Dynamic Output Controller with Sample-Time Gain Adjustment for a double inverted pendulum system stabilization

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Abstract: In this paper, we propose a control method to stabilize the double inverted pendulum on car system (DIPCS). The DIPCS is a prismatic type inverted pendulum system that has a base-link and two interconnected pendulums with different lengths. The proposed control methodology does not require the knowledge of the dynamic model and under assumption that not all state variables are available to be measured. Stabilization around the upper unstable equilibrium is one of the most important control problems for the DIPCS. Then, in order to stabilize the DIPCS at its upright position, it is used an adaptive dynamic feedback controller. Moreover, the adaptive approach is based on full dynamics of the output feedback controller. Finally, our proposal is such that guarantees the boundedness of the obtained deviations of error state function by means of the concept of the UUB (Uniform-ultimately bounded) stability. Since any bounded dynamics can be imposed inside of a multidimensional ellipsoid, we suggest the gain tuning of the adaptive controller providing that all possible trajectories of the DIPCS arrive into a small size ellipsoid.

Keywords: Robust control, double inverted pendulum on car system, dynamic output control, uniform-ultimately bounded stability.

1. INTRODUCTION

Acrobatic robots have been studied as a typical examples of underactuated mechanical systems, see e.g. Fantoni and Lozano (2001); Ordaz and Poznyak (2015); Spong (1994). The **DIPCS** (see Fig. 1) is a classical robotic-type underactuated system (**RT-US**). The dynamics can be represented in the standard second order differential equation as:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + F(q,\dot{q}) = \tau \qquad (1)$$

where q, $\dot{q}$, and $\ddot{q} \in \mathbb{R}^n$ are angular position, velocity and acceleration respectively, and n defines the degree of freedom (in this case $n = 3$). The inertia matrix is $D(q)$, the Coriolis and centripetal matrix is $C(q,\dot{q})$, $G(q)$ and $F(q,\dot{q})$ are the corresponding potential and tribology vector forces. Commonly $F(q,\dot{q})$ contains strong nonlinearities such as Armstrong friction, Maxwell slip effects, Viscous and Coulomb frictions, among other uncertainties. The mathematical model of this system can be found in Rubı et al. (2002). The set of joints which describes the movement of the **DISP** is given by one prismatic and two both revolute joints. The first one defines the linear position q_1 around the horizontal axis. The double pendulum positions are measured by the revolute joints between each pendulum, which give the angular position q_2 and q_3. The parameters of the **DIPCS** are: M_1 as the mass of the prismatic joint, $m_2 \neq m_3$ are the mass of second and third link, respectively. The lengths of each pendulum are given by $l_2 \neq l_3$, and g is the gravity constant. Physically, the positions $(q = \{q_1, q_2, q_3\},\ q_i \in \mathbb{R},\ i = 1,2,3)$ can be measured (for example) by resistive or optical sensors. They are available in both, rotational and linear joint. Leaving the velocity to be estimated by using numerical sensors. The velocity and acceleration information is often obtained by numerical differentiation of the quantized position signal. However, with an encoder pulse train, the sampled positions contain numerical errors which results in uncertainties on velocity estimation Kim and Lee (2008); Yu and Li (2006). Other velocity state estimations are based on nonconventional control analysis, like sliding mode or neural network pattern Polyakov and Poznyak (2009); Chan (1998). Besides this, the state estimation can be designed via output control design Merry et al. (2010). One of the most usual output control realization is based on linearized model where the control objective is to drive around to zero the error of state estimation. For this reason, in this work we assume that the set of velocities $\dot{q} \in \mathbb{R}^3$ is not available to be measured.

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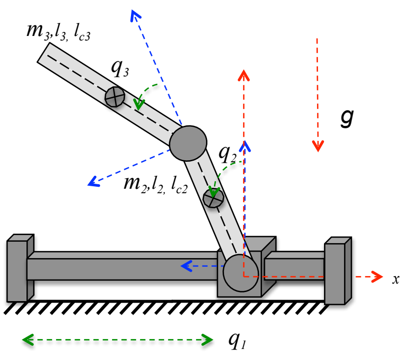

Fig. 1. Double Inverted Pendulum System (**DIPCS**).

