

Strict Lyapunov functions for model-reference adaptive control based on the Mazenc construction

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Abstract: We analyze the stability of systems stemming from direct model-reference adaptive control. Although the statements of stability themselves are well-established for many years now, we provide a direct stability analysis both for linear and nonlinear systems under conditions of persistency of excitation. Our proofs are short and constructive as we provide strict Lyapunov functions that have all the required properties as established by Barbashin/Krasovskii's seminal papers on uniform global asymptotic stability.

Keywords: Adaptive control systems, persistency of excitation, time-varying systems

1. INTRODUCTION

Ever since its origins in the 1960s, analysis and design of adaptive control systems has been a steering force in the advancement of control theory. Particularly popular are the speed-gradient adaptive control approach Fradkov [1990] and model-reference adaptive control Narendra and Annaswamy [1986], since they apply to a variety of physical systems. Although simple in nature and quite intuitive, these methods pose significant challenges to stability analysis, due to their inherent nonlinear and time-varying nature; even in the case when the plant to be controlled is linear time-invariant.

A commonly-used method to establish the convergence of tracking errors is based on properties of signals in \mathcal{L}_p spaces; the most popular of these is known as *Barbalát Lemma* Barbalát [1959]. Guaranteeing convergence of the parameter errors, on the other hand, is a more challenging task of analysis. The recurrent, sufficient and necessary, condition under various control schemes is known as persistency of excitation and it was introduced in Åström and Bohn [1965] in the context of identification of discrete-time linear systems. Evolving from such context to the realm of continuous-time systems was a considerable step undertaken from the landmark papers Anderson [1977]; Morgan and Narendra [1977a;b]. Since then, not only the study of persistency of excitation has not been exhausted but it has taken various alternative forms —see Narendra and Annaswamy [1987; 1989] and the more recent papers Panteley et al. [2001]; Lee [2003]; Loria et al. [1999] where definitions tailored for nonlinear systems were introduced. Analysis methods for *linear* systems often rely on the integration of the system's dynamics and hence, on the scrutiny of the state transition matrix Ioannou and Sun [1996]. One of the most notable tools is the concept of uniform complete observability and the observation

that this is invariant under output injection Anderson et al. [1986]. Other methods, which rely on the uniform integrability of state trajectories, lead to shorter proofs Loria and Panteley [2002]. Notably such methods may also be used to compute explicit convergence rates, both in the linear Loria [2004] and nonlinear cases Loria and Panteley [2004]. See also Brockett [2000] for the case of gradient systems.

Without doubt, the most *direct* method of analysis is that of Lyapunov's. Indeed, converse Lyapunov theorems and stability proofs established in the previous and many other references, guarantee the existence of Lyapunov functions for adaptive-control systems. Yet, their complexity is such that constructing a strict Lyapunov function for such systems has eluded the research community, at least until Mazenc et al. [2009a] where, to the best of our knowledge, the first strict Lyapunov function for nonlinear systems reminiscent of model-reference-adaptive control was proposed. The neat *Mazenc construction* method used therein was originally introduced in Mazenc [2003] and is described in great detail in Malisoff and Mazenc [2009]. See also Loria et al. [2019].

In this paper we focus on systems that appear in model-reference-adaptive control of linear plants and feedback linearizable systems. This is a more particular class of systems than the one considered in Mazenc et al. [2009a;b] and Loria et al. [2019], but, for the same reason, the Lyapunov functions that result from the Mazenc construction method are fairly simple: firstly, for linear time-varying systems, reminiscent of those studied in Anderson [1977]; Anderson et al. [1986]; Ioannou and Sun [1996]; Loria and Panteley [2002], we provide a simple *quadratic* Lyapunov function with which exponential stability is established. Then, in the more realistic case, that of nonlinear time-varying systems, we provide an “almost” quadratic strict Lyapunov function. More precisely, it has a quadratic lower-bound, and a polynomial upperbound that is related to the degree of the nonlinearities in the system. Furthermore, the total derivative has a negative quadratic upperbound. The interest of having such simple Lyapunov

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functions may not be overestimated; for instance, they are fundamental in the analysis of robustness, computation of input-output \mathcal{L}_p gains, and for the purpose of Lyapunov redesign in contexts of disturbance compensation or dynamic output feedback designs.

