ISSN: 2594-2492

# Synchronization of a fractional order chaotic system with attractors propagating on a line

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**Abstract:**
This paper addresses the problem of synchronization for a class of fractional order chaotic systems with a double-scroll attractor propagates on a line. Based on the stability theory of fractional order systems, some basic dynamical properties are studied, such as equilibrium points, Lyapunov exponents and strange attractors of the chaotic system. Using the active control approach the synchronization of incommensurate fractional order systems with different basins of attraction is achieved. The numerical results illustrate that fast synchronization can be achieved between fractional order chaotic systems.

*Keywords:* Chaotic behavior; Fractional order; Active control; Synchronization.

## 1. INTRODUCTION

Chaotic systems and their implementations have been studied during the last decades. In (Hsiao 2018, Lan et al. 2018, Rasmussen et al. 2017, Cicek et al. 2018, Abernethy and Gooding 2018) the sensitivity of chaotic systems to parameters and initial conditions are considered for many real-world applications such as data encryption, secure communication power systems, biology, circuit theory, control. Fractional calculus has been considered as the extension of the integer-order calculus to non-integer order calculus. It has also become a powerful tool to describe the dynamics of complex systems which appear frequently in several branches of science and engineering. For instance in the field of viscoelasticity, robotics, feedback amplifiers, electrical circuits, control theory, electro analytical chemistry, fractional multi-poles, electromagnetics, bioengineering, as shown in (Saadia and Rashdi 2018, Sun et al. 2018, Long et al. 2018, Ionescu and Kelly 2017, Guo et al. 2018).

The chaotic dynamics of fractional order systems began to attract the interest of the scientific community in recent years. The chaotic systems can also be modeled more accurately by non-integer differential equations. According to the Poincare-Bendixson theorem (Hirsch and Smale 1965), chaos can exist in a given continuous autonomous dynamical system specified by differential equations if it has three or more dimensions. However, by introducing fractional derivative with order $0 < q < 1$ into the well-known chaotic systems, many authors found that these systems with fractional version remain chaotic and associated with the advances in numerical methods for solving them and their electronic implementations as can be consulted in (Diethelm and Ford 2002, Garrapa 2018, Zambrano-Serrano et al. 2016, Li and Chen 2004, Grigorenko and Grigorenko 2003, Pham et al. 2018, Munoz-Pacheco et al. 2018) and (Petras 2011).

Since (Pecora and Carroll 1990) have shown that chaotic systems can be synchronized by introducing an appropriate coupling. The notion of synchronization of chaos has become an important research area in nonlinear science, not only for understanding the complicated phenomena in various fields but also due to its potential applications especially in secure communication and image encryption. Two indistinguishable chaotic systems, starting from non-identical initial values, would evolve in time to completely different trajectories because of the sensitive dependence of chaotic systems to their initial values. The aim of synchronizing chaos is to ensure that the states track the desired trajectory. A variety of approaches have been proposed to deal with synchronization of chaotic systems, these included, backstepping, adaptive, and active controls as mentioned in (Shukla and Sharma 2017, Singh et al. 2017). Chaos synchronization using Active control method was proposed by (Bai and Lonngren 1997). If the nonlinearity of the system is known, an active controller can be easily designed according to the given conditions of the chaotic system to achieve synchronization globally, and it is treated as one of the most interesting control strategy for its simplicity (Singh et al. 2017).

With the inspiration from the above discussions, In this paper, a fractional order system with a double-scroll attractor propagates on a line is presented, based on modifying a parameter, the position of the chaotic attractor in the phase plane is modified. Also a dynamic analysis of the system will be presented. Moreover, we consider the problem of chaos synchronization of incommensurate fractional order systems. Preliminaries are discussed in Section 2. In Section 3, systems' descriptions are given. Section 4 contains the synchronization which is achieved using Active control method. Numerical simulation and results are carried out in Section 5. which is followed by a Conclusion given in Section 6.