On the other hand, the control problem of **RT-US** is divided in two different procedures. The first one is swing from stable equilibria to unstable one (or upright configuration). The second one is related to balance at the unstable equilibrium (stabilize around its unstable configuration). For decades the swing-up of **RT-US** is taken by the classical control based on energy analysis, which gives good results Fantoni and Lozano (2001); Rubı et al. (2002); Spong (1994). In order to stabilize systems, a classical Linear Quadratic Regulator, $\mathcal{H}$-infinity, neural network, fuzzy or robust control are implemented on them. For the case when the dynamic model is not available, the classical neural network, or fuzzy control are well implemented. So, in the literature review, the compensators based on fuzzy and neural network, for this class of systems, do not present any analytical formalism to conclude any type of stability in Lyapunov sense. Other control strategy is the so called Dynamic Output Feedback (see for example Azhmyakov et al. (2013)). Therefore, these techniques do not consider the external disturbances analysis or the case when not all state variables are available to be measured. In the classical works, the authors assume that the dynamic model is completely known and take into account the linearization of an ideal model. In the real life, this assertion is not true, because the mechanical systems have tribology effects, parameter uncertainties, and other uncertainties. Instead to use a classical state estimation, the Dynamic Output Feedback algorithm allows to control a process without a complete availability of state variables. Moreover, by using sophisticated processing algorithms on the model, we can apply robust techniques (see for example Azhmyakov et al. (2013); Nazin et al. (2007)). Then the question is the following. Based on previous facts and by using the classical stability theory without the knowledge of the mathematical model, is it possible to design a control law for stabilization of the **DIPCS**?

In order to give an answer to the previous question, in this paper we work with a singular class of functions called *Quasi-Lipschitz* functions (**QL-F**). The study of this class of functions is a typical case of external disturbances rejection presented on the Sliding-Mode control theory. Bellow, we present a definition for the functions of a class of **QL-F**. This class of functions permits to work with parametric uncertainties, unmodeled dynamics, external bounded perturbations and dynamic uncertainties Nazin et al. (2007); Ordaz and Poznyak (2015). In order to define the class of **QL-F**, we require two scalar and one matrix coefficients. One of them, defines the uncertainty, unknown dynamics and external perturbation characteristics in terms of its upper bound. The other one, is like Lipschitz constant (we present this parameter characteristic on the next section). The third coefficient is a matrix, this defines a function deviation which is represented in Quasi-linear form. The selection of these parameters is not trivial, because in order define them, we require the knowledge of mathematical model Ordaz and Poznyak (2015). Then, by former assumptions, we consider that the parameters of **QL-F** are unknown too (at least, the matrix parameter). So, the consideration of this class of functions permits to rewrite the nonlinear dynamic model as Perturbed Quasi-linear one.

1.1 Main contribution

The goal of this work concerns with finding the unknown parameters which describe the **QL-F** nonlinear system (the perturbed quasi linear dynamics format). Moreover, to stabilize the system we use a specific sample time gain adjustment based on Dynamic Output feedback. By using online information during the process, the controller (in some sense) is a class of learning or adaptive feedback. But the difference between time varying and adaptive control is given by the previous knowledge of some deviation gain matrix. Now, in this work, we present this characteristic. The deviation gain matrix is obtained by minimization of a specific energy function. Actually, this procedure is developed such that the system trajectories remain into a bounded invariant set. This process is known as Attractive ellipsoid Method (**AEM**).

1.2 Paper outline

The outline of this work is as follows: In Section 2 we present some important concepts, definitions used and the nonlinear uncertain model. The Problem formulation is presented in Section 3. Next section, presents the main paper contribution, the control algorithm and its stability analysis. The numerical aspects are presented in Section 5. Section 6 presents the **DIPCS** as illustrative example. Finally we give the work conclusions.

2. SOME DEFINITIONS AND CONCEPTS

In this section we present some concepts used in this work, the system estate space representation in the standard Quasi linear format.

2.1 Quasi-Lipschitz functions

Definition 1. (The class $\mathcal{C}$ of quasi-Lipschitz functions). A vector function $g : \mathbb{R}^n \to \mathbb{R}^n$ is said to be from the class

$\mathcal{C}(G,\delta_1,\delta_2)$ $(G\in\mathbb{R}^{n\times n};\ \delta_1,\delta_2\geq 0)$ of **quasi-Lipschitz functions** if for any $x\in\mathbb{R}^n$ it satisfies the inequality

$$\|g(x)-Gx\|^2\leq\delta_1+\delta_2\|x\|^2.$$

Notice that

- the growth rate of $g(x)$ as $\|x\|\to\infty$ is not faster than linear;

- if $\delta_1=0$ the class $\bar{\mathcal{C}}(G,0,\delta_2)$ is the class of Lipschitz functions.

