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Quad-rotor Control Using Velocity Field Method

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Abstract: The purpose of this work is to show the control of a quad-rotor aircraft using the velocity field method, which is a suitable strategy to generate the trajectory that the quad-rotor must follow. The proposed field is two-dimensional, because of that, we must impose some restrictions: the altitude (z) and the yaw angle (ψ) are fixed at a desired value, hence the system is restricted to behave as two independent systems of four integrators in cascade for x and y coordinates. The nested saturation controller is used to track the desired velocities in both: $x-\theta$ and $y-\phi$ subsystems. This controller is exponentially stable, thus, a correct tracking could be guaranteed. Simulations results show the effectiveness of this proposal.

Keywords: quad-rotor, tracking trajectory, velocity field.

1. INTRODUCTION

Quad-rotors commonly referred as *drones* are unmanned aerial vehicles (UAV) that have taken importance in the scientific community because they are a good test base for the implementations of several control techniques. One of the advantages of this aircraft is the size, making them suitable to work in a laboratory under desired conditions.

This work proposes a velocity field method strategy, where the UAV follows a predefined vector field, thus regulating the velocity the vehicle reaches the desired trajectory. This method has been applied mostly in robotic arms as shown by Moreno & R.Kelly (2003)-Moreno (2007), in Fukui & Wada (2016) for a therapeutic $n-$link manipulator, using the passive velocity field controller developed by Li (1995) , another application is developed in Pérez-D'Arpino et al. (2008) using a vision system the desired trajectory is generated with the objective to avoid obstacles, also in Narikiyo & Kawanishi (2017) the method is used for an exoskeleton. We present an easy way to generate the velocity flow. It consists on the weighted sum of two fields, namely: the approaching field that points from any position to the closest point of the desired trajectory; and the tangential field, that is anywhere tangential to the closest point.

In order to guarantee the vehicle to follow the proposed velocity field, we have used nested saturation control. Contributions to this field were made by several authors, just to mention, a stabilization of a quad-rotor is shown in Castillo et al. (2004)-Sanchez et al. (2008) implementing the nested saturation controller Teel (1992), quad-rotor formations are presented in García-Delgado et al. (2012) implementing the well know potential field method and obstacle avoidance with the same technique is achieved in García-Delgado & Dzul (2009).

In the following sections we focus in the velocity field to control the movement of the quad-rotor in Euclidean plane, in Section 2 the dynamic model of a six degrees of freedom body representing the quad-rotor is determined, the control strategy is developed in Section 3 using Proportional-Derivative (PD) controllers for altitude and yaw and the nested saturation algorithm for the roll and pitch angles, in Section 4 we present the proposed velocity field to be followed, whereas the results of the implementation of the velocity field with nested saturation controller are shown and discussed in Section 5, finally the conclusions of the paper and an appendix showing derivatives of the velocity function, are presented.

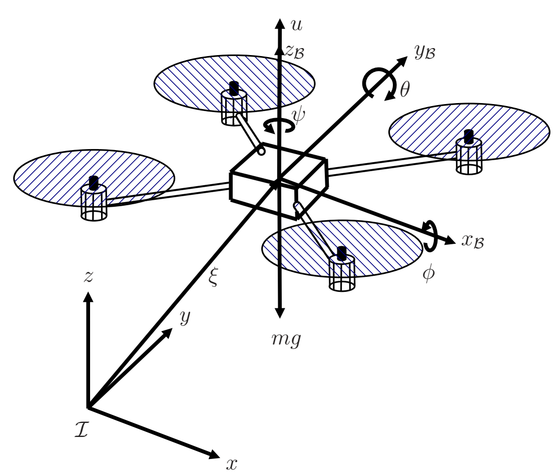

Fig. 1. Quad-rotor helicopter outline

2. DYNAMIC MODEL

The Euler-Lagrange formalism is implemented for the dynamic model of the quad-rotor. This model is obtained representing the aircraft as a rigid body in space attached to one force and three torques. The motors dynamics and the propeller flexibility are despised.

Consider a quad-rotor helicopter as shown in Figure 1. The generalized coordinates of the system are

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix}$$

where $\boldsymbol{\xi} = (x, y, z)^T \in \mathbb{R}^3$ denotes the center of mass position of the quad-rotor, relative to the inertia frame $\mathcal{I}$, and $\boldsymbol{\eta} = (\phi, \theta, \psi) \in \mathbb{R}^3$ represents the orientation of the body frame, expressed in Euler's angles.

The dynamic model of the quad-rotor is:

$$m\ddot{x} = u(c_\psi s_\theta c_\phi + s_\psi s_\phi) \quad (1)$$
$$m\ddot{y} = -u(c_\psi s_\phi - s_\psi s_\theta c_\phi) \quad (2)$$
$$m\ddot{z} = uc_\theta c_\phi - mg \quad (3)$$
$$\ddot{\phi} = \tilde{\tau_\phi} \quad (4)$$
$$\ddot{\theta} = \tilde{\tau_\theta} \quad (5)$$
$$\ddot{\psi} = \tilde{\tau_\psi} \quad (6)$$

with m being the mass of the vehicle, g the gravitational constant, $s_a = \sin a$, $c_a = \cos a$, and

$$\tilde{\boldsymbol{\tau}} = [\tilde{\tau_\phi}, \tilde{\tau_\theta}, \tilde{\tau_\psi}]^T = \mathbb{J}^{-1}(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\dot{\boldsymbol{\eta}})$$

where $\tilde{\boldsymbol{\tau}}$ is a vector with the quad-rotor torques, $\mathbb{J}$ represents the inertial matrix and $\boldsymbol{C}$ stands for the Coriolis matrix.

