the dynamic property is

$$U'(\tau) = -[U'(-\tau)]^T = -\sum_{j=0}^{m} A_j^T U(\tau + jh) - \sum_{j=0}^{m-1} \int_{-h}^{0} G_j(\theta - jh)^T U(\tau - \theta + jh)d\theta, \tag{7}$$

and the algebraic property can be rewritten as

$$-W = \sum_{j=0}^{m} \left[U(-jh)A_j + A_j^T U(jh)\right] + \sum_{j=0}^{m-1} \left\{ \int_{-h}^{0} U(\theta - jh)G_j(\theta - jh)d\theta + \int_{-h}^{0} G_j(\theta - jh)^T U(-\theta + jh)d\theta \right\}. \tag{8}$$

The Lyapunov matrix is the unique solution of the two points boundary problem, defined by properties (4-8), provided that the Lyapunov condition holds (the characteristic equation of (1) has no eigenvalues that are symmetric with respect to the imaginary axis).
This is the foundation of the exponential stability criterion for systems with pointwise and distributed delays presented in Egorov et al. (2017) .

Theorem 1. System (1) is exponentially stable if and only if the Lyapunov condition holds and for every natural number $r \geq 2$,

$$\left\{ U\left(\frac{j-i}{r-1}H\right)\right\}_{i,j=1}^{r} > 0. \tag{9}$$

Moreover, if the Lyapunov condition holds and system (1) is unstable, there exists r such that

$$\left\{ U\left(\frac{j-i}{r-1}H\right)\right\}_{i,j=1}^{r} \ngtr 0.$$

3. LYAPUNOV MATRIX CONSTRUCTION

The semi-analytic construction of the Lyapunov matrix for kernels of the type (2) which satisfy the condition (3) is addressed in this section.

3.1 Delay free auxiliary system

We define $2m$ auxiliary matrices corresponding to the multiple constant delays,

$$Y_i(\tau) = U(\tau + ih), \qquad -m \leq i \leq m-1. \tag{10}$$

In the case where the scalar functions $g_{0,i}(\theta)$ of the kernels G_0 and G_1 are repeated, we need to define auxiliary matrices for each scalar function. For example, if $m = 2$, for each $g_{0,i}(\theta)$ we must define the following matrices

Table 1. Auxiliary matrices

$G_0(\theta)$	$G_1(\theta - h)$
$Z_1 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta+h)d\theta$	$Z_3 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta)d\theta$
$Z_2 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta)d\theta$	$Z_4 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta-h)d\theta$
$J_1 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta-h)d\theta$	$J_3 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta)d\theta$
$J_2 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta-2h)d\theta$	$J_4 = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta-h)d\theta.$

The fact of having repeated scalar functions in both kernels G_0 and G_1 allows a significant reduction of the auxiliary matrices Z_i and J_i conclude from Table 1 that $Z_2 = Z_3$ and $J_1 = J_4$, so, we only need 6 auxiliary matrices, as follows

$$Z_1(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta+h)d\theta$$
$$Z_2(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta)d\theta$$
$$Z_3(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta-h)d\theta$$
$$J_1(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta)d\theta$$
$$J_2(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta-h)d\theta$$
$$J_3(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta-2h)d\theta.$$

The case where $m = 3$ requires 18 auxiliary matrices, but as we noted earlier, because of the repeated scalar functions, it is possible to reduce the number of auxiliary matrices to 10. For $m = 4$, the initial 32 auxiliary matrices are reduced to 14. Next, we properly write this observations in a general form.
The functions $g_{0,i}(\theta)$ define auxiliary matrices

$$Z_{i,p}(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau+\theta+ph)d\theta, \; -(m-1) \leq p \leq m-1,$$
$$J_{i,p}(\tau) = \int_{-h}^{0} g_{0,i}(\theta)U(\tau-\theta+ph)d\theta, \; -m \leq p \leq m-2, \tag{11}$$

where $0 \leq i \leq \tilde{k}$. The number of auxiliary matrices is $6m - 2 + (4m-2)\tilde{k}$.