2.2 Attractive ellipsoids

Definition 2. (attractive ellipsoid). We say that the ellipsoid

$$\mathcal{E}(0,P)=\{x\in\mathbb{R}^n: x^\intercal Px\leq 1,\ P=P^\intercal>0\}$$

(with the center in the origin and with the corresponding ellipsoidal matrix P) is **attractive** for some dynamic system if for its trajectory $\{x\}_{t\geq 0}$ the following property holds:

$$\limsup_{t\to\infty} x^\intercal(t)Px(t)\leq 1.$$

In fact, all trajectories $\{x\}_{t\geq 0}$ of a dynamic system remain bounded if for this system there exists an attractive ellipsoid $\mathcal{E}(0,P)$.

2.3 Uncertain nonlinear model

Consider the nonlinear dynamic system given by

$$\begin{array}{l}\dot{x}=f(x)+Bu+\zeta_x(t),\ t\in\mathbb{R}_+\\ y=h(x)+\zeta_y(t),\ x(0)=x_0\in\mathbb{R}^n,\end{array}\qquad(2)$$

where: $x\in\mathbb{R}^n$ is the state vector, $y\in\mathbb{R}^p$ is the output system, $f:\mathbb{R}^n\to\mathbb{R}^n$ and $h:\mathbb{R}^n\to\mathbb{R}^p$ are **uncertain nonlinear vector-functions** participating in the right-hand side of dynamics and output system (2) respectively, $B\in\mathbb{R}^{n\times m}$ is the matrix realizing the actuator-mapping, $u\in\mathbb{R}^m$ is the control input at time t, $\zeta_x(t)\in\mathbb{R}^n$ and $\zeta_y(t)\in\mathbb{R}^p$ are external perturbations.

In this work we suppose that the functions f and h of (2) are quasi-Lipschitz, fulfilling the next assertion:

$$f\subset\bar{\mathcal{C}}(A,c_1,c_2),\ h\subset\bar{\mathcal{C}}(C,c_3,c_4),$$

moreover, the external perturbations $\zeta_x(t)$ and $\zeta_y(t)$ are assumed to be bounded:

$$\|\zeta_x(t)\|^2\leq c_5<\infty,\ \|\zeta_y(t)\|^2\leq c_6<\infty,$$

which define that the system (2) can be represented in the *quasi-linear format* as follows

$$\begin{array}{l}\dot{x}=Ax+Bu+\xi_x(x,t),\quad x(0)=x_0,\\ y=Cx+\xi_y(x,t),\end{array}\qquad(3)$$

with

$$\begin{array}{l}\xi_x(x,t):=\Delta f(x)+\zeta_x(t),\ \Delta f(x):=f(x)-Ax,\\ \xi_y(x,t):=\Delta h(x)+\zeta_y(t),\ \Delta h(x):=h(x)-Cx.\end{array}\qquad(4)$$

Furthermore, here the uncertain terms $\xi_x^\intercal(x,t)$ and $\xi_y^\intercal(x,t)$ satisfy for any $t\geq 0$ and any x, the following inequality:

$$\|\xi_x(x,t)\|^2\leq d_1+d_2\|x\|^2,\ \|\xi_y(x,t)\|^2\leq d_3+d_4\|x\|^2.\qquad(5)$$

where d_i, $i=1,...,4$, are fixed bounded scalars.

3. PROBLEM FORMULATION

The goal of this paper is to provide robust stability on the system (2) by using the full order output dynamic system. Then, we need to design the control gain matrices $\mathbf{C}\in\mathbb{R}^{m\times n}$ and $\mathbf{D}\in\mathbb{R}^{m\times p}$ of the full order output compensator

$$u=\mathbf{C}x_r+\mathbf{D}y,\qquad(6)$$

where $x_r\in\mathbb{R}^n$ has the following dynamic based on the (2) output

$$\dot{x}_r=\mathbf{A}x_r+\mathbf{B}y,\qquad(7)$$

with $\mathbf{A}\in\mathbb{R}^{n\times n}$ and $\mathbf{B}\in\mathbb{R}^{n\times p}$. Furthermore, the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ matrices also should be designed such that the system trajectories x in closed loop with (6) arrive into an attractive ellipsoid (if it exists) of a minimal possible *size* (in fact, the minimization of the trace of the inverse ellipsoid matrix). Now, in order to solve this problem we assume the next statements:

i) For some reason we do not know some characteristics of the **quasi-Lipschitz functions**. Actually, we suppose that matrix A is unknown but we have an estimate $\hat{A}_k$ with its corresponding matrix dimensions, moreover, we know matrix $\mathbf{C}$ and the scalar parameters c_i, $i=1,...,4$.

ii) In order to arrive the system trajectories into an ellipsoid of minimal size, we select the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ as constant over sample time as $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$ and $\mathbf{D}_i$ on the time intervals $t\in[t_{i-1},t_i)$, for all $i=1,2,....$ The above matrices are estimated by online information based on previous time interval measurements.

iii) The original nonlinear system (2) is assumed to be controllable and observable (see for example Isidori (1995)); it is assumed that the pair (A,B) is controllable and the pair (A,C) is observable.