## 2. MATHEMATICAL BACKGROUND

Fractional order calculus is a generalization of integration and differentiation to non-integer order, denoted by the operator $Dt^q$, where $q \in R$ is the fractional order. This operator is a notation for taking both the fractional derivative and the fractional integral of a function into a single expression and can be formally defined as

$$
{}_aD_t^q f = \begin{cases} \dfrac{d^q f}{dt^q}, & q > 0, \\ f, & q = 0, \\ \displaystyle\int_a^t f(d\tau)^q, & q < 0. \end{cases} \quad (1)
$$

where $f$ is a function of time. There are several different definitions for the fractional differential operator that can be adopted for (1). Hereafter, we consider the fractional derivative operator $d^q/dt^q$ , with $q < 1$, to be Caputo's derivative with starting point $a = 0$ defined by

$$
D_t^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, \quad (2)
$$

where $m$ is an integer number and $\Gamma(\cdot)$ is the Gamma function. Caputo's derivative of order $\alpha$ is a formal generalization of the integer derivative under Laplace transformation and is widely used in the engineering field (Sun et al. 2018).

### *2.1 Fractional predictor-corrector algorithm*

The numerical method used in this work to compute the solution of the fractional order systems is the Adams-Bashforth-Moulton predictor-corrector scheme, reported in (Diethelm and Ford 2002, Garrapa 2018). The predictor-corrector scheme is based on the Caputo fractional differential operator (2) which allows us to specify both homogeneous and inhomogeneous initial conditions.

Consider the following fractional differential equation:

$$
\begin{aligned} D^\alpha y(t) &= f(t, y(t)), \quad 0 \le t \le T; \\ y^{(k)}(0) &= y_0^{(k)}, \qquad k = 0, 1, \ldots, n-1. \end{aligned} \quad (3)
$$

The solution of (3) is given by an integral equation of Volterra type as

$$
y(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} y_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} f(z, y(z)) dz. \quad (4)
$$

How it is showed in (Diethelm and Ford 2002), there is a unique solution of (3) on some interval $[0, T]$, thence we are interested in a numerical solution on the uniform grid $\{t_n = nh | n = 0, 1, \ldots, N\}$ with some integer $N$ and stepsize $h = T/N$, then (4) can be replaced by a discrete form to get the corrector as follows

$$
\begin{aligned} y_h(t_{n+1}) = &\sum_{k=0}^{\lceil \alpha \rceil - 1} y_0^k \frac{t^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_h^p(t_{n+1})\right) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, y_h(t_j)\right), \end{aligned} \quad (5)
$$

where

$$
a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, \quad j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ \quad -2(n-j+1)^{\alpha+1}, \quad 1 \le j \le n, \\ 1, \quad j = n+1, \end{cases} \quad (6)
$$

Moreover, the predictor has the following structure

$$
y_h^p(t_{n+1}) = \sum_{k=0}^{\lceil \alpha \rceil - 1} y_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)), \quad (7)
$$

with $b_{j,n+1}$ defined by

$$
b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j^\alpha)). \quad (8)
$$

The error of this approximation is given by

$$
\max_{j=0,1,\ldots N} |y(t_j) - y_h(t_j))| = \mathcal{O}(h^P), \quad (9)
$$

where $P = \min(2, 1+\alpha)$.

### *2.2 Stability of fractional order systems*

Starting from Eqs. (1) and (2), it is possible to study the stability of fractional order systems. A fractional order differential equation with $0 < q < 1$ typically presents a stability region that is larger than that of the same equation with integer order $q = 1$.

*Proposition 1 The roots of the equation $f(x) = 0$ are called the equilibria of the fractional order differential system $D^q x = f(x)$, where $x = (x_1, x_2, \ldots, x_n)^T \in R$, $f(x) \in R$ and $D^q x = (D^{q_1} x_1, D^{q_2} x_2, \ldots, D^{q_n} x_n)^T$, $q_i \in R^+$, $i = 1, 2, \ldots, n$.*

 

*Theorem 1.* Consider the following $n-$dimensional fractional order system

$$\begin{aligned} D^{q_1}x_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ D^{q_2}x_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ &\vdots \\ D^{q_n}x_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{aligned} \tag{10}$$

where all $q_i$'s are rational numbers between 0 and 1. Assume $m$ be the lowest common multiple of the denominators $u_i$'s of $q_i$'s, where $q_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i, v_i \in Z^+$ for $i = 1, 2, \ldots, n$. Define:

$$\Delta(\lambda) = \begin{bmatrix} \lambda^{mq_1} - a_{11} & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & \lambda^{mq_2} - a_{22} & \ldots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \ldots & \lambda^{mq_n} - a_{nn} \end{bmatrix}. \tag{11}$$