3. CONTROL STRATEGY

For the aircraft control we assume that the altitude an yaw angles are fixed at some desired value.

3.1 Altitude and Yaw Control

Let us define $\psi_d = 0 \ \forall \ t > 0$, such that, the quad-rotor maintains a zero degree yaw angle during the flight. The control is obtained through

$$\tilde{\tau}_\psi = -k_{p\psi}\tilde{\psi} - k_{v\psi}\dot{\psi} \quad (7)$$

where $\tilde{\psi} = (\psi - \psi_d)$, $k_{p\psi}$ and $k_{v\psi}$ denote the proportional and derivative constants of the PD controller.

In a similar way the control of a constant altitude can be achieved applying the control input.

$$u = (r_1 + mg)\frac{1}{\cos\theta\cos\phi}$$

where $r_1 = -k_{pz}\tilde{z} - k_{vz}\dot{z}$ we obtain

$$u = [(-k_{pz}\tilde{z} - k_{vz}\dot{z}) + mg]\frac{1}{\cos\theta\cos\phi} \quad (8)$$

where $\tilde{z} = z_d - z$ represents the altitude error, k_{pz} and k_{vz} are positive constants related to the PD controller. The roll and pitch angles (ϕ, θ) must be small to avoid singularities in (8), these angles are obtained by the nested saturation controller.

3.2 Roll and Pitch Control

To stabilize the quad-rotor, the nested saturation controller is used. This method was proposed by Teel (1992). It is used to stabilize a chain of integrators in cascade, and is exponentially stable for both, regulation and tracking trajectory.

Consider the quad-rotor model (1)-(6), under the control inputs (8) and (7). After a finite time, $z \to z_d$, and $\dot{z} = \psi = \dot{\psi} = 0$, and considering to be restricted to small angles θ and ϕ, such that $\sin(\phi) \approx \phi$ and $\cos(\phi) \approx 1$, and so for θ. The system (1)-(6) can be seen as a reduced system:

$$\ddot{x} = g\theta \quad (9)$$
$$\ddot{\theta} = \tilde{\tau}_\theta \quad (10)$$
$$\ddot{y} = -g\phi \quad (11)$$
$$\ddot{\phi} = \tilde{\tau}_\phi \quad (12)$$

where both subsystems are integrators in cascade

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ g\theta \\ \dot{\theta} \\ \tilde{\tau_\theta} \end{bmatrix} \quad \text{and} \quad \frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \\ \phi \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -g\phi \\ \dot{\phi} \\ \tilde{\tau_\phi} \end{bmatrix} \quad (13)$$

Let us call the subsystems state variables $\boldsymbol{x}$ and $\boldsymbol{y}$, so that

$$\boldsymbol{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} y \\ \dot{y} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

With the purpose to achieve tracking trajectory, it is necessary to stabilize the error of each variable, this is

$$\boldsymbol{x_e} = \begin{bmatrix} e_x \\ \dot{e}_x \\ e_\theta \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} x - x_d \\ \dot{x} - \dot{x}_d \\ \theta - \theta_d \\ \dot{\theta} - \dot{\theta}_d \end{bmatrix} \quad \text{and} \quad \boldsymbol{y_e} = \begin{bmatrix} e_y \\ \dot{e}_y \\ e_\phi \\ \dot{e}_\phi \end{bmatrix} = \begin{bmatrix} y - y_d \\ \dot{y} - \dot{y}_d \\ \phi - \phi_d \\ \dot{\phi} - \dot{\phi}_d \end{bmatrix} \quad (14)$$

Each subsystem can be represented as $\dot{\boldsymbol{x}} = \boldsymbol{A}_x\boldsymbol{x} + \boldsymbol{B}_x u$, with $\boldsymbol{x} \in \mathbb{R}^n, u \in \mathbb{R}$, then a linear transformation exist $\boldsymbol{z}_x = \boldsymbol{T}_{zx}\boldsymbol{x_e}$ that maps (13) in $\dot{\boldsymbol{z}} = \boldsymbol{A}_z\boldsymbol{z} + \boldsymbol{B}_z u$, where

$$\boldsymbol{A}_z = \begin{bmatrix} 0 & k_2 & k_3 & \dots & k_n \\ 0 & 0 & k_3 & \dots & k_n \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & k_n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \boldsymbol{B}_z = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

where the elements $k_1, \dots, k_n \neq 0 \in \mathbb{R}$.

In this particular case the transformation matrix elements $\boldsymbol{T}_{zx}$ are:

$$\boldsymbol{T}_{zx} = \begin{bmatrix} \frac{k_2k_3k_4}{g} & \frac{k_2k_3 + k_2k_4 + k_3k_4}{g} & k_2 + k_3 + k_4 & 1 \\ 0 & \frac{k_3k_4}{g} & k_3k_4 & 1 \\ 0 & 0 & k_4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (15)$$

and the vector $\boldsymbol{x}_e$ is that from (14). In a similar way $\boldsymbol{z}_y = \boldsymbol{T}_{zy}\boldsymbol{y}$, is defined $\boldsymbol{T}_{zy}$ as:

$$
\boldsymbol{T}_{zy} = \begin{bmatrix} -\dfrac{k_2k_3k_4}{g} & -\dfrac{k_2k_3+k_2k_4+k_3k_4}{g} & k_2+k_3+k_4 & 1 \\ 0 & -\dfrac{k_3k_4}{g} & k_3k_4 & 1 \\ 0 & 0 & k_4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (16)
$$