Under previous assumptions, and using the extended dynamics $z^\intercal=[x_r^\intercal,\ x^\intercal]$ we can define a full order dynamic output feedback system in closed loop as:

$$\dot{z}=(\mathcal{B}\Theta_i\mathcal{C}+\mathcal{A}_i)\,z+\mathcal{B}\Theta_i\xi_1+\xi_2,\quad z(0)=z_0,\qquad(8)$$

where the uncertainties $\xi_1^\intercal:=[0,\ \xi_y^\intercal(x,t)]$, $\xi_2^\intercal:=[0_n^\intercal,\ \xi_x^\intercal(x,t)]$, and the matrices

$$\mathcal{B}:=\begin{bmatrix}I&0\\0&B\end{bmatrix},\ \ \mathcal{C}:=\begin{bmatrix}I&0\\0&C\end{bmatrix},$$

$$\mathcal{A}_i:=\begin{bmatrix}0_{n\times n}&0_{n\times n}\\0_{n\times n}&\hat{A}_k\end{bmatrix},\ \Theta_i:=\begin{bmatrix}\mathbf{A}_i&\mathbf{B}_i\\\mathbf{C}_i&\mathbf{D}_i\end{bmatrix},$$

with Θ_i given on time interval $t\in(t_{i-1},t_i]$. Remember that the ellipsoid $\mathcal{E}\left(0,\bar{P}\right)$ is a compact set, with $\bar{P}>0$, and it is named positively **invariant** if any trajectory initiated in this set, remains inside of the set during all future time. Now, we are ready to formulate the problem to solve.

Problem formulation. *Based on the available information* $\{x,x_r,\hat{A}_k\}_{t\geq 0}$ and Θ_0 *the problem is to design the* *sequence* $\{\Theta_i\}_{i=1,2,...}$ *of the gain matrices* $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$ and $\mathbf{D}_i$ *providing for any plant with uncertainties from the class* $\bar{\mathcal{C}}(G,\delta_1,\delta_2)$ *the existence of an attractive ellipsoid of a minimal possible size (traditionally, the size* $\mathcal{E}\left(0,\bar{P}\right)$ is associated with the trace of the ellipsoid matrix $\bar{P}$. This problem can be formulated by the following minimization problem:

$$\begin{array}{l}\min \operatorname{tr}\left\{\bar{P}_i\right\},\\ \text{subject to } 0<\bar{P}_i, \quad \Theta_i\,(i=1,2,...)\,.\end{array} \tag{9}$$

The *UUB-property* of the uncertain extended system (8), is guaranteed if the minimization problem (9) is feasible.

4. ON THE FULL ORDER DYNAMIC OUTPUT CONTROL

In this section we present the basic idea of designing an output controller providing a *good* robust performance for the extended system (8) under perturbations or uncertainties $\xi_x(t)$ and $\xi_y(t)$ satisfying (3) based on **AEM** concept.

Now, because not all state variables are available to be measured, we use next fact:

$$\bar{x}:=\bar{x}(t)=h^{-1}\left[x(t)-x(t-h)\right],\ 0<h\ll 1,$$

for all time interval t_i, so that $\dot{x}(t)=\bar{x}(t)+\Delta(t)$, where $\Delta(t)$ is the current error of the Euler approximation. Substituting this on the original one

$$\xi=\xi_x+B\mathbf{D}_i\xi_y=\bar{x}+\Delta-(A+B\mathbf{D}_iC)\,x-B\mathbf{C}_ix_r,$$

hence,

$$\delta=\dot{x}-Ax=\xi-\Delta$$

is the *joint uncertainty* term, and

$$\dot{x}:=\mathbf{E}x+B\mathbf{C}_ix_r+\bar{x},\ \mathbf{E}:=-\left(A+B\mathbf{D}_iC\right)$$

or in discrete form as:

$$x_{k+1}:=\mathbf{E}x_k+B\mathbf{C}_ix_{rk}+\bar{x}_k$$

is the measurable vector available at any discrete time $t_k\subseteq t\in[t_{i-1},t_i)$.