We stress that in this note we focus on the “classical” method of direct model-reference adaptive control, as described, for instance, in Narendra and Annaswamy [1989]; Khalil [1996]. Hence, numerous more “modern” MRAC controllers, such as for instance that proposed in Guzman and Moreno [2011] which guarantees finite-time convergence, are beyond scope.

The remainder of this paper is organized as follows. For the sake of motivation and clarity of exposition, in Section 2 we revise the well-known direct-adaptive-control problem, which is otherwise well documented in many textbooks, such as Sastry and Bodson [1989]; Narendra and Annaswamy [1989]; Ioannou and Sun [1996]. In Section 3 we present our main results and we conclude with some remarks in Section 4.

2. MOTIVATION

Consider an innocuous single-input-single-output linear autonomous system,

$$\dot{x} = Ax + Bu \quad (1a)$$

$$y = Cx \quad (1b)$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$, and the pair (A,B) is controllable. For simplicity, let A and B be in the controllability canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \\ -a_1 & \cdots & -a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Suppose it is desired for the system (1) to behave as the reference model

$$\dot{x}^* = A^*x^* \quad (2)$$

where A^* is designed to be Hurwitz and its poles are chosen according to some desired performance. This problem may also be posed as that of stabilization of the trajectory $t \mapsto x^*$, which is solution of (2). To that end, we define the tracking errors $e := x - x^*$ and we define a new matrix A_{cl} with its poles carefully chosen to be stable. This determines the target closed-loop dynamics

$$\dot{e} = A_{cl}e \quad (3)$$

which determines the *transient* performance in closed loop while the reference model (2) describes the dynamics of the system in “steady” state.

Now, to achieve the target closed-loop dynamics (3) we subtract (2) from (1a) to find

$$\dot{e} = Ax - A^*x^* + Bu$$

from which it is clear that the control input

$$Bu := A^*x^* - Ax + A_{cl}e \quad (4)$$

leads to our objective. Since the system is in canonical controllable form, (4) is equivalent to

$$u = [a_1^* \cdots a_n^*]x^* + [a_1 \cdots a_n]x + [a_{cl1} \cdots a_{cln}]e$$

where a_i , a_i^* , and a_{cli} correspond, respectively, to the coefficients of the last row of A, A^* , and A_{cl} . Now, this control law may also be written as

$$u = \theta^{*\top}x^* - \theta^\top x + Ke \quad (5)$$

where

$$\theta^* := \begin{bmatrix} a_1^* \\ \vdots \\ a_n^* \end{bmatrix}, \quad \theta := \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad K := \begin{bmatrix} a_{cl1} \\ \vdots \\ a_{cln} \end{bmatrix}. \quad (6)$$

Clearly, the implementation of (5) relies on the knowledge of θ . In the case that the latter is unknown, it is common to use adaptive control and to invoke the certainty-equivalence principle. Namely, one replaces $\theta^\top x$ with $\hat{\theta}^\top x$ in the controller and uses an adaptation law to update $\hat{\theta}$ in function of the measured output $C^\top x$. That is, we redefine

$$u = \theta^{*\top}x^* - \theta^\top x + Ke - \tilde{\theta}^\top x,$$

where $\tilde{\theta} := \hat{\theta} - \theta$, in (1a). We see that the closed-loop dynamics now is

$$\dot{e} = A_{cl}e - Bx^\top \tilde{\theta}. \quad (7)$$

At this point, we introduce the function

$$V_1(e) := e^\top Pe, \quad P = P^\top > 0 \quad (8)$$

where P is solution to the Lyapunov equation $A_{cl}^\top P + PA_{cl} = -Q$, for any given $Q = Q^\top > 0$. Such P exists (and may be easily computed) because A_{cl} has been *designed* to be Hurwitz. Then, we consider the function

$$V_2(e, \tilde{\theta}) = V_1(e) + \frac{1}{\gamma} |\tilde{\theta}|^2, \quad \gamma > 0. \quad (9)$$