Lemma 2. Let $U(\tau)$ be a Lyapunov matrix of system (1), associated with a symmetric matrix W. Then the auxiliary matrices (10), (11) satisfy the system of linear differential equations,

$$\left\{\begin{array}{l} Y_i'(\tau) = \displaystyle\sum_{j=0}^{m} Y_{i-j}(\tau)A_j + \sum_{p=0}^{m-1}\left\{\sum_{j=0}^{\tilde{k}} Z_{j,i-p}(\tau)C_{p,j}\right\}, \\ \qquad 0 \le i \le m-1, \\ Y_i'(\tau) = -\displaystyle\sum_{j=0}^{m} A_j^T Y_{i+j}(\tau) - \sum_{p=0}^{m-1}\left\{\sum_{j=0}^{\tilde{k}} C_{p,j}^T J_{j,i+p}(\tau)\right\}, \\ \qquad -m \le i \le -1, \\ Z_{i,p}'(\tau) = g_{0,i}(0)Y_p(\tau) - g_{0,i}(-h)Y_{p-1}(\tau) \\ \qquad - \displaystyle\sum_{k=0}^{\tilde{k}} \alpha_{0,i}^k Z_{k,p}(\tau), \\ \qquad 0 \le i \le \tilde{k}, \quad -(m-1) \le p \le m-1, \\ J_{i,p}'(\tau) = -g_{0,i}(0)Y_p(\tau) + g_{0,i}(-h)Y_{p+1}(\tau) \\ \qquad + \displaystyle\sum_{k=0}^{\tilde{k}} \alpha_{0,i}^k J_{k,p}(\tau), \\ \qquad 0 \le i \le \tilde{k}, \quad -m \le p \le m-2, \end{array}\right. \tag{12}$$

the boundary conditions for $m \ge 1$ are

$$\left\{\begin{array}{l} Y_i(0) = Y_{i-1}(h), \qquad -m+1 \le i \le m-1, \\ Z_{i,-(m-1)}(0) = J_{i,m-2}^T(h), \qquad 0 \le i \le \tilde{k}, \\ Z_{i,0}(0) = \displaystyle\int_0^h g_{0,i}(\theta-h)Y_{-1}(\theta)d\theta, \quad 0 \le i \le \tilde{k}, \\ \displaystyle\sum_{j=0}^{m}\left[Y_{-j}(0)A_j + A_j^T Y_{j-1}(h)\right] + \sum_{j=0}^{m-1}\left\{\sum_{i=0}^{\tilde{k}} Z_{i,-j}(0)C_{j,i} \right. \\ \qquad \left. + \displaystyle\sum_{i=0}^{\tilde{k}} C_{j,i}^T J_{i,j-1}(h)\right\} = -W, \\ \qquad \text{In addition only for } m \ge 2, \\ Z_{i,p}(0) = Z_{i,p-1}(h), \quad 0 \le i \le \tilde{k}, \quad -(m-2) \le p \le m-1, \\ J_{i,p}(0) = J_{i,p-1}(h), \quad 0 \le i \le \tilde{k}, \quad -(m-1) \le p \le m-2, \\ J_{i,0}(0) = \displaystyle\int_0^h g_{0,i}(-\theta)Y_0(\theta)d\theta, \quad 0 \le i \le \tilde{k}. \end{array}\right. \tag{13}$$

Proof. The system of equations (12) follows from the dynamic properties (4), (7) and definitions (10) and (11). The boundary conditions are given by the algebraic property (8) and definitions (10) and (11).

Lemma 3. If $g_{0,i}(\theta) = 1$, for some i there is equivalence between the auxiliary matrices $J_{i,p}(\tau)$ and $Z_{i,p}(\tau)$ hence the system (12), (13) can be further reduced.

Proof. It is easy to see that

$$\int_{-h}^{0} U(\tau+\theta+(p+1)h)d\theta = \int_{-h}^{0} U(\tau-\theta+ph)d\theta.$$

Theorem 4. Given a time delay system (1) where matrices $G_j(\theta - jh)$ are of the form (2) and satisfy (3). Then, there exists a solution

$$\begin{aligned} &Y_i(\tau), \quad -m \le i \le m-1 \\ &Z_{i,p}(\tau),\ 0 \le i \le \tilde{k}, -(m-1) \le p \le m-1, \\ &J_{i,p}(\tau),\ 0 \le i \le \tilde{k}, -m \le p \le m-2 \end{aligned}$$

of the delay free system of matrix equations (12) such that $U(\tau) = Y_0(\tau)$, $\tau \in [0,\ h]$. The boundary conditions (13) are satisfied by the solution as well.

4. ILLUSTRATIVE EXAMPLES

The Lyapunov matrix construction is carried out in this section for two examples. To show its usefulness we determine the stability region of the system with the help of the stability condition (9).

