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Stability Analysis Using Block-Triangular Decomposition for Linear Time-Delay Systems

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Abstract: In this paper, we propose a block-triangular decomposition of linear systems subject to a simple delay in the state-space realization. Under certain conditions, the proposed decomposition allows applying some stability criteria already available.

Keywords: Linear equations, time-delay, stability analysis, matrix triangularization, coordinate transformations.

In this work, we are concerned with the stability analysis of linear time-delay systems with a single delay τ, given by

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) \tag{1}$$

where $x \in \mathbb{R}^n$, $A,\ B \in \mathbb{R}^{n\times n}$, and $\tau \in \mathbb{R}_{>0}$.

The stability analysis of (1) has received a lot of attention, and several results are already available [Hayes (1950); Malek-Zavarei and Jamshidi (1987); Sakata (1998); Cahlon and Schmidt (2000, 2001); Niculescu (2001); Fridman (2014)]. If $B = 0$, then its stability can be directly established if all the eigenvalues of A lie in the left side of the complex plane. If $A = 0$, the stability of (1) can be checked using the following result.

Theorem 1. (Hara and Sugie (1996)). The zero solution of $\dot{x} = B\,x(t - \tau)$ is asymptotically stable if and only if the real eigenvalues of B denoted by $-\lambda_1, -\lambda_2, ..., -\lambda_l$ are in

$$0 < \lambda_j < \frac{\pi}{2\tau}\ (j = 1, \ldots, l); \tag{2}$$

and the complex eigenvalues of B denoted by $-\rho_k(\cos\theta_k \pm i\sin\theta_k)$ satisfy

$$0 < \rho_k\tau < \frac{\pi}{2} - |\theta_k|\ \ (k = 1, \ldots, m); \tag{3}$$

where $i^2 = -1$ and $n = l + 2m$.

In the scalar case ($n = 1$), we have the following result.

Theorem 2. (Hayes (1950)). The zero solution of (1) is asymptotically stable if and only if the system (1) is in the union of the regions

$$\{(A, B) :\ -A = B\cos\alpha,\ 0 < -B\tau\sin\alpha < \alpha,\ 0 < \alpha < \pi\}, \tag{4}$$

and

$$\{(A, B) :\ A < -B \leq -A\}\,. \tag{5}$$

Even though the general case is still an open problem, there exists a necessary and sufficient condition for checking the stability of a particular class of systems of the form (1), for which a nonsingular matrix solution $Q(0)$ of the following nonlinear algebraic matrix equation exists:

$$[\exp(A + Q(0))\ \tau]Q(0) = B. \tag{6}$$

Theorem 3. (Malek-Zavarei and Jamshidi (1987)). Consider a system of the form (1), for which $Q(0) \in \mathbb{R}^{n\times n}$ is a nonsingular solution of (6). Then a necessary and sufficient condition for asymptotic stability of (1) is that

$$\left\|\lambda_i\{BQ^{-1}(0)\}\right\| < 1, \tag{7}$$

for all $i = 1, \ldots, n$.

Remark 1. Note that this is not a general solution since there does not always exist a solution to the equation (6) for systems whose origin is asymptotically stable. For example, the system

$$\dot{x}(t) = 0.5x(t) - x(t - 1), \tag{8}$$

is asymptotically stable but (6) has no solution $Q(0)$.

In this work, we propose a block-triangular decomposition that can be used to analyze the stability of some higher-dimensional linear time-delay systems, by decomposing it in low-order subsystems for which stability criteria are already available.