A direct computation shows that, by setting

$$\dot{\hat{\theta}} := \gamma x B^\top P e, \quad (10)$$

the derivative of V_2 along the trajectories of the closed-loop system,

$$\dot{e} = A_{cl}e - Bx^\top \tilde{\theta} \quad (11a)$$

$$\dot{\hat{\theta}} = \gamma x B^\top P e, \quad (11b)$$

yields

$$\dot{V}_2 \leq -q_m |e|^2, \quad q_m := \lambda_{\min}(Q). \quad (12)$$

Standard arguments, based on Barbalat’s lemma, may now be invoked to conclude that $e \rightarrow 0$ hence, that $x \rightarrow x^*$. Furthermore, to ensure that $\tilde{\theta} \rightarrow 0$ it is also well known that the regressor $\phi := x B^\top$ must be persistently exciting that is, that there must exist μ and $T > 0$ such that

$$\int_t^{t+T} \phi(\tau) \phi(\tau)^\top \geq \mu \quad \forall t \geq 0. \quad (13)$$

Sufficiency may be established if, in addition, there exists $\phi_M > 0$ such that

$$\max \{ |\phi|_\infty, |\dot{\phi}|_\infty \} \leq \phi_M \quad (14)$$

where

$$|\phi|_\infty := \sup_{t \geq 0} |\phi(t)|.$$

More precisely, the following well-known statement, which may be found in several texts on adaptive control, such as Narendra and Annaswamy [1989]; Sastry and Bodson [1989]; Anderson et al. [1986]; Ioannou and Sun [1996], is often invoked in the literature:

Lemma 1. Consider the *linear* time-varying system

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{\theta}} \end{bmatrix} = \begin{bmatrix} A & B\phi(t)^\top \\ -\phi(t)C & 0 \end{bmatrix} \begin{bmatrix} e \\ \hat{\theta} \end{bmatrix}. \quad (15)$$

Assume that the triple (A, B, C) is strictly positive real, ϕ is absolutely continuous bounded, and $\dot{\phi}$ is bounded almost everywhere. Then, the origin is globally exponentially stable if and only if (13) holds. \square

Unfortunately, in spite of the clear explanations found, *e.g.*, in Narendra and Annaswamy [1989]; Khalil [1996], it is often overlooked in the literature that the system (11) is not linear autonomous anymore and, consequently, Lemma 1 cannot be invoked “off-the-shelf” to analyze the stability of the origin for (11). Indeed, if we compare (11) with (15), we see that $A = A_{cl}$, $B = -B$, $C = -\gamma B^\top P$, and $\phi = x$. For the purpose of stability analysis this poses an important technical problem since x is the *state* of the original plant and ϕ in Lemma 1 is (meant to be) a function of time.

In order to carry on with a qualitative analysis of the solutions of (15), there are two possibilities: the first is to define $\phi(t) := x(t)$ and the second is $\phi(t, e) := [e + x^*(t)]$, but in either case the function ϕ does not depend on time *only*, as required in Lemma 1. In the first case ϕ is a functional of the original plant’s trajectories $x(t, t_o, x_o)$, which depend on the initial conditions (t_o, x_o) —see [Khalil, 1996, p. 626]. In the second case, ϕ is defined as function that depends both on time and (generally nonlinearly) on the state—see Narendra and Annaswamy [1989].

These facts have several crucial consequences. Firstly, if for the purpose of analysis we choose to use $\phi(t) := x(t)$, unless persistency of excitation is imposed to hold with the same μ and T for all initial conditions, *uniform* convergence may not be guaranteed for the *nonlinear* system (11)—see Loría and Panteley [2002]. Furthermore, even if uniform global asymptotic stability is possible—see Morgan and Narendra [1977b]; Narendra and Annaswamy [1989], *global exponential* stability is out of reach—see Gibson and Annaswamy [2015]. Yet, uniform exponential stability on any compact may still be obtained Loría [2004]; Loría and Panteley [2004].

If neither uniformity nor rate of convergence are of the essence, one can be content with establishing (non-uniform) convergence by invoking standard arguments that rely on Barbalát lemma and output injection analysis, as it is common in adaptive-control literature—Ioannou and Sun [1996]. Roughly speaking, the output-injection argument is the following: since $x = e + x^*(t)$, the closed-loop system (11) may be written as

$$\dot{e} = A_{cl}e - Bx^*(t)^\top \tilde{\theta} - Be^\top \tilde{\theta} \quad (16a)$$

$$\dot{\tilde{\theta}} = \gamma x^*(t) B^\top P e + \gamma e B^\top P e \quad (16b)$$

and, since $e \rightarrow 0$, it is expected that the output-injection terms

$$K(t, e) := \begin{bmatrix} -Be^\top \tilde{\theta} \\ \gamma e B^\top P e \end{bmatrix}$$

also vanish. On the other hand, the system (16) with $K \equiv 0$, is exactly of the form (15) and Lemma 1 may be invoked by imposing a condition of persistency of excitation on the *reference* trajectory $x^*(t)$. Hence, if the speed of convergence of $K(t, e(t))$ is “high enough”, it may also be concluded that $\tilde{\theta}(t)$ and $e(t) \rightarrow 0$. A rigorous, but lengthy proof, relying on the concept of uniform complete observability and integration of solutions, is provided for *linear* systems in Anderson et al. [1986], see also Ioannou and Sun [1996]. The necessary and sufficient conditions for uniform convergence are also emphasized in Loría and Panteley [2002] via shorter proofs.