And the vector $\boldsymbol{y}_e$ is given in (14), where the control law according the conventional nested saturation function, for the case of thacking trajectory is

$$
u = x_d^{(n)} - \sigma_{b_n}(k_n z_n + \sigma_{b_{n-1}}(k_{n-1}z_{n-1} + \cdots + \sigma_{b_1}(k_1 z_1)))
$$

with

$$
\sigma_{b_n}(s) = \begin{cases} -b, & \text{if } s < -b \\ s, & \text{if } -b \leq s \leq b \\ b, & \text{if } s > b \end{cases}
$$

where b is a positive constant, therefore, the controller for the roll angle is

$$
\tilde{\tau}_\phi = \ddot{\phi}_d - \sigma_{b4}(k_4 z_{4x} + \sigma_{b3}(k_3 z_{3x} + \sigma_{b2}(k_2 z_{2x} + \sigma_{b1}(k_1 z_{1x})))) \quad (17)
$$

and for the pitch angle is

$$
\tilde{\tau}_\theta = \ddot{\theta}_d - \sigma_{b4}(k_4 z_{4y} + \sigma_{b3}(k_3 z_{3y} + \sigma_{b2}(k_2 z_{2y} + \sigma_{b1}(k_1 z_{1y})))) \quad (18)
$$

with b_1, b_2, b_3 and b_4 as the limits of the saturation function.

4. VELOCITY FIELD

The objective for the velocity field method in this particular case is to follow a predefined trajectory. For this method it is necessary to calculate two vector fields, *approaching field* and *tangential field*. The approaching field is defined by the vectors that aim directly to the trajectory, in which each vector $\boldsymbol{V}_{ac}$ is obtained as the normalized subtraction of the closest point to the trajectory, as proposed by Pérez-D'Arpino et al. (2008). The trajectory to follow is a circle of radius r_{tr} and the center at the point (o_x, o_y), expressed by

$$
x_{tr} = o_x + r_{tr}\cos(\alpha), \quad \alpha \in [0, 2\pi] \quad (19)
$$

$$
y_{tr} = o_y + r_{tr}\sin(\alpha), \quad \alpha \in [0, 2\pi] \quad (20)
$$

In order to calculate the approaching field, it is necessary to find the closest point from any position of the workspace to the trajectory. This closest point is calculated by

$$
\min\left(\sqrt{(x - x_{tr})^2 + (y - y_{tr})^2}\right) \quad (21)
$$

where x_{tr} and y_{tr} are the points conforming the trajectory. Let us define the vectors:

$$
\bar{\boldsymbol{\xi}} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} x - o_x \\ y - o_y \end{bmatrix}, \qquad \tilde{\boldsymbol{\xi}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} x_{cl} - x \\ y_{cl} - y \end{bmatrix}
$$

where $\bar{x}, \bar{y}$ are the difference between any point of the workspace and the center of the circular trajectory, and $\tilde{x}, \tilde{y}$ are the position errors from any point of the workspace and its closest point within the desired trajectory.

The coordinates of the closest point to the trajectory are

$$
x_{cl} = o_x + r_{tr}\cos(\alpha_{cl})
$$

$$
y_{cl} = o_y + r_{tr}\sin(\alpha_{cl})
$$

where α_{cl} denotes the angle of the closest point to the trajectory. Taking the Equations (19)-(20) to calculate x_{cl} and y_{cl}, we can define the distance between the actual position and the closest point in the trajectory as

$$
\|\tilde{\boldsymbol{\xi}}\| = \sqrt{\tilde{x}^2 + \tilde{y}^2} \quad (22)
$$

The angle α_{cl} is obtained from the derivative of the distance $\|\tilde{\boldsymbol{\xi}}\|$ with respect to α_{cl}, or $d\|\tilde{\boldsymbol{\xi}}\|/d\alpha_{cl} = 0$, and it is

$$
\alpha_{cl} = \operatorname{atan2}(\bar{y}, \bar{x})
$$

The approaching field is defined as

$$
\boldsymbol{V}_{ac} = \frac{\tilde{\boldsymbol{\xi}}}{\|\tilde{\boldsymbol{\xi}}\|} \quad (23)
$$

Let us denote the partial derivatives of x_{cl} and y_{cl} as v_{x_c} and v_{y_c}.

$$
\boldsymbol{v}_c = \begin{bmatrix} v_{x_c} \\ v_{y_c} \end{bmatrix} = \begin{bmatrix} -r_{tr}\sin\alpha_{cl} \\ r_{tr}\cos\alpha_{cl} \end{bmatrix}
$$

These values are necessary to generate the tangential field as follows

$$
\boldsymbol{V}_{tg} = \frac{\boldsymbol{v}_c}{\|\boldsymbol{v}_c\|} \quad (24)
$$

The velocity field is obtained performing the normalized weighted sum

$$
\boldsymbol{V} = \frac{F_1\boldsymbol{V}_{ac} + F_2\boldsymbol{V}_{tg}}{\|F_1\boldsymbol{V}_{ac} + F_2\boldsymbol{V}_{tg}\|} \quad (25)
$$

where F_1 and F_2 are functions of the Euclidean distance between the point in space and the desired trajectory, which are given by:

$$
F_1 = \frac{2}{1 + e^{-\gamma\|\tilde{\boldsymbol{\xi}}\|}} - 1, \quad F_2 = 1 - F_1 \quad (26)
$$

The behavior of this functions allows us to obtain a field in which the directions of the tangential vectors prevails over the approaching vectors in the trajectory vicinity, when the opposite effect occurs far from the trajectory, this behavior is affected by modifying the constant γ. In Figure 2 is shown a velocity field generated with a value $\gamma = 0.4$, $o_x = 40$, $o_y = 30$ and $r_{tr} = 10$.