The main idea is to first consider the delayed state as a control input, and if the pair (A, B) is not controllable, then there exists a transformation $T_1^{-1}x(t) = \begin{bmatrix} x_1(t) \\ z_1(t) \end{bmatrix}$ that decomposes the system as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{z}_1(t) \end{bmatrix} = \begin{bmatrix} A_3 & A_2 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ z_1(t) \end{bmatrix} + \begin{bmatrix} B_3 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ z_1(t-\tau) \end{bmatrix}, \quad (9)$$

resulting in a non-delayed dynamics (which would correspond to z_1), which is analyzed by computing the eigenvalues of A_1, and a mixed-dynamics part (which correspond to x_1). Such T_1 can be constructed using a basis of the controllable matrix $[B\ AB ... A^{n-1}B]$ and the required vectors to complete an invertible matrix, as it is the standard procedure to separate the controllable and uncontrollable subsystems. In case that the pair (A, B) is controllable, the subsystem z_1 has zero dimension, and the transformation T_1 is the identity matrix. It should be noted that the dynamics of z_1 does not depend on the delay, and its stability can be tested directly. If z_1 is asymptotically stable, the stability of (9) is equivalent to the stability of

$$\dot{x}_1(t) = A_3 x_1(t) + B_3 x_1(t-\tau). \quad (10)$$

It is known that the pair (A_3, B_3) is controllable, but now it is the non-delayed state $x_1(t)$ that can be considered as a control input. If the pair (B_3, A_3) is not controllable, then there exists a transformation $\bar{T}_1^{-1}x_1 = \begin{bmatrix} \xi_1(t) \\ \eta_1(t) \end{bmatrix}$ that decomposes the system as

$$\begin{bmatrix} \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\eta}_1(t) \end{bmatrix} \\ \dot{z}_1(t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{\xi_1} & A_{\eta_1} \\ 0 & 0 \end{bmatrix} & A_2 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi_1(t) \\ \eta_1(t) \end{bmatrix} \\ z_1(t) \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} B_{\xi_1} & B_{\eta_1} \\ 0 & B_1 \end{bmatrix} & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi_1(t-\tau) \\ \eta_1(t-\tau) \end{bmatrix} \\ z_1(t-\tau) \end{bmatrix}. \quad (11)$$

If z_1 is asymptotically stable, the stability of (11) is determined by analyzing a pure-delayed dynamics given by

$$\dot{\eta_1}(t) = B_1 \eta_1(t-\tau), \quad (12)$$

and a mixed-dynamics part

$$\dot{\xi_1}(t) = A_{\xi_1}\xi_1(t) + B_{\xi_1}\xi_1(t-\tau). \quad (13)$$

This process is repeated until no further decomposition is possible which results in a block-triangular form that is composed by non-delayed dynamics, pure-delayed dynamics, and a mixed-dynamics part. In this way the system (1) has been decomposed in a simpler form for the stability analysis. This result is summarized in the following algorithm.

Decomposition algorithm for stability analysis

The system $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ can be decomposed in a simpler form for the stability analysis with the following procedure.

Step 1.