4.1 Example 1

Consider the delay linear system,

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + A_2x(t-2h) + \int_{-h}^{0} G_0(\theta)x(t+\theta)d\theta + \int_{-h}^{0} G_1(\theta-h)x(t+\theta-h)d\theta, \tag{14}$$

the kernels G_0 and G_1 have repeated scalar functions, as follows

$$G_0(\theta) = \theta e^{-a_0\theta}C_{0,0} + e^{-a_0\theta}C_{0,1},$$
$$G_1(\theta-h) = \theta e^{-a_0\theta}C_{1,0} + e^{-a_0\theta}C_{1,1},$$

$$C_{0,0} = \begin{bmatrix} 0 & 0 \\ -1.408k_0 & 0 \end{bmatrix},\ C_{0,1} = \begin{bmatrix} 0 & 0 \\ 2.7896k_0 + 1.76k_1 & 2.2k_0 \end{bmatrix},$$
$$C_{1,0} = \begin{bmatrix} 0 & 0 \\ 0 & -1.76k_0 \end{bmatrix},\ C_{1,1} = \begin{bmatrix} 0 & 0 \\ 0 & 3.487k_0 + 2.2k_1 \end{bmatrix},$$

$$\text{and} \quad A_0 = \begin{bmatrix} 0.2 & 0 \\ 6.4088k_0 + 3.4871k_1 & -2 + k_0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & k_1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where k_0 and k_1 are the design parameters. In this case $m = 2$ and $\tilde{k} = 1$, so the number of auxiliary matrices is $6(2) - 2 + (4(2) - 2) = 16$. We define 4 auxiliary matrices corresponding to multiple constant delays,

$$Y_1(\tau) = U(\tau+h), \qquad Y_{-1}(\tau) = U(\tau-h),$$
$$Y_0(\tau) = U(\tau), \qquad Y_{-2}(\tau) = U(\tau-2h),$$

We also associate 12 auxiliary matrices corresponding to the scalar functions

$$Z_{0,1}(\tau) = \int_{-h}^{0} \theta e^{-a_0\theta}U(\tau+\theta+h)d\theta,\ Z_{1,1}(\tau) = \int_{-h}^{0} e^{-a_0\theta}U(\tau+\theta+h)d\theta,$$

$$Z_{0,0}(\tau) = \int_{-h}^{0} \theta e^{-a_0\theta}U(\tau+\theta)d\theta, \quad Z_{1,-1}(\tau) = \int_{-h}^{0} e^{-a_0\theta}U(\tau+\theta-h)d\theta,$$

$$Z_{0,-1}(\tau) = \int_{-h}^{0} \theta e^{-a_0\theta}U(\tau+\theta-h)d\theta,\ Z_{1,0}(\tau) = \int_{-h}^{0} e^{-a_0\theta}U(\tau+\theta)d\theta,$$

$$J_{0,0}(\tau) = \int_{-h}^{0} \theta e^{-a_0\theta}U(\tau-\theta)d\theta,\ J_{1,-2}(\tau) = \int_{-h}^{0} e^{-a_0\theta}U(\tau-\theta-2h)d\theta,$$

$$J_{0,-1}(\tau) = \int_{-h}^{0} \theta e^{-a_0\theta}U(\tau-\theta-h)d\theta, J_{1,-1}(\tau) = \int_{-h}^{0} e^{-a_0\theta}U(\tau-\theta-h)d\theta,$$

$$J_{0,-2}(\tau) = \int_{-h}^{0} \theta e^{-a_0\theta}U(\tau-\theta-2h)d\theta,\ J_{1,0}(\tau) = \int_{-h}^{0} e^{-a_0\theta}U(\tau-\theta)d\theta.$$

By Lemma 2, we obtain the delay free system of linear matrix differential equations,