Then the zero solution of system (10) is globally asymptotically stable in the Lyapunov sense if all roots $\lambda$'s of the equation $\det(\Delta(\lambda)) = 0$ satisfy $|\arg(\lambda)| > \pi/2m$. $\Delta(s)$ is called the characteristic matrix and $\det(\Delta(\lambda))$ is called the characteristic polynomial of systems (10), (Petras 2011).

*Theorem 2.* The equilibrium point $E^*$ is asymptotically stable if and only if the instability measure

$$\rho = (\pi/2m) - \min_i\{\arg(\lambda_i)\}, \tag{12}$$

is strictly negative, i.e., $\rho < 0$. Where $\lambda_i$'s are roots of equations: $\det(\mathrm{diag}([\lambda^{mq_1} \quad \lambda^{mq_2} \ldots \lambda^{mq_n}]) - \partial f/\partial x|_{x=E^*}) = 0$, $\forall E^* \in \Omega$ (Petras 2011, Munoz-Pacheco et al. 2018).

If $\rho = 0$ have the geometric multiplicity one, then $E^*$ is stable.

*Remark* If $\rho$ is positive, then $E^*$ is unstable and the system may exhibit chaotic behavior (Petras 2011, Munoz-Pacheco et al. 2018).

## 3. MODEL DESCRIPTIONS

Recently, (Munoz-Pacheco et al. 2018) proposed a new fractional order chaotic system with a variable double-scroll attractor on a line, as follows

$$\begin{aligned} D^{q_1}x &= z + x(y - a), \\ D^{q_2}y &= 1 - |x|, \\ D^{q_3}z &= -x - z. \end{aligned} \tag{13}$$

Where $(x, y, z)$ are the state variables, $a$ is a real parameter, and $(q_1, q_2, q_3) \in [0, 1]$ are the fractional orders. In the fractional order chaotic system (13) the Caputo definition (2) has been used. In order to find the numerical solutions of fractional order system (13), we have applied the Adams-Bashforth-Moulton predictor-corrector algortihm of subsection 2.1

In order to obtain equilibrium points, keep the left-hand side of system (13) be zero and then the system's equation can be written as

$$\begin{aligned} 0 &= z + x(y - a), \\ 0 &= 1 - |x|, \\ 0 &= -x - z. \end{aligned} \tag{14}$$

The new fractional order system has only two unstable equilibrium points, which are denoted by $E^* = (x^*, y^*, z^*)$, Therefore $E_1 = (1, 1+a, -1)$ and $E_2 = (-1, 1+a, 1)$. From (14) we observed the relation of the parameter $a$ with the equilibrium point i.e., when the value of $a$ is a real value, a chaotic attractor is observed with propagation on a line. For investigating the stability and type of these two equilibrium points, we consider the Jacobian matrix corresponding to different equilibria and calculate their eigenvalues. The Jacobian matrix of system (13) is given by

$$J = \begin{pmatrix} y^* - a & x^* & 1 \\ -sgn(x^*) & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

The eigenvalues as shown in Table 1. According to Theorem 1 the fractional order system is asymptotically stable if $q < 0.8180$.

Table 1. Eigenvalues of the Jacobian matrix for equilibria $E_1$ and $E_2$

| | $E_1$ | $E_2$ |
|---|---|---|
| $\lambda_1$ | $-0.6823$ | -0.6823 |
| $\lambda_2$ | $0.3412 + 1.1615i$ | $0.3412 + 1.1615i$ |
| $\lambda_3$ | $0.3412 - 1.1615i$ | $0.3412 - 1.1615i$ |

According to Theorem 1 in previous section and by setting $m = 100$, the eigenvalues $\lambda_i$ are attained as follows

$$\det(\mathrm{diag}(\lambda^{mq_1}, (\lambda^{mq_1 2}, \ldots, (\lambda^{mq_n}))) - J|_{\mathbf{x}^*}) = \tag{15}$$

$$\lambda^{255} - \lambda^{170} + \lambda^{168} + \lambda^{87} + 1 = 0. \tag{16}$$

As was demonstrated in (Petras 2011), only roots in the first Riemann's sheet satisfy $-\pi/m < \phi < \pi/m$ with $\phi = |\arg(\lambda)|$ and therefore have a physical meaning. Roots with $|\arg(\lambda)| > \pi/m$ are not physical.