- If the pair (A, B) is controllable, define $x_1(t) = x(t)$. Otherwise, if the pair (A, B) is not controllable there exists a transformation $\begin{bmatrix} x_1(t) \\ z_1(t) \end{bmatrix} = T_1^{-1}x(t)$ that decomposes the system as
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{z}_1(t) \end{bmatrix} = \begin{bmatrix} A_{3,1} & A_{2,1} \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ z_1(t) \end{bmatrix} + \begin{bmatrix} B_{3,1} & B_{2,1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ z_1(t-\tau) \end{bmatrix}.$$
whose stability is equivalent to the stability of
$$\begin{aligned} \dot{x}_1(t) &= A_{3,1}x_1(t) + B_{3,1}x_1(t-\tau), \\ \dot{z}_1(t) &= A_1 z_1(t), \end{aligned} \quad (14)$$
where $z_1 \in \mathbb{R}^{s_1}$, $A_1 \in \mathbb{R}^{s_1 \times s_1}$, and $x_1 \in \mathbb{R}^{\ell_1}$, $A_{3,1}, B_{3,1} \in \mathbb{R}^{(\ell_0 - s_1)\times(\ell_0 - s_1)}$, with $\ell_0 = n$.
If $\dim(z_1) > 0$, and A_1 is not Hurwitz, the system (14) is not stable. Stop.
- If $\dim(x_1) < 2$, or the pair $(B_{3,1}, A_{3,1})$ is controllable, then the algorithm has converged. Set $\xi_1(t) = x_1(t)$ and stop.
- Otherwise, if the pair $(B_{3,1}, A_{3,1})$ is not controllable there exists a transformation $\bar{T}_1^{-1}x_1 = \begin{bmatrix} \xi_1(t) \\ \eta_1(t) \end{bmatrix}$ that decomposes system (14) as
$$\begin{bmatrix} \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\eta}_1(t) \end{bmatrix} \\ \dot{z}_1(t) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{\xi_1} & A_{\eta_1} \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi_1(t) \\ \eta_1(t) \end{bmatrix} \\ z_1(t) \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} B_{\xi_1} & B_{\eta_1} \\ 0 & B_1 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi_1(t-\tau) \\ \eta_1(t-\tau) \end{bmatrix} \\ z_1(t-\tau) \end{bmatrix}, \quad (15)$$
whose stability is, in turn, equivalent to the stability of
$$\begin{aligned} \dot{\xi}_1(t) &= A_{\xi_1}\xi_1(t) + B_{\xi_1}\xi_1(t-\tau), \\ \dot{\eta}_1(t) &= B_1\eta_1(t-\tau), \\ \dot{z}_1(t) &= A_1 z_1(t), \end{aligned} \quad (16)$$
where $\eta_1 \in \mathbb{R}^{p_1}$, $B_1 \in \mathbb{R}^{p_1 \times p_1}$, and $\xi_1 \in \mathbb{R}^{\ell_1}$, $A_{\xi_1}, B_{\xi_1} \in \mathbb{R}^{\ell_1 \times \ell_1}$, with $\ell_1 = \ell_0 - s_1 - p_1$.

Step k.

- From Step $k-1$ we have defined
$$\begin{aligned} \dot{\xi}_{k-1}(t) &= A_{\xi_{k-1}}\xi_{k-1}(t) + B_{\xi_{k-1}}\xi_{k-1}(t-\tau), \\ \dot{\eta}_i(t) &= B_i\eta_i(t-\tau), \\ \dot{z}_i(t) &= A_i z_i(t) \qquad i = 1, \dots, k-1. \end{aligned} \quad (17)$$
where $\eta_i \in \mathbb{R}^{p_i}$, $B_i \in \mathbb{R}^{p_i \times p_i}$, $z_i \in \mathbb{R}^{s_i}$, $A_i \in \mathbb{R}^{s_i \times s_i}$ and $\xi_{k-1} \in \mathbb{R}^{\ell_{k-1}}$, $A_{\xi_{k-1}}, B_{\xi_{k-1}} \in \mathbb{R}^{\ell_{k-1} \times \ell_{k-1}}$.
- If $\dim(\xi_{k-1}) < 2$ or the pair $(A_{\xi_{k-1}}, B_{\xi_{k-1}})$ is controllable, then the algorithm has converged. Stop.
- Otherwise, if the pair $(A_{\xi_{k-1}}, B_{\xi_{k-1}})$ is not controllable there exists a transformation $T_k^{-1}\xi_{k-1}(t) = \begin{bmatrix} x_k(t) \\ z_k(t) \end{bmatrix}$ that decomposes the system (17) as

$$\begin{bmatrix}\dot{x}_k(t)\\ \dot{z}_k(t)\end{bmatrix}=\begin{bmatrix}A_{3,k} & A_{2,k}\\ 0 & A_k\end{bmatrix}\begin{bmatrix}x_k(t)\\ z_k(t)\end{bmatrix}+\begin{bmatrix}B_{3,k} & B_{2,k}\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_k(t-\tau)\\ z_k(t-\tau)\end{bmatrix}$$
$$\begin{aligned}\dot{\eta}_i(t) &= B_i\eta_i(t-\tau),\\ \dot{z}_i(t) &= A_i z_i(t) \qquad\qquad i=1,\ldots,k-1,\end{aligned}$$