Corollary 1. Suppose that matrix A matrix is unknown. Here matrix A is given by classical general matrix recursion in the extended form:

$$x_{k+1}=\mathbf{F}\mathbf{z}_k+\xi_k,\ \mathbf{F}:=[\mathbf{E},B\mathbf{C}_i],\mathbf{z}=[x_k^{\intercal},\ x_{rk}^{\intercal}]. \tag{10}$$

Then, the Least Mean Squares $\mathbf{F}_k$ of the matrix $\mathbf{F}$ is uniquely defined and is given by

$$\begin{array}{c}\mathbf{F}_n=H_nZ_n^{-1}\\ H_n:=\sum_{j=1}^{n}\mathbf{z}_jx_{j+1}^{\intercal},\ Z_n:=\sum_{j=0}^{n}\mathbf{z}_n\mathbf{z}_n^{\intercal}>0.\end{array} \tag{11}$$

By the previous result, the matrix A can be estimated in discrete form as $\hat{A}_k$. And for every time interval the estimated matrix remains constant as $\hat{A}_k$. Since we need that all extended system trajectories arrive into an ellipsoid of minimal size, then, we need to guarantee that the matrix estimation remains bounded. Moreover, we do not know the matrix gains $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$. In order to formulate a result which includes some solution of the unknown matrices, we use an storage function as follows:

$$V(z)=z^{\intercal}P_iz, \tag{12}$$

where $P_i:=\operatorname{diag}([R_i,\ \mathcal{P}_i])$, and $R_i,\ \mathcal{P}_i\in\mathbb{R}^{n\times n}$ are positive definite matrices. Matrix $\hat{A}$ is used in order to give a set of solutions (P_i,Θ_i) in terms of available information. Moreover, on the designed algorithm this fact gives some learning process.

Proposition 1. Due to the time interval $t\in[t_{i-1},t_i)$, for $i\in\mathbb{N}$ and for some given initial condition of its interval. If there exists a collection $(P_i,\Theta_i,Q_i,\beta_i,\alpha_i)$ such that it satisfies the matrix inequality $W_i<0$, with:

$$\begin{array}{c}W_i=\begin{bmatrix}\Omega_i, & \Theta_i^{\intercal}\mathcal{B}^{\intercal}P_i^{\intercal}, & P_i\\ P_i\mathcal{B}\Theta_i, & -\varepsilon_{1,i}I, & 0\\ P_i, & 0, & -\varepsilon_{2,i}I\end{bmatrix},\\ \Omega_i=P_i\left(\mathcal{B}\Theta_i\mathcal{C}+\bar{\mathcal{A}}_i+\frac{\alpha}{2}I\right)+\left(\mathcal{B}\Theta_i\mathcal{C}+\bar{\mathcal{A}}_i+\frac{\alpha}{2}I\right)^{\intercal}P_i+Q_i,\\ Q=\begin{bmatrix}0 & 0\\ 0 & (d_2\varepsilon_{1,i}+2d_4\varepsilon_{2,i})I\end{bmatrix},\end{array} \tag{13}$$

where $Q\in\mathbb{R}^{2n\times 2n}$ is positive definite matrix, $\varepsilon_{1,i}$, $\varepsilon_{2,i}$, $\alpha_i\in\mathbb{R}$ are positive too. Then, for the storage function (12), the next assertion holds

$$\begin{array}{c}\dot{V}_i\leq\dfrac{\beta_i}{\alpha_i}+\left(V_{i-1}-\dfrac{\beta_i}{\alpha_i}\right)\exp\left[-\alpha_i\left(t+t_{i-1}\right)\right],\\ \beta_i:=\varepsilon_{1,i}d_1+2\varepsilon_{2,i}d_3.\end{array}$$

On the other hand, it is well-known that the concept of an energetic function was rigorously formalized by means of the Lyapunov stability theory as well as the notion of a positive invariant set.

Now, we can rename $V(z)$ on time interval $t\in[t_{i-1},t_i)$ as $V_i\left(z\right)$. Here we just note that if the former proposition holds, it implies that, the storage function (12) is not obligatory monotonically non increasing. In other words, $V_i\left(z\right)$ is not a Lyapunov function for the considered system. At least for this time-interval.

It would be of interest to prove an assertion about the existence of a family of attractive ellipsoids. To make this ellipsoid of a minimal possible size, it is sufficient to select the free parameters $(P_i,\Theta_i,Q_i,\beta_i,\alpha_i)$ as a solution of the following optimization problem (it follows that a "bigger" $\bar{P}$ provides a "smaller" $z\left(t\right)$):

$$\begin{array}{l}\max\left\{\dfrac{\alpha_i}{\beta_i}\operatorname{tr}P_i\right\},\\ \text{subject to }\left\{\begin{array}{l}0<\alpha_i,\varepsilon_{1,i},\ \varepsilon_{2,i},\\ 0<P_i=P_i^{\intercal},\ W<0,\ \Theta_i.\end{array}\right.\end{array} \tag{14}$$

Notice that this optimization problem is a nonlinear optimization problem, subject to bilinear matrix inequality (**BMI**), with fixed $\alpha_i,\varepsilon_{1,i},\varepsilon_{2,i}$. Finally the set of solutions $(P_i,\Theta_i,Q_i,\beta_i,\alpha_i)$ of the optimization problem (14) gives the best solution to stabilize the extended system (8).