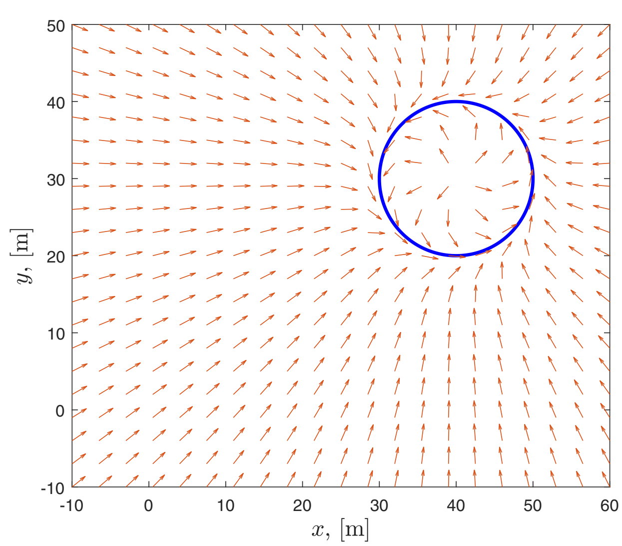

Fig. 2. Velocity field of Equation (25), with desired trajectory at position (40, 30) and radio $r_{tr} = 10$m.

Type	Constant	Value
Quad-rotor Model	m	0.5 kg
	g	9.81 m/s^2
NS Controller constants	k_1	1
	k_2	1
	k_3	1
	k_4	1
NS Controller Boundaries	b_1	0.1
	b_2	0.2
	b_3	0.4
	b_4	0.8
PD Gains for z	k_{pz}	0.5
	k_{vz}	0.8
PD gains for ψ	$k_{p\psi}$	0.374
	$k_{v\psi}$	0.8

Table 1. Constants used in simulations.

The velocity field vector is expressed by (25). It represents the desired velocity that the vehicle must achieve at each instant. In order to track the trajectory, with the nested saturation controller, according to state vectors (14) and control laws (17) and (18), it is necessary to compute three successive derivatives of the velocity field. They are developed in Appendix A.

5. SIMULATION RESULTS

The results were obtained through simulations in Simulink of MATLAB$_{®}$. The experiments were performed with the following initial conditions $x(0) = 0, y(0) = 0, z(0) = 1$ and $\psi(0) = 0$, and the constants shown in Table 1.

Given that the desired trajectory does not need a specific position at any particular time, and it is instead a desired flow, the position error can be suppressed in the state vectors $\boldsymbol{x_e}$ and $\boldsymbol{y_e}$, from (14), that is

$$\boldsymbol{x_e} = \begin{bmatrix} 0 \\ \dot{e}_x \\ e_\theta \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{x} - \dot{x}_d \\ \theta - \theta_d \\ \dot{\theta} - \dot{\theta}_d \end{bmatrix} \quad \text{and} \quad \boldsymbol{y_e} = \begin{bmatrix} 0 \\ \dot{e}_y \\ e_\phi \\ \dot{e}_\phi \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{y} - \dot{y}_d \\ \phi - \phi_d \\ \dot{\phi} - \dot{\phi}_d \end{bmatrix} \tag{27}$$

The desired variables are obtained from

$$\begin{bmatrix} \dot{x}_d \\ \theta_d \\ \dot{\theta}_d \\ \ddot{\theta}_d \end{bmatrix} = \begin{bmatrix} V_x \\ \dot{V}_x/g \\ \ddot{V}_x/g \\ \dddot{V}_x/g \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dot{y}_d \\ \phi_d \\ \dot{\phi}_d \\ \ddot{\phi}_d \end{bmatrix} = \begin{bmatrix} V_y \\ -\dot{V}_y/g \\ -\ddot{V}_y/g \\ -\dddot{V}_y/g \end{bmatrix}$$

So that, we can use the nested saturation controller in (17)-(18).

Figure 3 shows the trajectory followed by the drone. The initial position is marked with a cross. Then, the vehicle is carried by the velocity field flow towards the trajectory. It reaches the trajectory in a soft way and remains there thereafter.

Figure 4 shows the position error according to (22). This value represents the distance of the vehicle with respect to the closest point of the desired trajectory. It can be seen that the vehicle reaches the trajectory in around 25 seconds and the error maintains low thereafter.

Figure 5 depicts the quad-rotor speed error, this is, the norm of the velocity error vector $[\dot{e}_x, \quad \dot{e}_y]^T$. With this value, it is possible to see whether the velocity field is properly followed. Indeed, it can be seen that the velocity

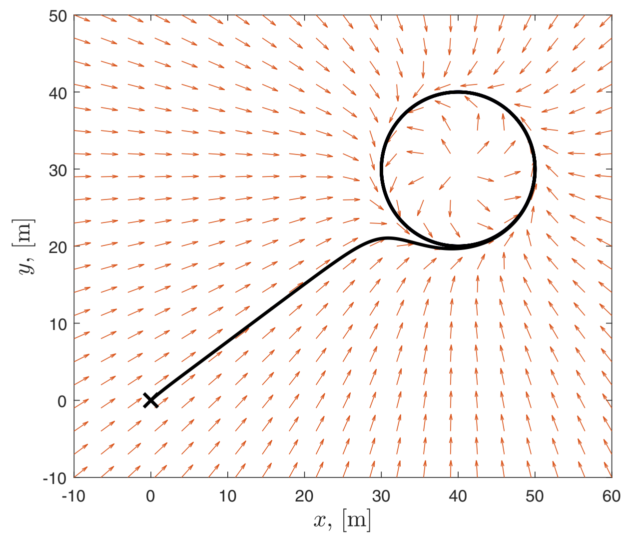

Fig. 3. Trajectory of the vehicle using a velocity field.