An alternative, and intuitive, argument relies on the fact that a function ϕ does not lose its property of persistency of excitation when a (sufficiently fast) decaying *signal*

$t \mapsto \epsilon$ is added to it. More precisely, $\tilde{\phi}(t) = \phi(t) + \epsilon(t)$ is persistently exciting if so is ϕ , while $\epsilon \rightarrow 0$ and $\epsilon \in \mathcal{L}_2$ —Narendra and Annaswamy [1989]; Ioannou and Sun [1996]. Based on this fact, one may be tempted to use Lemma 1 for the *nonlinear* system (11) by arguing that $x(t)$ is persistently exciting because $x(t) = x^*(t) + e(t)$. While such reasoning has been extensively used in the literature, and may lead to establish convergence of both, $e(t)$ and $\tilde{\theta}(t)$, there is no guarantee that such convergence is uniform in the initial conditions.

3. MAIN RESULTS

Our first main result is a proof of Lemma 1 alternative to that found in some of the previous references. Secondly, we provide a direct proof of uniform global asymptotic stability for strictly-passive systems

$$\dot{e} = Ae + B(t, e)\tilde{\theta} \quad (17a)$$

$$\dot{\tilde{\theta}} = -\gamma B(t, e)^\top P e \quad (17b)$$

where $P = P^\top$ is such that $A^\top P + PA = -Q$ where $Q = Q^\top \geq q_m I$. These systems cover, in particular, (16). Our contributions lie in providing strict Lyapunov functions for both cases that is, Lyapunov functions that are positive definite radially unbounded and with negative definite derivative. As expected, only for the system (15) global exponential stability is concluded.

3.1 Linear systems

Lemma 2. Let $B_o(t) := B(t, 0)$ and consider the system

$$\dot{e} = Ae + B_o(t)\tilde{\theta} \quad (18a)$$

$$\dot{\tilde{\theta}} = -\gamma B_o(t)^\top P e, \quad \gamma > 0 \quad (18b)$$

Assume that there exist a_M, b_M, μ , and $T > 0$ such that $|A| \leq a_M$, $B_o(t)$ satisfies

$$\max \{ |B_o|_\infty, |\dot{B}_o|_\infty \} \leq b_M \quad \text{a.e.} \quad (19)$$

and

$$\int_t^{t+T} B_o(\tau)^\top B_o(\tau) d\tau \geq \mu I \quad \forall t \geq 0. \quad (20)$$

Then, for sufficiently large values of $c > 0$, the function

$$V(t, e, \tilde{\theta}) = c \left[e^\top P e + \frac{1}{\gamma} |\tilde{\theta}|^2 \right] - e^\top B_o(t)\tilde{\theta} - \frac{1}{4} \tilde{\theta}^\top \left[\int_t^\infty e^{t-\tau} B_o(\tau)^\top B_o(\tau) d\tau \right] \tilde{\theta}$$

□

is positive definite, radially unbounded, and satisfies

$$\sigma_1 |z|^2 \leq V(t, e, \tilde{\theta}) \leq \sigma_2 |z|^2 \quad (21)$$

$$\dot{V}(t, e, \tilde{\theta}) \leq -\sigma_3 |z|^2 \quad (22)$$

where $z := [e^\top \tilde{\theta}^\top]^\top$ and $\sigma_i > 0$.