$$\begin{aligned}
Y_1'(\tau) &= Y_1(\tau)A_0 + Y_0(\tau)A_1 + Y_{-1}(\tau)A_2 + Z_{0,1}(\tau)C_{0,0}\\
&\quad + Z_{1,1}(\tau)C_{0,1} + Z_{0,0}(\tau)C_{1,0} + Z_{1,0}(\tau)C_{1,1}\\
Y_0'(\tau) &= Y_0(\tau)A_0 + Y_{-1}(\tau)A_1 + Y_{-2}(\tau)A_2 + Z_{0,0}(\tau)C_{0,0}\\
&\quad + Z_{1,0}(\tau)C_{0,1} + Z_{0,-1}(\tau)C_{1,0} + Z_{1,-1}(\tau)C_{1,1}\\
Y_{-1}'(\tau) &= -A_0^TY_{-1}(\tau) - A_1^TY_0(\tau) - A_2^TY_1(\tau) - C_{0,0}^TJ_{0,-1}(\tau)\\
&\quad - C_{0,1}^TJ_{1,-1}(\tau) - C_{1,0}^TJ_{0,0}(\tau) - C_{1,1}^TJ_{1,0}(\tau)\\
Y_{-2}'(\tau) &= -A_0^TY_{-2}(\tau) - A_1^TY_{-1}(\tau) - A_2^TY_0(\tau) - C_{0,0}^TJ_{0,-2}(\tau)\\
&\quad - C_{0,1}^TJ_{1,-2}(\tau) - C_{1,0}^TJ_{0,-1}(\tau) - C_{1,1}^TJ_{1,-1}(\tau)\\
Z_{0,1}'(\tau) &= he^{a_0h}Y_0(\tau) - Z_{1,1}(\tau) + a_0Z_{0,1}(\tau)\\
Z_{0,0}'(\tau) &= he^{a_0h}Y_{-1}(\tau) - Z_{1,0}(\tau) + a_0Z_{0,0}(\tau)\\
Z_{0,-1}'(\tau) &= he^{a_0h}Y_{-2}(\tau) - Z_{1,-1}(\tau) + a_0Z_{0,-1}(\tau)\\
J_{0,0}'(\tau) &= -he^{a_0h}Y_1(\tau) + J_{1,0}(\tau) - a_0J_{0,0}(\tau)\\
J_{0,-1}'(\tau) &= -he^{a_0h}Y_0(\tau) + J_{1,-1}(\tau) - a_0J_{0,-1}(\tau)\\
J_{0,-2}'(\tau) &= -he^{a_0h}Y_{-1}(\tau) + J_{1,-2}(\tau) - a_0J_{0,-2}(\tau)\\
Z_{1,1}'(\tau) &= Y_1(\tau) - e^{a_0h}Y_0(\tau) + a_0Z_{1,1}(\tau)\\
Z_{1,0}'(\tau) &= Y_0(\tau) - e^{a_0h}Y_{-1}(\tau) + a_0Z_{1,0}(\tau)\\
Z_{1,-1}'(\tau) &= Y_{-1}(\tau) - e^{a_0h}Y_{-2}(\tau) + a_0Z_{1,-1}(\tau)\\
J_{1,0}'(\tau) &= e^{a_0h}Y_1(\tau) - Y_0(\tau) - a_0J_{1,0}(\tau)\\
J_{1,-1}'(\tau) &= e^{a_0h}Y_0(\tau) - Y_{-1}(\tau) - a_0J_{1,-1}(\tau)\\
J_{1,-2}'(\tau) &= e^{a_0h}Y_{-1}(\tau) - Y_{-2}(\tau) - a_0J_{1,-2}(\tau). \qquad (15)
\end{aligned}$$

According to (13), the boundary conditions are,

$$\begin{aligned}
Y_1(0) &= Y_0(h)\\
Y_0(0) &= Y_{-1}(h)\\
Y_{-1}(0) &= Y_{-2}(h)\\
Z_{0,1}(0) &= Z_{0,0}(h)\\
Z_{0,0}(0) &= \int_0^h (\theta - h)e^{-a_0(\theta-h)}Y_{-1}(\theta)d\theta\\
J_{0,0}(0) &= J_{0,-1}(h)\\
J_{0,-1}(0) &= J_{0,-2}(h)\\
Z_{1,1}(0) &= Z_{1,0}(h)\\
Z_{1,0}(0) &= \int_0^h e^{-a_0(\theta-h)}Y_{-1}(\theta)d\theta\\
J_{1,0}(0) &= J_{1,-1}(h)\\
J_{1,-1}(0) &= J_{1,-2}(h)\\
Z_{0,-1}(0) &= J_{0,0}^T(h)\\
Z_{1,-1}(0) &= J_{1,0}^T(h)\\
J_{0,0}(0) &= -\int_0^h \theta e^{a_0\theta}Y_0(\theta)d\theta\\
J_{1,0}(0) &= \int_0^h e^{a_0\theta}Y_0(\theta)d\theta
\end{aligned}$$

$$\begin{aligned}
&A_0^TY_0(0) + Y_0(0)A_0 + A_1^TY_0(h) +\\
&Y_{-1}(0)A_1 + A_2^TY_1(h) + Y_{-2}(0)A_2\\
&\vdots\\
&+Z_{0,0}(0)C_{0,0} + C_{0,0}^TJ_{0,-1}(h)\\
&+Z_{1,0}(0)C_{0,1} + C_{0,1}^TJ_{1,-1}(h)\\
&+Z_{0,-1}(0)C_{1,0} + C_{1,0}^TJ_{0,0}(h) \qquad (16)\\
&+Z_{1,-1}(0)C_{1,1} + C_{1,1}^TJ_{1,0}(h) = -W.
\end{aligned}$$