By solving (15), $\lambda_{1,2} = 1.0021 \pm 0.0152j$ with $\phi = 0.0152$ are the only roots satisfying $-0.0314 < \phi < 0.0314$.

In order to generate chaos in system (13), the instability measure defined by Theorem 2. The instability measure is $\rho = 0.0004$ for $\lambda_{1,2}$, and therefore the proposed fractional order system (13) satisfies the necessary condition for exhibiting a double-scroll chaotic attractor. Additionally, the minimum fractional order, where the chaotic behavior can be found, is $q > 0.818$. The phase portraits of chaotic incommensurate fractional system with a variable double-scroll attractor on a line (13) are shown in Fig 1. When $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, and setting different values of parameter $a$ like $a = -8$, $a = -4$, $a = 0$, and $a = 4$. Also, the double-scroll chaotic attractor is only observed when the system (13) is defined in the fractional

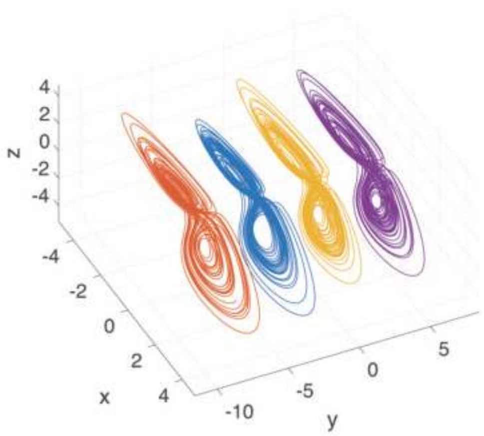


Fig. 1. Chaotic attractor of incommensurate fractional-order system with $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, and different values of parameter $a = -8$, $a = -4$, $a = 0$, and $a = 4$.

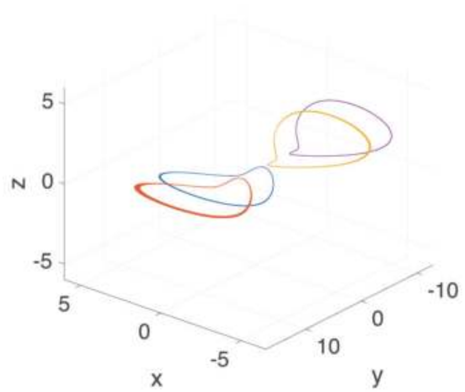


Fig. 2. Periodic attractor of incommensurate fractional-order system with $q_1 = 1$, $q_2 = 1$, $q_3 = 1$, and different values of parameter $a = -8$, $a = -4$, $a = 0$, and $a = 4$.

order domain. Conversely, a periodic attractor is obtained for integer-order as shown in Fig. 2. Additionally, this dynamics is also verified by computing the Lyapunov spectrum since a positive Lyapunov exponent is a firm of chaos. Figure 3 shows the Lyapunov spectrum for system. The chaotic behavior is valid for fractional orders $q_i \in [0.818, 0.985]$.

## 4. SYNCHRONIZATION BETWEEN INCOMMENSURATE FRACTIONAL ORDER SYSTEMS WITH ATTRACTORS PROPAGATING ON A LINE

In this section we study the synchronization between incommensurate fractional order systems. Assuming that the fractional order system (13) with $a_1 = 0$ drives the fractional order system (13) with $a_2 = 4$, we define the drive and response system as follows

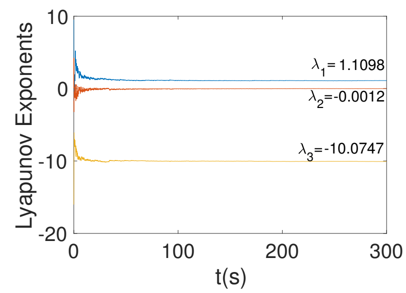


Fig. 3. Lyapunov exponent spectrum of the system (13).