whose stability is equivalent to the stability of

$$\begin{aligned}\dot{x}_k(t) &= A_{3,k}x_k(t)+B_{3,k}x_k(t-\tau),\\ \dot{z}_k(t) &= A_k z_k(t),\\ \dot{\eta}_i(t) &= B_i\eta_i(t-\tau),\\ \dot{z}_i(t) &= A_i z_i(t) \qquad\qquad i=1,\ldots,k-1.\end{aligned}$$

where $z_k \in \mathbb{R}^{s_k}$, $A_k \in \mathbb{R}^{s_k\times s_k}$ and $x_k \in \mathbb{R}^{\ell_{k-1}-s_k}$, $A_{3,k}, B_{3,k} \in \mathbb{R}^{(\ell_{k-1}-s_k)\times(\ell_{k-1}-s_k)}$.

- If $\dim(x_k) < 2$ or the pair $(B_{3,k}, A_{3,k})$ is controllable then the algorithm has converged. Set $\xi_k(t) = x_k(t)$ and stop.
- Otherwise, if the pair $(B_{3,k}, A_{3,k})$ is not controllable there exists a transformation $\bar{T}_k^{-1}x_k(t) = \begin{bmatrix}\xi_k(t)\\ \eta_k(t)\end{bmatrix}$ that decomposes the system (18) as

$$\begin{bmatrix}\dot{\xi}_k(t)\\ \dot{\eta}_k(t)\end{bmatrix}=\begin{bmatrix}A_{\xi_k} & A_{\eta_k}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi_k(t)\\ \eta_k(t)\end{bmatrix}+\begin{bmatrix}B_{\xi_k} & B_{\eta_k}\\ 0 & B_k\end{bmatrix}\begin{bmatrix}\xi_k(t-\tau)\\ \eta_k(t-\tau)\end{bmatrix}$$
$$\begin{aligned}\dot{z}_k(t) &= A_k z_k(t),\\ \dot{\eta}_i(t) &= B_i\eta_i(t-\tau),\\ \dot{z}_i(t) &= A_i z_i(t) \qquad\qquad i=1,\ldots,k-1,\end{aligned}$$

whose stability is equivalent to the stability of

$$\dot{\xi}_k(t) = A_{\xi_k}\xi_k(t)+B_{\xi_k}\xi_k(t-\tau), \tag{18}$$
$$\dot{\eta}_i(t) = B_i\eta_i(t-\tau), \tag{19}$$
$$\dot{z}_i(t) = A_i z_i(t) \qquad\qquad i=1,\ldots,k. \tag{20}$$

where $\eta_i \in \mathbb{R}^{p_i}$, $B_i \in \mathbb{R}^{p_i\times p_i}$, $z_i \in \mathbb{R}^{s_i}$, $A_i \in \mathbb{R}^{s_i\times s_i}$ and $\xi_k \in \mathbb{R}^{\ell_k}$, $A_{\xi_k}, B_{\xi_k} \in \mathbb{R}^{\ell_k\times\ell_k}$, with $\ell_k = \ell_{k-1} - s_k - p_k$.

The algorithm will end in a finite number of steps $n^\star \le n$ since at every step, either the dimension of ξ_k reduces, or the algorithm converges.

Now, we can state our main result.

Theorem 4. The asymptotic stability of the zero solution of the system (1) is equivalent to the asymptotic stability of the zero solution of the associated system (18)–(20).

The proof can be directly obtained from the algorithm.

Remark 2. The stability of (19) can be tested using Theorem 1, while (20) is stable if and only if all A_i are Hurwitz.