We write the delay free system of differential equations (15) in vector form by means of Kronecker products properties. We represent all auxiliary matrices in the vectorized form as $y_i(\tau) = vec(Y_i(\tau))$. Then, the vectorization of system (15) is $\dot{R}(\tau) = LR(\tau)$ where

$$\begin{aligned}
R(\tau) = [&y_1(\tau),\ y_0(\tau),\ y_{-1}(\tau),\ y_{-2}(\tau),\ z_{0,1}(\tau),\\
&z_{0,0}(\tau),\ z_{0,-1}(\tau),\ j_{0,0}(\tau),\ j_{0,-1}(\tau),\ j_{0,-2}(\tau),\\
&z_{1,1}(\tau),\ z_{1,0}(\tau),\ z_{1,-1}(\tau),\ j_{1,0}(\tau),\ j_{1,-1}(\tau),\\
&j_{1,-2}(\tau)]^T.
\end{aligned}$$

$R(\tau)$ is such that $R(\tau) = e^{L\tau}R(0)$, and it follows from the boundary condition (16) that

$$R(\tau) = e^{L\tau}[M + Ne^{Lh}]^{-1}\begin{bmatrix} 0 \\ -W \end{bmatrix}. \qquad (17)$$

We obtain the real matrices L, M, and N from the vectorization process and due to space limitations, their explicit form is omitted. Using (17) we construct $U(\tau)$ for $a_0 = 0.2$, $h = 1$, and $W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For $(k_0, k_1) = (0.25, -1.75)$, the elements of $U(\tau) = Y_0(\tau)$ and $U(\tau + h) = Y_1(\tau)$, $\tau \in [0, 1]$ are shown in Figure 1. We apply the necessary condition (9) that uses the Lyapunov matrix constructed to find the stability region of system (14) in the space of parameters (k_0, k_1). The region where this condition holds is depicted on Figure 2, for $r = 1$ and $r = 4$, respectively, the continuous lines correspond to the exact stability boundaries obtained using the D-partition method introduced by Neimark (1949). No further improvement is obtained for greater r.

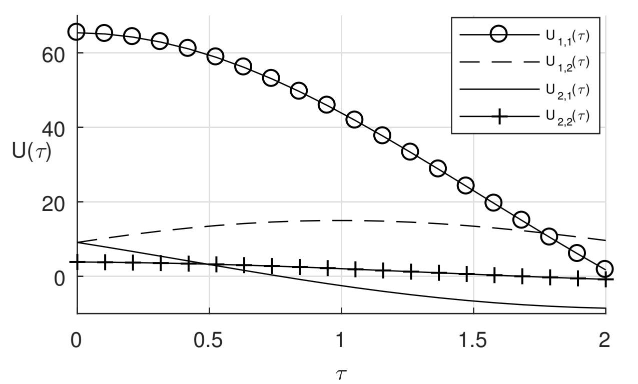


Fig. 1. Matrix $U(\tau)$ elements, example 1.

4.2 Example 2

Consider the delay linear system,

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + A_2x(t-2h) + \qquad (18)$$
$$\int_{-h}^0 G_0(\theta)x(t+\theta)d\theta + \int_{-h}^0 G_1(\theta-h)x(t+\theta-h)d\theta,$$

the kernels G_0 and G_1 have repeated scalar functions, as follows

$$G_0(\theta) = C_{0,0} + e^{\theta} C_{0,1},$$
$$G_1(\theta - h) = C_{1,0} + e^{\theta} C_{1,1},$$

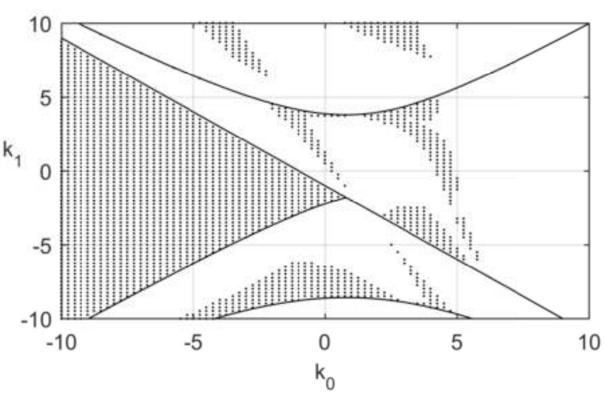
$$C_{0,0} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},\ C_{0,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$C_{1,0} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix},\ C_{1,1} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix},$$

and $A_0 = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & k_0 \end{bmatrix},\ A_1 = \begin{bmatrix} -0.8 & 0 \\ 0 & k_1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$