Proof. For the sake of clarity, we start by rewriting the function V as

$$V(t, e, \tilde{\theta}) = cV_2(e, \tilde{\theta}) + W_1(t, e, \tilde{\theta}) + \frac{1}{4}W_2(t, \tilde{\theta}) \quad (23)$$

$$W_1(t, e, \tilde{\theta}) := -e^\top B_o(t)\tilde{\theta} \quad (24)$$

$$W_2(t, \tilde{\theta}) := -\tilde{\theta}^\top \left[\int_t^\infty e^{t-\tau} B_o(\tau)^\top B_o(\tau) d\tau \right] \tilde{\theta} \quad (25)$$

where we recall that V_2 is defined in (9). For further development, we underline some useful bounds on these functions. Firstly, in view of (19), for any $\lambda_1 > 0$, we have

$$|W_1(t, e, \tilde{\theta})| \leq b_M |e| |\tilde{\theta}| \leq \frac{\lambda_1 b_M^2}{2} |e|^2 + \frac{1}{2\lambda_1} |\tilde{\theta}|^2.$$

Secondly, in view of (20),

$$-b_M^2 |\tilde{\theta}|^2 \leq W_2(t, \tilde{\theta}) \leq -\mu e^{-T} |\tilde{\theta}|^2. \quad (26)$$

Finally, we point out that P is positive definite so there exist p_m and $p_M > 0$ such that $p_m I \leq P \leq p_M I$. Therefore,

$$\frac{c}{2} [p_m |e|^2 + \frac{1}{\gamma} |\tilde{\theta}|^2] \leq V_2(e, \tilde{\theta}) \leq \frac{c}{2} [p_M |e|^2 + \frac{1}{\gamma} |\tilde{\theta}|^2]$$

and, for sufficiently high values of $c > 0$, that is, such that

$$c > \max \left\{ \frac{\lambda_1 b_M^2}{p_m}, \frac{1}{\lambda_1} + 2b_M^2, 2\mu e^{-T} \right\}, \quad (27)$$

we conclude that there exist $\sigma_1, \sigma_2 > 0$, such that V satisfies (21).

Next, we compute the total derivative of V along the trajectories of (18). We have, on one hand,

$$\begin{aligned} \dot{W}_1(t, e, \tilde{\theta}) &= Y_1(t, e, \tilde{\theta}) \\ Y_1(t, e, \tilde{\theta}) &:= -e^\top A^\top B_o(t) \tilde{\theta} - \tilde{\theta}^\top B_o(t)^\top B_o(t) \tilde{\theta} \\ &\quad - e^\top \dot{B}_o(t) \tilde{\theta} + \gamma e^\top B_o(t) B_o(t)^\top P e. \end{aligned} \quad (28)$$

Therefore, defining $\xi := B_o(t) \tilde{\theta}$, for any $\lambda_2 > 0$, we obtain

$$\dot{W}_1(t, e, \tilde{\theta}) \leq \frac{c_1}{2} |e|^2 - \frac{1}{2} |\xi|^2 + \frac{1}{2\lambda_2} |\tilde{\theta}|^2 \quad (29)$$

where

$$c_1 := a_M^2 + (\lambda_2 + 2\gamma p_M) b_M^2 \quad (30)$$

On the other hand, $\dot{W}_2(t, \tilde{\theta}) = Y_2(t, e, \tilde{\theta})$ where

$$\begin{aligned} Y_2(t, e, \tilde{\theta}) &:= 2\gamma \tilde{\theta}^\top M(t) B_o(t)^\top P e + W_2(t, \tilde{\theta}) + |\xi|^2, \\ M(t) &:= \int_t^\infty e^{t-\tau} B_o(\tau)^\top B_o(\tau) d\tau. \end{aligned}$$

That is, in view of (26),

$$\dot{W}_2(t, \tilde{\theta}) \leq |\xi|^2 - \mu e^{-T} |\tilde{\theta}|^2 + \frac{1}{\lambda_3} |\tilde{\theta}|^2 + \lambda_3 \gamma^2 b_M^6 p_M^2 |e|^2. \quad (31)$$

Thus, using (12), (29), (31), and setting $c > 0$ such that

$$c q_m \geq \frac{\lambda_3}{2} \gamma^2 b_M^6 p_M^2 + \lambda_1 b_M^2 + c_1,$$

and λ_2, λ_3 such that

$$\frac{\lambda_2 \lambda_3}{\lambda_2 + 2\lambda_3} \geq \frac{2}{\mu e^{-T}},$$

we see that

$$\dot{V}(t, e, \tilde{\theta}) \leq -\frac{1}{4} |\xi|^2 - \frac{c q_m}{2} |e|^2 - \frac{\mu e^{-T}}{8} |\tilde{\theta}|^2 \quad (32)$$

Hence, (22) holds with

$$\sigma_3 := \min \left\{ \frac{c q_m}{2}, \frac{\mu e^{-T}}{8} \right\}.$$

3.2 Feedback linearizable systems

Lemma 2 provides a direct proof of an otherwise well-known statement that appears in numerous textbooks. As we stressed in the previous section, however, it is inefficacious in the analysis of model-reference adaptive control systems, even when dealing with linear time-invariant plants since, in closed loop, they are of the form (17).