Remark 3. If $\dim(\xi_{n^\star}) = 1$, the stability of (18) can be tested using Theorem 2. If $n > \dim(\xi_{n^\star}) > 1$, then the stability test will be simpler to test than the one of system (1).

Remark 4. If $\dim(\xi_{n^\star}) > 1$, the stability of (18) can be tested using Theorem 3 if there exists a solution of (6) for $Q(0)$.

This result is now illustrated using some academic examples.

Example 1. Let us consider the system

$$\dot{x}(t) = Ax(t) + Bx(t-1), \tag{21}$$

with

$$A = \begin{bmatrix}0 & 0 & 1\\ -18 & 3 & 2\\ -24 & 4 & -5\end{bmatrix}$$

and

$$B = \begin{bmatrix}1 & 1 & 1\\ 4 & 5 & 4\\ -2 & -1 & -2\end{bmatrix}.$$

Since the pair (A, B) is not controllable, we define the transformation $T_1^{-1}x = \begin{bmatrix}x_1(t)\\ z_1(t)\end{bmatrix}$ with

$$T_1 = \begin{bmatrix}1 & 1 & 0\\ 4 & 5 & 0\\ -2 & -1 & 1\end{bmatrix},$$

that decomposes the system (21) as

$$\left[\begin{array}{c}\dot{x}_1(t)\\ \hline \dot{z}_1(t)\end{array}\right]=\left[\begin{array}{cc|c}0 & 0 & 3\\ -2 & -1 & -2\\ \hline 0 & 0 & -1\end{array}\right]\left[\begin{array}{c}x_1(t)\\ \hline z_1(t)\end{array}\right]+\left[\begin{array}{cc|c}-1 & 0 & 1\\ 4 & 5 & 0\\ \hline 0 & 0 & 0\end{array}\right]\left[\begin{array}{c}x_1(t-1)\\ \hline z_1(t-1)\end{array}\right].$$

Where $A_{3,1} = \begin{bmatrix}0 & 0\\ -2 & -1\end{bmatrix}$ and $B_{3,1} = \begin{bmatrix}-1 & 0\\ 4 & 5\end{bmatrix}$, then the pair $(A_{3,1}, B_{3,1})$ is controllable, but we can verify the controllability of the pair $(B_{3,1}, A_{3,1})$, which is not controllable. Now define the transformation $\bar{T}_1^{-1}x_1 = \begin{bmatrix}\xi_1(t)\\ \eta_1(t)\end{bmatrix}$ with

$$\bar{T}_1 = \begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix}.$$

So the system with this transformation is

$$\begin{bmatrix}\dot{\xi}_1(t)\\ \dot{\eta}_1(t)\\ \dot{z}_1(t)\end{bmatrix}=\begin{bmatrix}-1 & 1 & 1\\ 0 & 0 & 3\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \eta_1(t)\\ z_1(t)\end{bmatrix}+\begin{bmatrix}5 & -2 & 0\\ 0 & -1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1(t-1)\\ \eta_1(t-1)\\ z_1(t-1)\end{bmatrix}. \tag{22}$$

whose stability is equivalent to

$$\begin{aligned}\dot{\xi}_1(t) &= -\xi_1(t)+5\xi_1(t-1),\\ \dot{\eta}_1(t) &= -\eta_1(t-1),\\ \dot{z}_1(t) &= -z_1(t),\end{aligned} \tag{23}$$

Beginning the stability analysis, the state z_1 is asymptotically stable. Then, the state η_1 according with Theorem 1, is asymptotically stable. The state ξ_1 does not comply with Theorem 2 so the system (21) is not asymptotically stable.