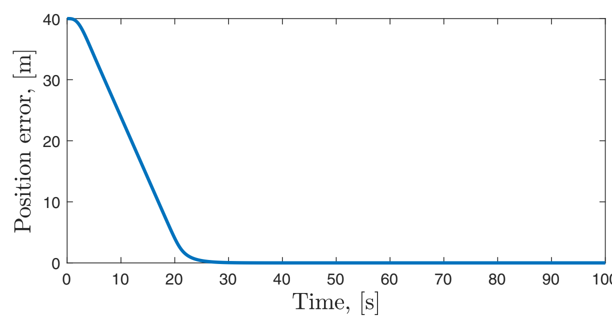

Fig. 4. Position error, $\|\tilde{\boldsymbol{\xi}}\|$.

reference is reached in approximately 7 seconds, and the vehicle keeps following the velocity reference.

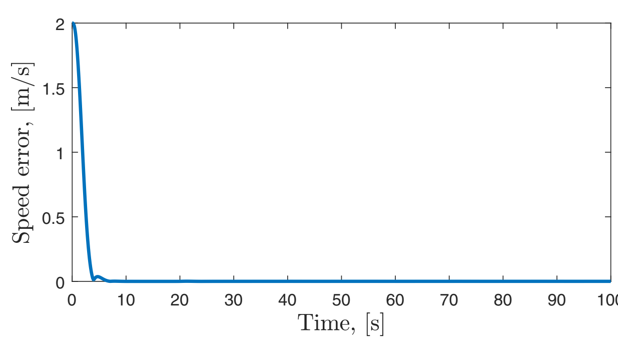

Fig. 5. Speed error.

6. CONCLUSIONS

The most important contributions in this work are the generation of a velocity field that gives reference of velocity to reach and to move into a circular trajectory, and the implementation of an exponentially stable control law, namely nested saturation. For this particular application, the nested saturation controller shows a proper control response. In the preview section of simulation results, we show that the combination of these strategies carries to a good scheme to track a trajectory.

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Appendix A. VELOCITY FIELD DERIVATIVES

Let us define the actual position vector as

$$\boldsymbol{\xi}_{tr} = \begin{bmatrix} x \\ y \end{bmatrix}$$

and the temporal derivatives by way of

$$\dot{\boldsymbol{\xi}}_{tr} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, \quad \ddot{\boldsymbol{\xi}}_{tr} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}, \quad \dddot{\boldsymbol{\xi}}_{tr} = \begin{bmatrix} \dddot{x} \\ \dddot{y} \end{bmatrix}$$

A.1 First Derivative of the Velocity Field

In order to simplify equations, let us perform the product "$\otimes$" of two bidimentional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as

$$c = \boldsymbol{a} \otimes \boldsymbol{b} = \det([\boldsymbol{a}\ \boldsymbol{b}]) = a_1 b_2 - a_2 b_1$$

so that the result is scalar and $\boldsymbol{a} \otimes \boldsymbol{a} = 0$.

Then, the derivative of α_{cl} is expressed as

$$\dot{\alpha}_{cl} = \frac{\bar{x}\dot{y} - \bar{y}\dot{x}}{\bar{x}^2 + \bar{y}^2} = \frac{\bar{x}\dot{y} - \bar{y}\dot{x}}{\|\bar{\boldsymbol{\xi}}_{tr}\|^2} = \frac{\bar{\boldsymbol{\xi}}_{tr} \otimes \dot{\boldsymbol{\xi}}_{tr}}{\bar{\boldsymbol{\xi}}_{tr}^T \bar{\boldsymbol{\xi}}_{tr}}$$

The time derivatives of the desired trajectory equations are

$$\dot{x}_{tr} = -r_{tr}\dot{\alpha}\sin\alpha$$
$$\dot{y}_{tr} = r_{tr}\dot{\alpha}\cos\alpha$$

Therefore, the derivatives of the points x_{cl} and y_{cl} is

$$\dot{x}_{cl} = -r_{tr}\dot{\alpha}_{cl}\sin\alpha_{cl} = \dot{\alpha}_{cl} v_{x_c}$$
$$\dot{y}_{cl} = r_{tr}\dot{\alpha}_{cl}\cos\alpha_{cl} = \dot{\alpha}_{cl} v_{y_c}$$

Let us define an equation for the velocity error as

$$\dot{\tilde{\boldsymbol{\xi}}}_{tr} = \tilde{\boldsymbol{v}} = \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} \dot{x}_{cl} - \dot{x} \\ \dot{y}_{cl} - \dot{y} \end{bmatrix}$$

The approaching field derivative is

$$\dot{\boldsymbol{V}}_{ap} = \frac{\dot{\tilde{\boldsymbol{\xi}}}_{tr} - (\boldsymbol{V}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr})\boldsymbol{V}_{ap}}{\|\tilde{\boldsymbol{\xi}}_{tr}\|}$$