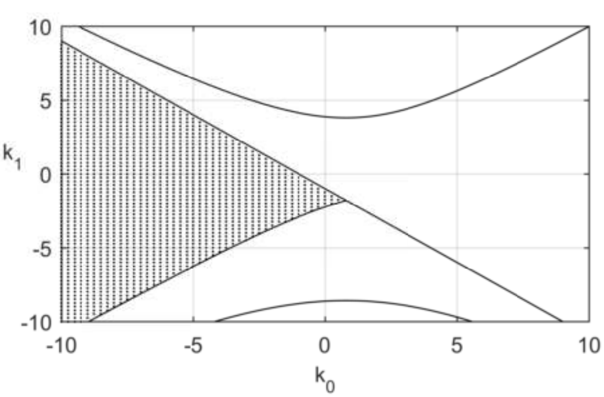
as we have done in the past example k_0 and k_1 are the design parameters. In this example $m = 2$ and $\tilde{k} = 1$ the number of auxiliary matrices is $6(2) - 2 + (4(2) - 2) = 16$. We define 4 auxiliary matrices corresponding to multiple constant delays,

$$Y_1(\tau) = U(\tau + h), \qquad Y_{-1}(\tau) = U(\tau - h),$$
$$Y_0(\tau) = U(\tau), \qquad Y_{-2}(\tau) = U(\tau - 2h),$$

and 12 auxiliary matrices corresponding to the scalar functions. Note that, there is a scalar function equal to 1, by Lemma 3, the variables are reduced to 9, as follows



(a) r=1



(b) r=4

Fig. 2. Stability region, system (14)

$$Z_{0,1}(\tau) = \int_{-h}^{0} U(\tau + \theta + h) d\theta,\ J_{1,0}(\tau) = \int_{-h}^{0} e^{\theta} U(\tau - \theta) d\theta,$$
$$Z_{0,0}(\tau) = \int_{-h}^{0} U(\tau + \theta) d\theta, \qquad J_{1,-1}(\tau) = \int_{-h}^{0} e^{\theta} U(\tau - \theta - h) d\theta,$$
$$Z_{0,-1}(\tau) = \int_{-h}^{0} U(t + \theta - h) d\theta,\ J_{1,-2}(\tau) = \int_{-h}^{0} e^{\theta} U(\tau - \theta - 2h) d\theta,$$
$$Z_{1,1}(\tau) = \int_{-h}^{0} e^{\theta} U(\tau + \theta + h) d\theta,$$
$$Z_{1,0}(\tau) = \int_{-h}^{0} e^{\theta} U(\tau + \theta) d\theta,$$
$$Z_{1,-1}(\tau) = \int_{-h}^{0} e^{\theta} U(\tau + \theta - h) d\theta.$$

By Lemma 2, we obtain the delay free system of linear matrix differential equations,

$$\begin{aligned}
Y_1'(\tau) &= Y_1(\tau)A_0 + Y_0(\tau)A_1 + Y_{-1}(\tau)A_2 + Z_{0,1}(\tau)C_{0,0} \\
&\quad + Z_{1,1}(\tau)C_{0,1} + Z_{0,0}(\tau)C_{1,0} + Z_{1,0}(\tau)C_{1,1} \\
Y_0'(\tau) &= Y_0(\tau)A_0 + Y_{-1}(\tau)A_1 + Y_{-2}(\tau)A_2 + Z_{0,0}(\tau)C_{0,0} \\
&\quad + Z_{1,0}(\tau)C_{0,1} + Z_{0,-1}(\tau)C_{1,0} + Z_{1,-1}(\tau)C_{1,1} \\
Y_{-1}'(\tau) &= -A_0^T Y_{-1}(\tau) - A_1^T Y_0(\tau) - A_2^T Y_1(\tau) - C_{0,0}^T Z_{0,0}(\tau) \\
&\quad - C_{0,1}^T J_{1,-1}(\tau) - C_{1,0}^T Z_{0,1}(\tau) - C_{1,1}^T J_{1,0}(\tau) \\
Y_{-2}'(\tau) &= -A_0^T Y_{-2}(\tau) - A_1^T Y_{-1}(\tau) - A_2^T Y_0(\tau) - C_{0,0}^T Z_{0,-1}(\tau) \\
&\quad - C_{0,1}^T J_{1,-2}(\tau) - C_{1,0}^T Z_{0,0}(\tau) - C_{1,1}^T J_{1,-1}(\tau) \\
Z_{0,1}'(\tau) &= Y_1(\tau) - Y_0(\tau) \\
Z_{0,0}'(\tau) &= Y_0(\tau) - Y_{-1}(\tau) \\
Z_{0,-1}'(\tau) &= Y_{-1}(\tau) - Y_{-2}(\tau) \\
Z_{1,1}'(\tau) &= Y_1(\tau) - e^{-h} Y_0(\tau) - Z_{1,1}(\tau) \\
Z_{1,0}'(\tau) &= Y_0(\tau) - e^{-h} Y_{-1}(\tau) - Z_{1,0}(\tau) \\
Z_{1,-1}'(\tau) &= Y_{-1}(\tau) - e^{-h} Y_{-2}(\tau) - Z_{1,-1}(\tau) \\
J_{1,0}'(\tau) &= -Y_0(\tau) + e^{-h} Y_1(\tau) + J_{1,0}(\tau) \\
J_{1,-1}'(\tau) &= -Y_{-1}(\tau) + e^{-h} Y_0(\tau) + J_{1,-1}(\tau) \\
J_{1,-2}'(\tau) &= -Y_{-2}(\tau) + e^{-h} Y_{-1}(\tau) + J_{1,-2}(\tau).
\end{aligned} \tag{19}$$