Example 2. Consider the system

$$\dot{x}(t) = Ax(t) + Bx(t-2); \tag{24}$$

where

$$A = \begin{bmatrix}-7 & -2 & 3\\ -5 & -4 & 3\\ -5 & -4 & 3\end{bmatrix},$$

$$B = \begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 1\\ -0.25 & 1.75 & 0.5\end{bmatrix}.$$

The pair (A,B) is not controllable, so we define a transformation $T_1^{-1}x = \begin{bmatrix} x_1(t) \\ z_1(t) \end{bmatrix}$ with

$$T_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ -0.25 & 1.75 & 0 \end{bmatrix},$$

that decomposes the system (24) as

$$\left[\begin{array}{c} \dot{x}_1(t) \\ \hline \dot{z}_1(t) \end{array}\right] = \left[\begin{array}{cc|c} -2.25 & -11.25 & -15 \\ -0.75 & -3.75 & -5 \\ \hline 0 & 0 & -2 \end{array}\right] \left[\begin{array}{c} x_1(t) \\ \hline z_1(t) \end{array}\right] +$$

$$\left[\begin{array}{cc|c} -1.25 & 9.75 & 1 \\ -0.25 & 2.75 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1(t-2) \\ \hline z_1(t-2) \end{array}\right]. \tag{25}$$

Where $A_{3,1} = \begin{bmatrix} -2.25 & -11.25 \\ -0.75 & -3.75 \end{bmatrix}$ and $B_{3,1} = \begin{bmatrix} -1.25 & 9.75 \\ -0.25 & 2.75 \end{bmatrix}$, then the pair $(A_{3,1}, B_{3,1})$ is controllable, but the pair $(B_{3,1}, A_{3,1})$ is not. So a transformation $\bar{T}_1^{-1}x_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix}$ with

$$\bar{T}_1 = \begin{bmatrix} -2.25 & 1 \\ -0.75 & 0 \end{bmatrix},$$

decomposes the system (25) as

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\eta}_1(t) \\ \dot{z}_1(t) \end{bmatrix} = \begin{bmatrix} -6 & 1 & 6.6667 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \eta_1(t) \\ z_1(t) \end{bmatrix} + \begin{bmatrix} 2 & 0.33 & 0 \\ 0 & -0.5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1(t-2) \\ \eta_1(t-2) \\ z_1(t-2) \end{bmatrix}.$$

whose stability is equivalent to

$$\begin{aligned} \dot{\xi}_1(t) &= -6\xi_1(t) + 2\xi_1(t-2), \\ \dot{\eta}_1(t) &= -0.5\eta_1(t-2), \\ \dot{z}_1(t) &= -z_1(t). \end{aligned} \tag{26}$$

The state z_1 is asymptotically stable, then the state η_1 according with Theorem 1 is asymptotically stable, and finally, the state ξ_1 complies with Theorem 2 so the system (24) is asymptotically stable.

Example 3. For the system

$$\dot{x}(t) = Ax(t) + Bx(t-1); \tag{27}$$

with

$$A = \begin{bmatrix} -2 & -1 & -1 \\ -2 & -1 & -1 \\ -1.5 & 0.5 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.5 & 1.5 & -1 \\ 2.5 & 0.5 & -1 \\ 3.5 & -0.5 & -1 \end{bmatrix}.$$

In this case the pair (A,B) is controllable, so the transformation T_1 is the identity matrix in $T_1^{-1}x = \begin{bmatrix} x_1 \\ z_1 \end{bmatrix}$. The subsystem z_1 is of dimension zero so the system (27) stays as

$$\dot{x}_1(t) = \begin{bmatrix} -2 & -1 & -1 \\ -2 & -1 & -1 \\ -1.5 & 0.5 & 3 \end{bmatrix} x_1(t) + \begin{bmatrix} 1.5 & 1.5 & -1 \\ 2.5 & 0.5 & -1 \\ 3.5 & -0.5 & -1 \end{bmatrix} x_1(t-1), \tag{28}$$

where $A_{3,1} = A$ and $B_{3,1} = B$, then the pair $(A_{3,1}, B_{3,1})$ is controllable, but the pair (B_3, A_3) is not. So there exists a transformation $\bar{T}_1^{-1}x_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix}$ with