Defining an equation for the tangential accelerations vector

$$\dot{\boldsymbol{v}}_c = \begin{bmatrix} \dot{v}_{x_c} \\ \dot{v}_{y_c} \end{bmatrix} = \begin{bmatrix} -\dot{\alpha}_{cl} v_{y_c} \\ \dot{\alpha}_{cl} v_{x_c} \end{bmatrix}$$

The time deivative of the tangential field is

$$\dot{\boldsymbol{V}}_{tg} = \frac{\dot{\boldsymbol{v}}_c - (\boldsymbol{V}_{tg}^T \dot{\boldsymbol{v}}_c)\boldsymbol{V}_{tg}}{\|\boldsymbol{v}_c\|}$$

Let us define

$$h(\tilde{\boldsymbol{\xi}}_{tr}) = -\gamma\|\tilde{\boldsymbol{\xi}}_{tr}\|, \quad \text{and} \quad f(\tilde{\boldsymbol{\xi}}_{tr}) = e^{h(\tilde{\boldsymbol{\xi}}_{tr})} \tag{A.1}$$

We can rewrite (26) as

$$F_1 = \frac{2}{1 + f(\tilde{\boldsymbol{\xi}}_{tr})} - 1, \quad F_2 = 1 - F_1$$

Considering the functions $f(\tilde{\boldsymbol{\xi}}_{tr})$ and $h(\tilde{\boldsymbol{\xi}}_{tr})$ of (A.1), we have

$$\dot{h}(\tilde{\boldsymbol{\xi}}_{tr}) = -\gamma(\boldsymbol{V}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr}),$$
$$\dot{f}(\tilde{\boldsymbol{\xi}}_{tr}) = e^{h(\tilde{\boldsymbol{\xi}}_{tr})}\dot{h}(\tilde{\boldsymbol{\xi}}_{tr}) = f(\tilde{\boldsymbol{\xi}}_{tr})\dot{h}(\tilde{\boldsymbol{\xi}}_{tr}) \tag{A.2}$$

and the derivative of Equation (26) can be written as

$$\dot{F}_1 = \frac{-2\dot{f}(\tilde{\boldsymbol{\xi}}_{tr})}{[1 + f(\tilde{\boldsymbol{\xi}}_{tr})]^2}, \quad \dot{F}_2 = -\dot{F}_1 \tag{A.3}$$

Let us define the velocity field numerator vector as

$$\boldsymbol{V} = F_1\boldsymbol{V}_{ap} + F_2\boldsymbol{V}_{tg}$$

The time derivative is

$$\dot{\boldsymbol{V}}_a = F_1\dot{\boldsymbol{V}}_{ap} + F_2\dot{\boldsymbol{V}}_{tg} + \dot{F}_1\boldsymbol{V}_{ap} + \dot{F}_2\boldsymbol{V}_{tg}$$

Then, the time derivative of the velocity field is

$$\dot{\boldsymbol{V}} = \frac{\dot{\boldsymbol{V}}_a - (\boldsymbol{V}^T\dot{\boldsymbol{V}}_a)\boldsymbol{V}}{\|\boldsymbol{V}_a\|}$$

A.2 Second Derivative of the Velocity Field

The second derivative of α_{cl} is

$$\ddot{\alpha}_{cl} = \frac{\bar{\boldsymbol{\xi}}_{tr} \otimes \ddot{\boldsymbol{\xi}}_{tr} - 2(\bar{\boldsymbol{\xi}}_{tr}^T \dot{\boldsymbol{\xi}}_{tr})\dot{\alpha}_{cl}}{\bar{\boldsymbol{\xi}}_{tr}^T \bar{\boldsymbol{\xi}}_{tr}}$$

The derivative of the points $\dot{x}_{cl}$ and $\dot{y}_{cl}$ is

$$\ddot{x}_{cl} = \ddot{\alpha}_{cl} v_{x_c} + \dot{\alpha}_{cl} \dot{v}_{x_c}$$
$$\ddot{y}_{cl} = \ddot{\alpha}_{cl} v_{y_c} + \dot{\alpha}_{cl} \dot{v}_{y_c}$$

The velocity error derivative of $\tilde{\boldsymbol{v}}$ is

$$\ddot{\tilde{\boldsymbol{\xi}}}_{tr} = \begin{bmatrix} \ddot{x}_{cl} - \ddot{x} \\ \ddot{y}_{cl} - \ddot{y} \end{bmatrix}$$

The second derivative of the approaching field defined in (23) is

$$\ddot{\boldsymbol{V}}_{ap} = \frac{\ddot{\tilde{\boldsymbol{\xi}}}_{tr} - (\dot{\boldsymbol{V}}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr} + \boldsymbol{V}_{ap}^T \ddot{\tilde{\boldsymbol{\xi}}}_{tr})\boldsymbol{V}_{ap} - 2(\boldsymbol{V}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr})\dot{\boldsymbol{V}}_{ap}}{\|\tilde{\boldsymbol{\xi}}_{tr}\|}$$

Let us define an equation for the derivative of tangential accelerations

$$\ddot{\boldsymbol{v}}_c = \begin{bmatrix} \ddot{v}_{x_c} \\ \ddot{v}_{y_c} \end{bmatrix} = \begin{bmatrix} -\ddot{\alpha}_{cl} v_{y_c} - \dot{\alpha}_{cl} \dot{v}_{y_c} \\ \ddot{\alpha}_{cl} v_{x_c} + \dot{\alpha}_{cl} \dot{v}_{x_c} \end{bmatrix}$$