To the best of our knowledge, the most direct statement on uniform global asymptotic stability for systems of the form (17) hence, which stands as a nonlinear counterpart of Lemma 2, is [Loría et al., 2000, Theorem 3]. Below we provide an alternative statement for such systems, which provides a polynomial strict Lyapunov function.

For the purpose of motivation, let us consider a fully feedback-linearizable system, in canonical form,

$$\dot{x}_1 = x_2 \quad (33a)$$

⋮

$$\dot{x}_{n-1} = x_n \quad (33b)$$

$$\dot{x}_n = \Phi(x)^\top \theta + g(x)u \quad (33c)$$

where $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a regressor function and $\theta \in \mathbb{R}^m$ is a vector of unknown lumped parameters. Assume that the control goal is to design u such that this system behaves as the reference model

$$\dot{x}_1^* = x_2^* \quad (34a)$$

⋮

$$\dot{x}_{n-1}^* = x_n^* \quad (34b)$$

$$\dot{x}_n^* = f(x^*) \quad (34c)$$

or, in other words, to steer $x(t) \rightarrow x^*(t)$ where $x^*(t)$ is solution of (34). The feedback-linearizing control input that achieves this goal is $u := g(x)^{-1} [f(x^*) - \Phi(x)^\top \theta - Ke]$ with $K := [\kappa_1 \ \dots \ \kappa_n]$. In case the the parameters θ are unknown, we use the certainty-equivalence adaptive controller

$$u = g(x)^{-1} [f(x^*) - \Phi(x)^\top \hat{\theta} - Ke] \quad (35a)$$

$$\dot{\hat{\theta}} = \gamma \Phi(x) B^\top P e \quad (35b)$$

where P satisfies $A^\top P + PA = -Q$, given an arbitrary positive-definite symmetric matrix Q , and

$$A := \begin{bmatrix} 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \\ -\kappa_1 & \dots & -\kappa_n \end{bmatrix},$$

which is Hurwitz by design. Then, defining the error coordinates $e := x - x^*$, we see that the closed-loop system takes the form (17) with A as above and $B(t, e) := -B\Phi(e + x^*(t))^\top$.

Lemma 3. Consider the system (17) under the following assumptions:

- (1) the matrix A is Hurwitz, P is positive definite symmetric and $A^\top P + PA = -Q = -Q^\top$;
- (2) there exist p and $\rho_i > 0$ with $i \leq p$ such that,

$$|B(t, e) - B_o(t)| \leq \sum_{j=1}^p \rho_j |e|^j; \quad (36)$$

- (3) the function B_o defined in Lemma 2 satisfies (19) and (20).

Let α be also a polynomial function with positive coefficients and of order $q = [p/2] + 1$, where $[p/2]$ denotes the smallest integer larger than, or equal to, $p/2$. Then, the function \mathcal{V} defined as

$$\begin{aligned} \mathcal{V}(t, e, \tilde{\theta}) &= \alpha \circ \left(e^\top P e + \frac{1}{\gamma} |\tilde{\theta}|^2 \right) - e^\top B_o(t) \tilde{\theta} \\ &\quad + W_1(t, e, \tilde{\theta}) + W_2(t, \tilde{\theta}), \end{aligned} \quad (37)$$

where W_1 and W_2 are defined in (24) and (25) respectively, is a strict Lyapunov function for the system (17). \square

Proof. The Lyapunov function candidate \mathcal{V} may be rewritten as

$$\mathcal{V}(t, e, \tilde{\theta}) = \alpha(V_2(e, \tilde{\theta})) + W_1(t, e, \tilde{\theta}) + W_2(t, \tilde{\theta}). \quad (38)$$

By definition, $\alpha(V_2)$ is a polynomial function of order q and with positive coefficients then, let $k_i > 0$ so

$$\alpha(V_2) = \sum_{i=1}^q k_i V_2^i. \quad (39)$$

Therefore, by setting $k_1 \geq c$ where c is defined in (27), we see that $\mathcal{V}(t, e, \tilde{\theta}) \geq V(t, e, \tilde{\theta})$ where the latter is defined in (23). Hence $\mathcal{V}(t, e, \tilde{\theta}) \geq V(t, e, \tilde{\theta})$ is positive definite and radially unbounded. Actually, it satisfies

$$\sigma_2 |z|^2 + \alpha'(V_2) \geq \mathcal{V}(t, e, \tilde{\theta}) \geq V_2(t, e, \tilde{\theta}) \geq \sigma_1 |z|^2$$

where $\alpha'(V_2)$ is a polynomial of degree q .