5. NUMERICAL ASPECTS

The solution of the optimization problem given by Proposition 1 satisfies the set of **BMI**'s (under fixed scalar parameters) having the following structure

$$\mathcal{W}_i = \begin{bmatrix} \Gamma_i, & \Theta_i^\intercal \mathbf{P}_i^\intercal, & P_i \\ \mathbf{P}_i \Theta_i, & -\varepsilon_{1,i} I, & 0 \\ P_i, & 0, & -\varepsilon_{2,i} I \end{bmatrix} < 0, \tag{15}$$
$$\Gamma_i = \mathbf{P}_i \left(\Theta_i \mathcal{C} + \bar{\mathcal{A}}_i + \tfrac{\alpha}{2} I\right) + \left(\Theta_i \mathcal{C} + \bar{\mathcal{A}}_i + \tfrac{\alpha}{2} I\right)^\intercal \mathbf{P}_i + Q_i.$$

Using the, so-called, *regular form representation* for the quasi-linear model (8) (see, for example, Polyakov and Poznyak (2009), Section 19.4.3.2) and defining the non-singular matrix

$$\bar{\mathbf{G}} := \begin{bmatrix} I_{(n-m)\times(n-m)} & -B_1 B_2^{-1} \\ 0_{m\times(n-m)} & B_2^{-1} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{16}$$

let us try to find the matrix $\mathbf{P}_i$ as $\mathbf{P}_i = T^\intercal P_i T \mathcal{B}$ where T is a block-diagonal

$$T := \begin{bmatrix} I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \bar{\mathbf{G}} \end{bmatrix}. \tag{17}$$

By using the change of variables $X_i := \mathbf{P}_i \Theta_i$, $Y_i := \mathbf{P}_i$ and $Z_i = P_i$, then we can formulate the next lemma.

Lemma 1. The set of variables satisfying (15) is contained within the set of variables satisfying the following Linear Matrix Inequality, ***LMI*** (under fixed scalar parameters)

$$\bar{\mathcal{W}}_i \left(X_i, Y_i \mid \alpha, \varepsilon_{1,i}, \varepsilon_{2,i}\right) := \begin{bmatrix} \bar{\Gamma}_i, & X_i^\intercal, & Z_i \\ X_i, & -\varepsilon_{1,i} I, & 0 \\ Z_i, & 0, & -\varepsilon_{2,i} I \end{bmatrix} < 0, \tag{18}$$

$$\Gamma_i = X_i \mathcal{C} + Z_i \bar{\mathcal{A}}_i + \mathcal{C}^\intercal X_i^\intercal + \bar{\mathcal{A}}_i^\intercal Z_i^\intercal + \alpha Z_i + Q_i,$$

with the elements subject to the block diagonal constraint $Z_i := \mathrm{diag}(Z_{i,1}, Z_{i,2}) > 0$.

In new variables X_i and Y_i the optimization problem (14) can be formulated as follows:

$$\max \left\{ \frac{\alpha_i}{\beta_i} \mathrm{tr}\left\{Z_i\right\} \right\}, \quad \text{subject to } \begin{cases} 0 < \alpha_i, \varepsilon_{1,i}, \ \varepsilon_{2,i}; \\ X_i, \ Y_i, \ 0 < Z_i = Z_i^\intercal, \ \bar{\mathcal{W}}_i < 0. \end{cases} \tag{19}$$

If X_i^*, Y_i^* and Z_i^* are the solution of considered constrained optimization problem, then, the optimal gain-matrices $\mathbf{A}_i^*$, $\mathbf{B}_i^*$, $\mathbf{C}_i^*$ and $\mathbf{D}_i^*$ can be found as:

$$\Theta_i^* := \begin{bmatrix} \mathbf{A}_i^*, & \mathbf{B}_i^* \\ \mathbf{C}_i^*, & \mathbf{D}_i^* \end{bmatrix} = \left\{\mathbf{P}_i^* \left(\mathbf{P}_i^*\right)^\intercal\right\}^{-1} \left(\mathbf{P}_i^*\right)^\intercal X_i^*. \tag{20}$$

Notice that this problem can be solved numerically using the MATLAB Toolbox SeDuMi and Yalmip. The numerical values X_i^*, Y_i^* and Z_i^* can be obtained by an iterative procedure as it is exposed in Ordaz and Poznyak (2015).