$$\begin{aligned} D^{q_1}x_1 &= z_1 + x_1(y_1 - a_1), \\ D^{q_2}y_1 &= 1 - |x_1|, \\ D^{q_3}z_1 &= -x_1 - z_1, \end{aligned} \tag{17}$$

and

$$\begin{aligned} D^{q_1}x_2 &= z_2 + x_2(y_2 - a_2) + u_1, \\ D^{q_2}y_2 &= 1 - |x_2| + u_2, \\ D^{q_3}z_2 &= -x_2 - z_2 + u_3, \end{aligned} \tag{18}$$

The unknown terms $u_1, u_2, u_3$ in (18) are active control functions to be determined. Define the error functions as

$$\begin{aligned} e_1 &= x_2 - x_1, \\ e_2 &= y_2 - y_1 \\ e_3 &= z_2 - z_1. \end{aligned} \tag{19}$$

Equation (19) together with (17) and (18) yields the error system

$$\begin{aligned} D^{q_1}e_1 &= z_2 + x_2(y_2 - a_2) - z_1 - x_1(y_1 - a_1) + u_1, \\ D^{q_2}e_2 &= -|x_2| + |x_1| + u_2, \\ D^{q_3}e_3 &= -x_2 - z_2 + x_1 + z_1 + u_1. \end{aligned} \tag{20}$$

We define active control functions $u_i$ as

$$\begin{aligned} u_1 &= V_1 - z_2 - x_2(y_2 - a_2) + z_1 + x_1(y_1 - a_1), \\ u_2 &= V_2 + |x_2| - |x_1|, \\ u_3 &= V_3 + x_2 + z_2 - x_1 - z_1. \end{aligned} \tag{21}$$

The terms $V_i$ are linear functions of the error terms $e_i(t)$ given by $V_1 = -e_1$, $V_2 = -e_2$ and $V_3 = -e_3$. With the choice of $u_i$ given by (21) the error system (20) becomes

$$\begin{aligned} D^{q_1}e_1 &= -e_1, \\ D^{q_2}e_2 &= -e_2, \\ D^{q_3}e_3 &= -e_3. \end{aligned} \tag{22}$$

It is clear from the stability analysis in section 2 that the system (22) is asymptotically stable. Hence we get the required synchronization.

 

## 5. SIMULATIONS AND RESULTS

Numerical simulations are given to visualize the synchronization between considered systems and to verify the effectiveness of the proposed method. Adams-Bashforth-Moulton method is used to solve the fractional order differential equations with time step size 0.015. Parameters of the driven system are taken as $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, $a = 0$ and response system $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, $a = 4$. The initial conditions $x_1(0) = 2, y_1(0) = 2, z_1(0) = 2$ and $x_2(0) = 3, y_2(0) = 4, z_2(0) = 3$ respectively. The initial error is $[e_1(t), e_2(t), e_3(t)]^T = [1, 2, 1]^T$. The graphical result of the phase portrait of system (17) and (18) without applying active control approach is presented in Fig. 4. Figure 5 displays the phase portraits of drive system (17) and response system (18) applying active control approach. The graphical presentation of the synchronization through error analysis is depicted in Fig. 6. In Fig. 7-9, we present the synchronization in the phase space for each state-variables $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$.

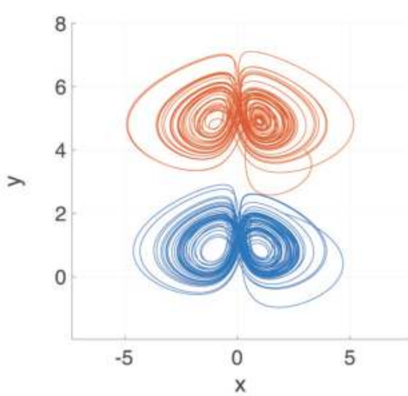


Fig. 4. Phase portraits of the fractional order system with $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, $a = 0$ for the blue attractor and $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, $a = 4$ for the orange attractor. The initial conditions $x_1(0) = 2, y_1(0) = 2, z_1(0) = 2$ and $x_2(0) = 3, y_2(0) = 4, z_2(0) = 3$ respectively. (Color online)

## 6. CONCLUSION

Based on active control theory, the synchronization between incommensurate fractional order systems with a double-scroll attractor propagates on a line was presented. Lyapunov exponent spectrum, phase portraits, instability measure and fractional order stability proved the chaos generation of the considered system. Numerical simulations proved the effectiveness of the proposed method.