Next, we proceed to compute the total derivative of \mathcal{V} . With that purpose in mind we start by stressing that

$$\frac{\partial \alpha}{\partial V_2} = \sum_{i=1}^q i k_i V_2^{i-1}. \quad (40)$$

which, in expanded form, corresponds to

$$\frac{\partial \alpha}{\partial V_2} = k_1 + 2k_2 \left[V_1(e) + \frac{1}{\gamma} |\tilde{\theta}|^2 \right] + 3k_3 \left[V_1(e) + \frac{1}{\gamma} |\tilde{\theta}|^2 \right]^2 + \dots$$

On the other hand, by virtue of Newton's Binomial Theorem,

$$\left[V_1(e) + \frac{1}{\gamma} |\tilde{\theta}|^2 \right]^j \geq V_1(e)^j + j V_1(e)^{j-1} \frac{1}{\gamma} |\tilde{\theta}|^2$$

hence,

$$\begin{aligned} \frac{\partial \alpha}{\partial V_2} &\geq \sum_{j=0}^{q-1} (j+1) k_{j+1} V_1(e)^j \\ &\quad + \sum_{j=0}^{q-2} (j+1)(j+2) k_{j+2} V_1(e)^j \left[\frac{1}{\gamma} |\tilde{\theta}|^2 \right] \end{aligned}$$

or, in more compact form,

$$\frac{\partial \alpha}{\partial V_2} \geq \sum_{j=0}^{q-1} \beta'_j V_1(e)^j + \sum_{j=0}^{q-2} \beta_j |e|^{2j} |\tilde{\theta}|^2 \quad (41)$$

where

$$\beta_j := \frac{1}{\gamma} (j+1)(j+2) k_{j+2} p_m \quad (42)$$

$$\beta'_j := (j+1) k_{j+1}. \quad (43)$$

Let us now compute the total derivative of \mathcal{V} along the trajectories of (17). Firstly, we remark that V_2 satisfies (12) hence, in view of (41), the total derivative of $\alpha(V_2)$ along the trajectories of (17) satisfies

$$\overline{\alpha(V_2)} = -q_m \frac{\partial \alpha}{\partial V_2} |e|^2$$

hence

$$\overline{\alpha(V_2)} \leq -q_m \sum_{j=0}^{q-2} \beta_j |e|^{2j+2} |\tilde{\theta}|^2 - q_m \sum_{j=0}^{q-1} \beta'_j |e|^{2j+2}. \quad (44)$$

Next, to compute the derivatives of W_1 and W_2 along the trajectories of (17), it is useful to write the dynamics in the *output-injection* form

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} Ae + B_o(t) \tilde{\theta} \\ -\gamma B_o(t)^\top P e \end{bmatrix} + \begin{bmatrix} [B(t, e) - B_o(t)] \tilde{\theta} \\ [B(t, e) - B_o(t)]^\top P e \end{bmatrix} \quad (45)$$

and we use the computations from the proof of Lemma 2.

On one hand, from (28), we obtain

$$\begin{aligned} \dot{W}_1 &= Y_1(t, e, \tilde{\theta}) + \gamma e^\top B_o(t) [B(t, e) - B_o(t)]^\top P e \\ &\quad - \tilde{\theta}^\top [B(t, e) - B_o(t)] B_o(t) \tilde{\theta}. \end{aligned} \quad (46)$$

where Y_1 is defined in (28) and, on the other hand, the total derivative of W_2 yields

$$\dot{W}_2(t, \tilde{\theta}) = Y_2(t, e, \tilde{\theta}) + 2\gamma \tilde{\theta}^\top M(t) [B(t, e) - B_o(t)]^\top P e. \quad (47)$$