Then the second derivative of the tangential field definend in (24) is

$$\ddot{\boldsymbol{V}}_{tg} = \frac{\ddot{\boldsymbol{v}}_c - (\dot{\boldsymbol{V}}_{tg}^T \dot{\boldsymbol{v}}_c + \boldsymbol{V}_{tg}^T \ddot{\boldsymbol{v}}_c)\boldsymbol{V}_{tg} - 2(\boldsymbol{V}_{tg}^T \dot{\boldsymbol{v}}_c)\dot{\boldsymbol{V}}_{tg}}{\|\boldsymbol{v}_c\|}$$

With the functions $\dot{f}(\tilde{\boldsymbol{\xi}}_{tr})$ and $\dot{h}(\tilde{\boldsymbol{\xi}}_{tr})$ of (A.2), we define

$$\ddot{h}(\tilde{\boldsymbol{\xi}}_{tr}) = -\gamma(\dot{\boldsymbol{V}}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr} + \boldsymbol{V}_{ap}^T \ddot{\tilde{\boldsymbol{\xi}}}_{tr}),$$
$$\ddot{f}(\tilde{\boldsymbol{\xi}}_{tr}) = \dot{f}(\tilde{\boldsymbol{\xi}}_{tr})\dot{h}(\tilde{\boldsymbol{\xi}}_{tr}) + f(\tilde{\boldsymbol{\xi}}_{tr})\ddot{h}(\tilde{\boldsymbol{\xi}}_{tr}) \quad \text{(A.4)}$$

and the derivative of (A.3) is written as

$$\ddot{F}_1 = \frac{-2\ddot{f}(\tilde{\boldsymbol{\xi}}_{tr})}{[1 + f(\tilde{\boldsymbol{\xi}}_{tr})]^2} + [1 + f(\tilde{\boldsymbol{\xi}}_{tr})]\dot{F}_1^2, \quad \ddot{F}_2 = -\ddot{F}_1 \quad \text{(A.5)}$$

The time derivative of the vector $\dot{\boldsymbol{V}}_a$ is

$$\ddot{\boldsymbol{V}}_a = F_1 \ddot{\boldsymbol{V}}_{ap} + F_2 \ddot{\boldsymbol{V}}_{tg} + 2\dot{F}_1 \dot{\boldsymbol{V}}_{ap} + 2\dot{F}_2 \dot{\boldsymbol{V}}_{tg} + \ddot{F}_1 \boldsymbol{V}_{ap} + \ddot{F}_2 \boldsymbol{V}_{tg}$$

Then, the second time derivative of the velocity field is

$$\ddot{\boldsymbol{V}} = \frac{\ddot{\boldsymbol{V}}_a - (\dot{\boldsymbol{V}}^T \dot{\boldsymbol{V}}_a + \boldsymbol{V}^T \ddot{\boldsymbol{V}}_a)\boldsymbol{V} - 2(\boldsymbol{V}^T \dot{\boldsymbol{V}}_a)\dot{\boldsymbol{V}}}{\|\boldsymbol{V}_a\|}$$

A.3 Third Derivative of the Velocity Field

The third derivative of α_{cl} is

$$\dddot{\alpha}_{cl} = \frac{\bar{\boldsymbol{\xi}}_{tr} \otimes \dddot{\boldsymbol{\xi}}_{tr} + \dot{\boldsymbol{\xi}}_{tr} \otimes \ddot{\boldsymbol{\xi}}_{tr} - 2(\bar{\boldsymbol{\xi}}_{tr}^T \ddot{\boldsymbol{\xi}}_{tr} + \dot{\boldsymbol{\xi}}_{tr}^T \dot{\boldsymbol{\xi}}_{tr})\dot{\alpha}_{cl}}{\bar{\boldsymbol{\xi}}_{tr}^T \bar{\boldsymbol{\xi}}_{tr}} - \frac{4(\bar{\boldsymbol{\xi}}_{tr}^T \dot{\boldsymbol{\xi}}_{tr})\ddot{\alpha}_{cl}}{\bar{\boldsymbol{\xi}}_{tr}^T \bar{\boldsymbol{\xi}}_{tr}}$$

The derivative of the points $\dot{x}_{cl}$ and $\dot{y}_{cl}$ is

$$\dddot{x}_{cl} = \dddot{\alpha}_{cl} v_{x_c} + 2\ddot{\alpha}_{cl} \dot{v}_{x_c} + \dot{\alpha}_{cl} \ddot{v}_{x_c}$$
$$\dddot{y}_{cl} = \dddot{\alpha}_{cl} v_{y_c} + 2\ddot{\alpha}_{cl} \dot{v}_{y_c} + \dot{\alpha}_{cl} \ddot{v}_{y_c}$$

The acceleration error derivative $\dot{\tilde{\boldsymbol{v}}}$ is

$$\dddot{\tilde{\boldsymbol{\xi}}}_{tr} = \begin{bmatrix} \dddot{x}_{cl} - \dddot{x} \\ \dddot{y}_{cl} - \dddot{y} \end{bmatrix}$$

The third derivative of the approaching field defined in (23) is

$$\dddot{\boldsymbol{V}}_{ap} = \frac{\dddot{\tilde{\boldsymbol{\xi}}}_{tr} - (\ddot{\boldsymbol{V}}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr} + 2\dot{\boldsymbol{V}}_{ap}^T \ddot{\tilde{\boldsymbol{\xi}}}_{tr} + \boldsymbol{V}_{ap}^T \dddot{\tilde{\boldsymbol{\xi}}}_{tr})\boldsymbol{V}_{ap}}{\|\tilde{\boldsymbol{\xi}}_{tr}\|} - \frac{3(\dot{\boldsymbol{V}}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr} + \boldsymbol{V}_{ap}^T \ddot{\tilde{\boldsymbol{\xi}}}_{tr})\dot{\boldsymbol{V}}_{ap} + 3(\boldsymbol{V}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr})\ddot{\boldsymbol{V}}_{ap}}{\|\tilde{\boldsymbol{\xi}}_{tr}\|}$$