According to (13), the boundary conditions are,

$$\begin{aligned}
Y_1(0) &= Y_0(h) \\
Y_0(0) &= Y_{-1}(h) \\
Y_{-1}(0) &= Y_{-2}(h) \\
Z_{0,1}(0) &= Z_{0,0}(h) \\
Z_{0,0}(0) &= \int_0^h Y_{-1}(\theta) d\theta \\
Z_{1,1}(0) &= Z_{1,0}(h) \\
Z_{1,0}(0) &= Z_{1,-1}(h) \\
J_{1,0}(0) &= J_{1,-1}(h) \\
J_{1,-1}(0) &= J_{1,-2}(h) \\
Z_{1,-1}(0) &= J_{1,0}^T(h) \\
&\vdots
\end{aligned}$$

$$Z_{1,0}(0) = \int_0^h e^{\theta - h} Y_{-1}(\theta) d\theta$$

$$J_{1,0}(0) = \int_0^h e^{-\theta} Y_0(\theta) d\theta$$

$$\begin{aligned} & A_0^T Y_0(0) + Y_0(0) A_0 + A_1^T Y_0(h) + \\ & Y_{-1}(0) A_1 + A_2^T Y_1(h) + Y_{-2}(0) A_2 \\ & + Z_{0,0}(0) C_{0,0} + C_{0,0}^T Z_{0,0}(h) + Z_{1,0}(0) C_{0,1} \\ & + C_{0,1}^T J_{1,-1}(h) + Z_{0,-1}(0) C_{1,0} + C_{1,0}^T Z_{0,1}(h) \\ & + Z_{1,-1}(0) C_{1,1} + C_{1,1}^T J_{1,0}(h) = -W. \end{aligned} \quad (20)$$

We apply again Kronecker product of matrices and the delay free system of differential equations (19) is $\dot{R}(\tau) = LR(\tau)$ where

$$\begin{aligned} R(\tau) = [& y_1(\tau),\ y_0(\tau),\ y_{-1}(\tau),\ y_{-2}(\tau),\ z_{0,1}(\tau), \\ & z_{0,0}(\tau),\ z_{0,-1}(\tau),\ z_{1,1}(\tau),\ z_{1,0}(\tau),\ z_{1,-1}(\tau), \\ & j_{1,0}(\tau),\ j_{1,-1}(\tau), j_{1,-2}(\tau)]^T, \end{aligned}$$

with $R(\tau) = e^{L\tau} R(0)$, and replacing the boundary condition (20) the solution is

$$R(\tau) = e^{L\tau} [M + N e^{Lh}]^{-1} \begin{bmatrix} 0 \\ -W \end{bmatrix}, \quad (21)$$

for $h = 0.01$, $W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $k_0 = 50$ and $k_1 = -53$, we show the elements of $U(\tau) = Y_0(\tau)$ and $U(\tau + h) = Y_1(\tau)$, $\tau \in [0, 0.01]$ in Figure 3. The stability region of system (14) in the space of parameters (k_0, k_1) is depicted on Figure 4, for $r = 1$ and $r = 4$, respectively.