6. ILLUSTRATIVE EXAMPLE

In this section we consider the **DISP** depicted in Fig.1. The control to be designed is intended to stabilize the pendulum in the vertical position using only the prismatic-force. The numerical results are applied in Matlab-Simulink. The mathematical model of the considered system can be presented as (2)-(8), where the **DIPCS** parameters and dynamic equation (1) are taken from Rubı et al. (2002). Notice that the corresponding dimensions are $n = 4$, $m = 1$ and $p = 3$.

For constant known matrix B, and the measured variables state are the positions (angles) $q_1 + \zeta_1(t)$, $q_2 + \zeta_2$ and $q_3 + \zeta_3(t)$ disturbed by noise. So, we have $y(t) = Cx(t) + \zeta(t)$ and

$$B = [0, \ 0, \ 0, \ 99.8646, \ -197.0170, \ 88.2812]^\intercal,$$
$$C = [I_{3\times3}, \ 0_{3\times3}].$$

It is not difficult to check that the "*unmodelled dynamics*" $\xi_x(t)$ belongs to the class $\mathcal{C}(A, d_1, d_2)$ and $\xi_y(t)$ belongs to the class $\mathcal{C}(C, d_3, d_4)$. Furthermore, we do not know the structure of the matrix A but we know its estimate $\hat{A}_i$, in order to apply the adaptive algorithm (10)-(11) we use the next initial matrix condition:

$$\hat{A}_0 = \begin{bmatrix} -7.0908 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11.11 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0105 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.46 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0546 \end{bmatrix}.$$

Notice that matrices $\hat{A}_0$, B, and the matrices C, $\hat{A}_0$, constitute a controllable and observable pairs, respectively. Applying this technique on the **DISP**, for the first iteration and taking as initial values $\alpha_0 = 0.45$, $\beta_{1,0} = 0.078$. In the second time interval $t_2 \in [5, 7.5)$ we obtain the following results; for $\beta_{2,0} = 0.008$, and applying the numerical algorithm we take $\alpha_1^* = 2.3$, $\varepsilon_{1,1}^* = 0.003$, $\varepsilon_{2,1}^* = 0.002$ and the estimated matrix

$$\hat{A}_2 = \begin{bmatrix} 0.12 & 0.023 & 0.165 & 1.012 & 0.00 & -0.01 \\ -0-15 & 0.56 & 0.1762 & -0.027 & 0.978 & 0.01 \\ 0.01 & -0.092 & -0.01 & 0.028 & 0.00 & 9.982 \\ 12.79 & -64.72 & -1.531 & 0.001 & -0.021 & -0.276 \\ 74.77 & 228.505 & -25.07 & 0.012 & -0.010 & 0.000 \\ -152.11 & -218.35 & 126.26 & 0.101 & 0.261 & -0.001 \end{bmatrix}.$$

Finally, the minimization procedure gives the next results

$$Z_{1,2}^* = \begin{bmatrix} 0.5552 & 0.5167 & 0.5782 & 0.1552 & 0.1107 & 0.0716 \\ 0.5167 & 0.4851 & 0.5422 & 0.1451 & 0.1036 & 0.0672 \\ 0.5782 & 0.5422 & 1.1723 & 0.2204 & 0.1721 & 0.1355 \\ 0.1552 & 0.1451 & 0.2204 & 0.0496 & 0.0369 & 0.0264 \\ 0.1107 & 0.1036 & 0.1721 & 0.0369 & 0.0278 & 0.0204 \\ 0.0716 & 0.0672 & 0.1355 & 0.0264 & 0.0204 & 0.0158 \end{bmatrix} \times 10^3,$$

$$Z_{2,2}^* = \begin{bmatrix} 0.9406 & 0.8806 & 1.3220 & 0.2984 & 0.2219 & 0.1580 \\ 0.8806 & 0.8244 & 1.2383 & 0.2795 & 0.2078 & 0.1480 \\ 1.3220 & 1.2383 & 2.0567 & 0.4399 & 0.3316 & 0.2432 \\ 0.2984 & 0.2795 & 0.4399 & 0.0968 & 0.0724 & 0.0523 \\ 0.2219 & 0.2078 & 0.3316 & 0.0724 & 0.0543 & 0.0394 \\ 0.1580 & 0.1480 & 0.2432 & 0.0523 & 0.0394 & 0.0288 \end{bmatrix} \times 10^4,$$