The third derivative of the tangential velocity field is

$$\dddot{\boldsymbol{v}}_c = \begin{bmatrix} \dddot{v}_{x_c} \\ \dddot{v}_{y_c} \end{bmatrix} = \begin{bmatrix} -\dddot{\alpha}_{cl} v_{y_c} - 2\ddot{\alpha}_{cl} \dot{v}_{y_c} - \dot{\alpha}_{cl} \ddot{v}_{y_c} \\ \dddot{\alpha}_{cl} v_{x_c} + 2\ddot{\alpha}_{cl} \dot{v}_{x_c} + \dot{\alpha}_{cl} \ddot{v}_{x_c} \end{bmatrix}$$

The third derivative of the tangential field defined in (24) is

$$\dddot{\boldsymbol{V}}_{tg} = \frac{\dddot{\boldsymbol{v}}_c - (\ddot{\boldsymbol{V}}_{tg}^T \dot{\boldsymbol{v}}_c + 2\dot{\boldsymbol{V}}_{tg}^T \ddot{\boldsymbol{v}}_c + \boldsymbol{V}_{tg}^T \dddot{\boldsymbol{v}}_c)\boldsymbol{V}_{tg}}{\|\boldsymbol{v}_c\|} - \frac{3(\dot{\boldsymbol{V}}_{tg}^T \dot{\boldsymbol{v}}_c + \boldsymbol{V}_{tg}^T \ddot{\boldsymbol{v}}_c)\dot{\boldsymbol{V}}_{tg} + 3(\boldsymbol{V}_{tg}^T \dot{\boldsymbol{v}}_c)\ddot{\boldsymbol{V}}_{tg}}{\|\boldsymbol{v}_c\|}$$

Through the functions $\ddot{f}(\tilde{\boldsymbol{\xi}}_{tr})$ and $\ddot{h}(\tilde{\boldsymbol{\xi}}_{tr})$ of (A.4), we define

$$\dddot{h}(\tilde{\boldsymbol{\xi}}_{tr}) = -\gamma(\ddot{\boldsymbol{V}}_{ap}^T \dot{\tilde{\boldsymbol{\xi}}}_{tr} + 2\dot{\boldsymbol{V}}_{ap}^T \ddot{\tilde{\boldsymbol{\xi}}}_{tr} + \boldsymbol{V}_{ap}^T \dddot{\tilde{\boldsymbol{\xi}}}_{tr}),$$
$$\dddot{f}(\tilde{\boldsymbol{\xi}}_{tr}) = \ddot{f}(\tilde{\boldsymbol{\xi}}_{tr})\dot{h}(\tilde{\boldsymbol{\xi}}_{tr}) + 2\dot{f}(\tilde{\boldsymbol{\xi}}_{tr})\ddot{h}(\tilde{\boldsymbol{\xi}}_{tr}) + f(\tilde{\boldsymbol{\xi}}_{tr})\dddot{h}(\tilde{\boldsymbol{\xi}}_{tr})$$

and the derivative of (A.5) is written as

$$\dddot{F}_1 = \frac{-2\dddot{f}(\tilde{\boldsymbol{\xi}}_{tr})}{[1 + f(\tilde{\boldsymbol{\xi}}_{tr})]^2} + 3[1 + f(\tilde{\boldsymbol{\xi}}_{tr})]\ddot{F}_1 \dot{F}_1 - [1 + f(\tilde{\boldsymbol{\xi}}_{tr})]\dot{F}_1^3 + \dot{f}(\tilde{\boldsymbol{\xi}}_{tr})\dot{F}_1^2,$$
$$\dddot{F}_2 = -\dddot{F}_1$$

The time derivative of $\ddot{\boldsymbol{V}}_a$ is

$$\dddot{\boldsymbol{V}}_a = F_1 \dddot{\boldsymbol{V}}_{ap} + F_2 \dddot{\boldsymbol{V}}_{tg} + 3\dot{F}_1 \ddot{\boldsymbol{V}}_{ap} + 3\dot{F}_2 \ddot{\boldsymbol{V}}_{tg} + 3\ddot{F}_1 \dot{\boldsymbol{V}}_{ap} + 3\ddot{F}_2 \dot{\boldsymbol{V}}_{tg} + \dddot{F}_1 \boldsymbol{V}_{ap} + \dddot{F}_2 \boldsymbol{V}_{tg}$$

Then, the third velocity field time derivative is

$$\dddot{\boldsymbol{V}} = \frac{\dddot{\boldsymbol{V}}_a - (\ddot{\boldsymbol{V}}^T \dot{\boldsymbol{V}}_a + 2\dot{\boldsymbol{V}}^T \ddot{\boldsymbol{V}}_a + \boldsymbol{V}^T \dddot{\boldsymbol{V}}_a)\boldsymbol{V}}{\|\boldsymbol{V}_a\|} - \frac{3(\dot{\boldsymbol{V}}^T \dot{\boldsymbol{V}}_a + \boldsymbol{V}^T \ddot{\boldsymbol{V}}_a)\dot{\boldsymbol{V}} + 3(\boldsymbol{V}^T \dot{\boldsymbol{V}}_a)\ddot{\boldsymbol{V}}}{\|\boldsymbol{V}_a\|}$$