$$\bar{T}_1 = \begin{bmatrix} -2 & -1 & 1 \\ -2 & -1 & 0 \\ -1.5 & 0.5 & 0 \end{bmatrix},$$

that decomposes the system (28) as

$$\left[\begin{array}{c} \dot{\xi}_1(t) \\ \hline \dot{\eta}_1(t) \end{array}\right] = \left[\begin{array}{cc|c} -4.1 & -0.3 & -1 \\ 0.7 & -1.9 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \xi_1(t) \\ \hline \eta_1(t) \end{array}\right] +$$

$$\left[\begin{array}{cc|c} 2.7 & 2.1 & -1.9 \\ -0.9 & -0.7 & 1.3 \\ \hline 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} \xi_1(t-1) \\ \hline \eta_1(t-1) \end{array}\right], \tag{29}$$

where $A_{\xi_1} = \begin{bmatrix} -4.1 & -0.3 \\ 0.7 & -1.9 \end{bmatrix}$ and $B_{\xi_1} = \begin{bmatrix} 2.7 & 2.1 \\ -0.9 & -0.7 \end{bmatrix}$, then the pair (B_{ξ_1}, A_{ξ_1}) is controllable, but the pair (A_{ξ_1}, B_{ξ_1}) is not. So a transformation $T_2^{-1}\xi_1 = \begin{bmatrix} x_2 \\ z_2 \end{bmatrix}$ with

$$T_2 = \begin{bmatrix} 2.7 & 1 \\ -0.9 & 0 \end{bmatrix},$$

can decompose the system (29) as

$$\begin{aligned} \begin{bmatrix} \dot{x}_2(t) \\ \dot{z}_2(t) \\ \dot{\eta}_1(t) \end{bmatrix} &= \begin{bmatrix} -4 & -0.7778 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2(t) \\ z_2(t) \\ \eta_1(t) \end{bmatrix} + \\ & \begin{bmatrix} 2 & 1 & -1.4444 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_2(t-1) \\ z_2(t-1) \\ \eta_1(t-1) \end{bmatrix}. \end{aligned} \tag{30}$$

This way the stability of

$$\begin{aligned} \dot{\xi}_2(t) &= -4x_2(t) + 2x_2(t-1) \\ \dot{z}_2(t) &= -2z_2(t), \\ \dot{\eta}_1(t) &= -\eta_1(t-1). \end{aligned} \tag{31}$$

is equivalent to the stability of the system (30). The state η_1 complies Theorem 1 so is asymptotically stable, then the state z_2 is asymptotically stable too. Then $\dim(x_2) < 2$ so $\xi_2 = x_2$ on (31) and it complies with Theorem 2 so the system (27) is asymptotically stable.

REFERENCES

Cahlon, B. and Schmidt, D. (2001). Asymptotic stability of linear delay differential equations. *Dynamic Systems and Applications*, 10(1), 63–88.

Cahlon, B. and Schmidt, D. (2000). On stability of systems of delay differential equations. *Journal of computational and applied mathematics*, 117(2), 137–158.

Fridman, E. (2014). *Introduction to time-delay systems: Analysis and control.* Springer.

Hara, T. and Sugie, J. (1996). Stability region for systems of differential-difference equations. *Funkcialaj Ekvacioj*, 39(1), 69–86.

Hayes, N. (1950). Roots of the transcendental equation associated with a certain difference-differential equation. *Journal of the London Mathematical Society*, 1(3), 226–232.

Malek-Zavarei, M. and Jamshidi, M. (1987). *Time-Delay Systems: Analysis, Optimization and Applications*. Elsevier Science Inc., New York, NY, USA.

Niculescu, S.I. (2001). *Delay effects on stability: a robust control approach*, volume 269. Springer Science & Business Media.

Sakata, S. (1998). Asymptotic stability for a linear system of differential-difference equations. *Funkcialaj Ekvacioj Serio Internacia*, 41, 435–450.