## ACKNOWLEDGEMENTS

This work was supported by CONACYT/Mexico, Proyecto Apoyado por el Fondo Sectorial de Investigación para la Educación [No. 258880]; VIEP-BUAP [2018]; and Plan de Trabajo CA [BUAP-CA-276]. E. Zambrano-Serrano acknowledges SEP-PRODEP/MEXICO by supporting the postdoctoral fellowship at BUAP (Grant No. 511-6/18-1242).

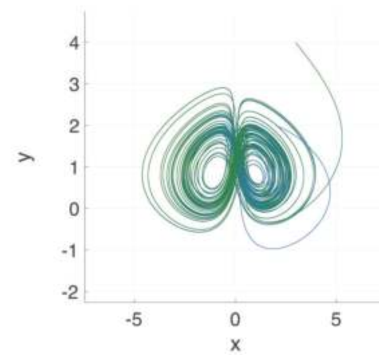


Fig. 5. Phase portraits of the fractional order system with $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, $a = 0$ for the blue attractor and $q_1 = 0.85$, $q_2 = 0.83$, $q_3 = 0.87$, $a = 4$ for the green attractor. The initial conditions $x_1(0) = 2, y_1(0) = 2, z_1(0) = 2$ and $x_2(0) = 3, y_2(0) = 4, z_2(0) = 3$ respectively applying active control approach. (Color online)

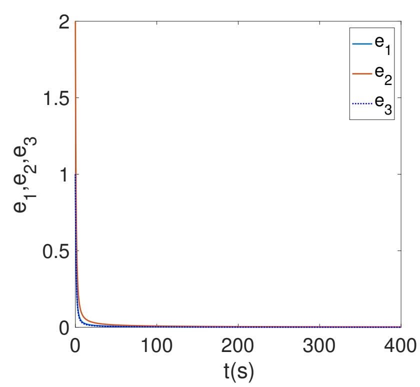


Fig. 6. Synchronization errors.

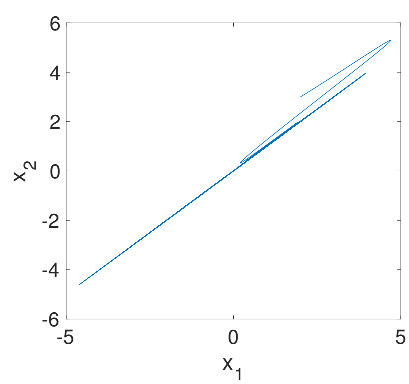


Fig. 7. Synchronization in the phase space $x_1 - x_2$.

 

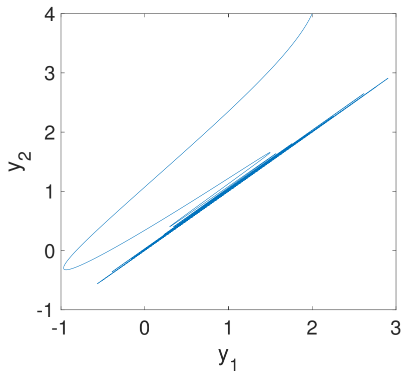


Fig. 8. Synchronization in the phase space $y_1 - y_2$.

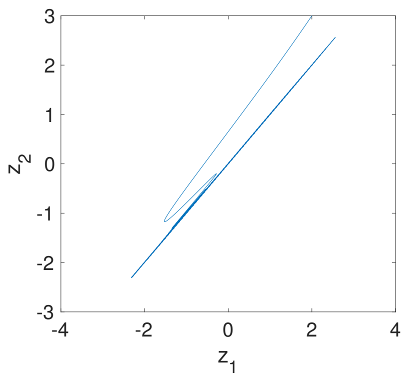


Fig. 9. Synchronization in the phase space $z_1 - z_2$.

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