We proceed now to bound the “output-injection” terms, that is, those containing $[B(t, e) - B_o(t)]$. To that end, let us consider the inequality (36); we remark that, for each $j \geq 3$ odd, we have

$$\rho_j |e|^j \leq \frac{\rho_j}{2} \left[|e|^{j-1} + |e|^{j+1} \right].$$

while, for any $\lambda_5 > 0$,

$$\rho_1 |e| \leq \frac{\rho_1}{2} \left[\frac{1}{\lambda_5} + \lambda_5 |e|^2 \right].$$

Therefore,

$$|B(t, e) - B_o(t)| \leq \sum_{j=0}^{\lceil p/2 \rceil} \delta_j |e|^{2j}$$

where

$$\delta_0 := \frac{\rho_1}{2\lambda_5}, \quad \delta_1 := \frac{\rho_1 \lambda_5 + 2\rho_2 + \rho_3}{2},$$

$$\delta_j := \frac{\rho_{2j-1} + 2\rho_{2j} + \rho_{2j+1}}{2}, \quad \forall j \geq 2$$

Using the latter and the definition $q := \lceil p/2 \rceil + 1$, we see that, on one hand, we have

$$\begin{aligned} \dot{W}_2(t, \tilde{\theta}) &\leq |\xi|^2 - \mu e^{-T} |\tilde{\theta}|^2 + \frac{1}{\lambda_3} |\tilde{\theta}|^2 + \lambda_3 \gamma^2 b_M^6 p_M^2 |e|^2 \\ &\quad + \frac{1}{\lambda_4} |\tilde{\theta}|^2 + \lambda_4 \gamma^2 b_M^4 p_M^2 \sum_{j=0}^{\lceil p/2 \rceil} \delta_j |e|^{2j} \end{aligned}$$

for which we also used (31). On the other hand, after (29), we obtain

$$\begin{aligned} \dot{W}_1(t, e, \tilde{\theta}) &\leq \left[\frac{c_1}{2} + \gamma b_M p_M \sum_{j=0}^{q-1} \delta_j |e|^{2j} \right] |e|^2 \\ &\quad + \left[b_M \sum_{j=0}^{q-1} \delta_j |e|^{2j} + \frac{1}{2\lambda_2} \right] |\tilde{\theta}|^2 - \frac{1}{2} |\xi|^2 \end{aligned} \quad (48)$$

Now, consider (44) and let

$$\beta'_0 \geq c_1 + \rho_1 \gamma b_M p_M + \frac{\lambda_3}{2} \gamma^2 b_M^6 p_M^2, \quad (49)$$

$$\beta'_j \geq \frac{[\gamma^2 b_M^4 p_M \lambda_4 / 4 + \gamma b_M p_M] \delta_j}{q_m}, \quad \forall j \in [1, q-1] \quad (50)$$

$$\beta_j \geq \frac{b_M \delta_{j+1}}{q_m}, \quad \forall j \in [0, q-2] \quad (51)$$

Thus, provided that λ_2 , λ_3 , and λ_5 are such that

$$\frac{\mu e^{-T}}{8} \geq \frac{\rho_1 b_M}{\lambda_5} + \frac{1}{\lambda_2} + \frac{1}{4\lambda_3},$$

$$\dot{\mathcal{V}}(t, e, \tilde{\theta}) \leq -\beta'_0 |e|^2 - \frac{\mu e^{-T}}{8} |\tilde{\theta}|^2 - \frac{1}{4} |\xi|^2. \quad \blacksquare$$

Remark 1. It is important to remark that even though \mathcal{V} satisfies a strict quadratic negative upperbound, one should not haste to conclude uniform global exponential stability of the origin, even for the system (11). Indeed, note that for this system the inequality (36) holds with $p = 1$, since $B(t, e)$ has linear growth. Hence, $\alpha(V_2)$ in (37) is of second order and, consequently, \mathcal{V} satisfies a quadratic lower-bound but not a quadratic upper-bound, as required to conclude uniform global exponential stability.

4. CONCLUSIONS

We have presented two Lyapunov functions for linear and nonlinear systems appearing in direct adaptive control. Although our results are restrictive in regards to the class of systems they may be considered as a building block in the analysis of more complex nonlinear time-varying systems. That is, in control problems where, otherwise, one needs to rely on converse Lyapunov theorems. For instance, they may be useful in the analysis of perturbed systems or in control redesign, notably in the context of dynamic output feedback control.

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