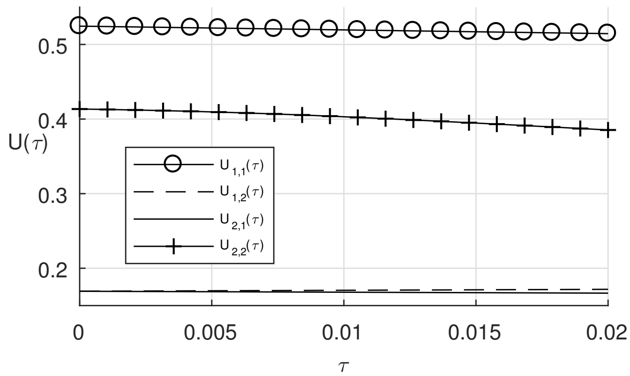
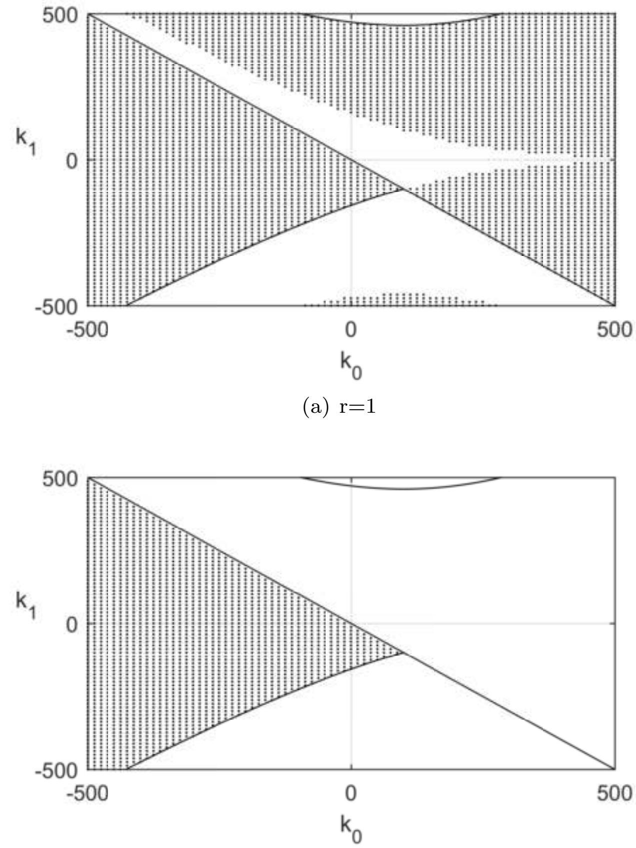


Fig. 3. Matrix $U(\tau)$ elements, example 2.

5. CONCLUSION

We show how to compute the Lyapunov matrix of a multiple distributed time-delay system with kernels including repeated scalar functions. The number of auxiliary matrices needed to define the delay free system of matrix equations and to construct the Lyapunov matrix are reduced in a meaningful way when kernels are repeated. Finally, we use this matrix to analyze the stability of the system.



(a) r=1

(b) r=4

Fig. 4. Stability region, system (18)

REFERENCES

Aliseyko, A. (2017). Lyapunov matrices for a class of time-delay systems with piecewise-constant kernel. *International Journal of Control*, 1–8.

Cuvas, C., Mondié, S., and Ochoa, G. (2015). Distributed delay systems with truncated gamma distribution: instability regions. *IFAC-PapersOnLine*, 48(12), 239–244.

Egorov, A.V., Cuvas, C., and Mondié, S. (2017). Necessary and sufficient stability conditions for linear systems with pointwise and distributed delays. *Automatica*, 80, 218–224.

Garcia-Lozano, H. and Kharitonov, V.L. (2004). Lyapunov matrices for time delay system with commensurate delays. *IFAC Proceedings Volumes*, 37(21), 91–95.

Huesca, E., Mondié, S., and Santos, O. (2009). Polynomial approximations of the Lyapunov matrix of a class of time delay systems. *IFAC Proceedings Volumes*, 42(14), 261–266.

Jarlebring, E., Vanbiervliet, J., and Michiels, W. (2009). Characterizing and computing the H_2 norm of time-delay systems by solving the delay lyapunov equation.

Kharitonov, V.L. (2006). Lyapunov matrices for a class of time delay systems. *Systems & Control Letters*, 55(7), 610–617.

Kharitonov, V.L. (2012). *Time-delay systems: Lyapunov functionals and matrices*. Springer Science & Business Media.

Neimark, J. (1949). D-subdivisions and spaces of quasi-polynomials. *Prikladnaya Matematika i Mekhanika*, 13, 349–380.