$$\mathbf{A}_2^* = \begin{bmatrix} 0.0066 & 0.0063 & 0.5236 & 0.0552 & 0.0514 & 0.0575 \\ 0.0043 & 0.0041 & 0.4917 & 0.0514 & 0.0483 & 0.0539 \\ -0.3524 & -0.3560 & 1.0881 & 0.0575 & 0.0539 & 0.1158 \\ -0.0348 & -0.0352 & 0.2026 & 0.0154 & 0.0144 & 0.0218 \\ -0.0347 & -0.0350 & 0.1591 & 0.0110 & 0.0103 & 0.0170 \\ -0.0384 & -0.0388 & 0.1265 & 0.0071 & 0.0067 & 0.0134 \end{bmatrix},$$

$$\mathbf{B}_2^* = \left[\, 36.6958, \ 29.0938, \ 466.5144, \ 55.7371, \ 45.6655, \ 51.9448 \,\right]^\intercal,$$
$$\mathbf{C}_2^* = -\left[\, 22.4329 \ 17.2984 \ 296.9763 \ 35.3490 \ 28.9540 \ 33.0489 \,\right],$$
$$\mathbf{D}_2^* = -\left[\, 36.6958 \ 29.0938 \ 466.5144 \,\right].$$

The simulation results are shown in figures 2, 3, 4 and 5, all of them are obtained without the use of the model (this assertion is given only for the control design, for numerical simulation we use the dynamics given on Rubı et al. (2002)).

The first plot presents the control signal for the process on the time $t \in [0, 20]$ seconds, Fig. 2. The initial conditions for this simulation are: $x(0) = [1.52, 0, -0.01, 01, 0, 0.01]$, $x_r(0) = [0, 0, 0, 0, 0, 0]$. The figures 3, 4, 5 depict the state variables. These pictures show the corresponding controlled trajectories $x(t)$ and their ellipsoids. The results show the control adaptation for the parametric estimation and how the attractive ellipsoid evolves respect its changes.

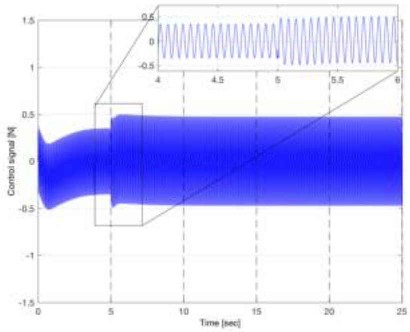

Fig. 2. Control signal.

7. CONCLUDING REMARKS

This paper presents an alternative control algorithm for robust control based on Dynamic Output Feedback for gain adjustment on sample time. For each time interval, this technique presents the property to remain constant the matrix gains. Due to the classical control algorithms are based on Quasi-Lipschitz functions properties, where the extension of the quasi-linear format is not well explained, here we present an alternative to obtain the quasi-linear format, in fact we use an online adjustment of the estimated matrix $\hat{A}$.

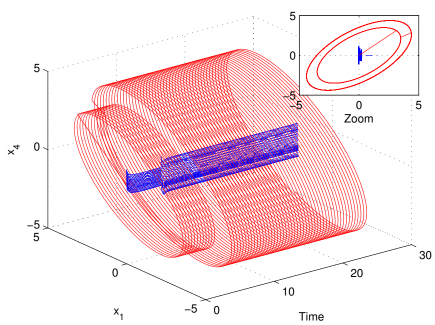

Fig. 3. Car trajectories and their corresponding sample time ellipsoid.

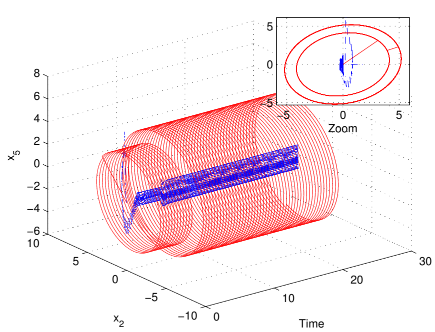

Fig. 4. First-link trajectories and their corresponding sample time ellipsoid.

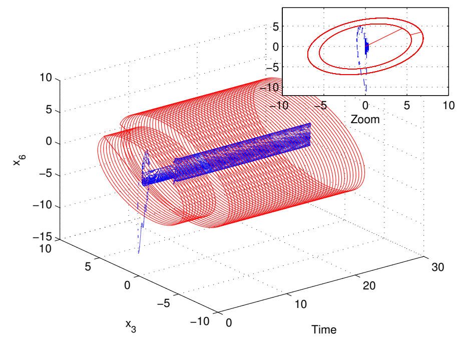

Fig. 5. Second-link trajectories and their corresponding sample time ellipsoid.

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