Memorias del Congreso Nacional de Control Automático
ISSN: 2594-2492

Virtual mechanical systems: an energy-based approach

Rodolfo Reyes-Báez [*,**] **Pablo Borja** [*,***]
Arjan van der Schaft [*,**] **Bayu Jayawardhana** [*,***]

[*] *Jan C. Willems Center for Systems and Control, University of Groningen, The Netherlands*
[**] *Bernoulli Institute of Mathematics, Computer Science and Artificial Intelligence, University of Groningen, P.O. Box 407, 9700 AK, Groningen, The Netherlands (r.reyes-baez@ieee.org, a.j.van.der.schaft@rug.nl)¿*
[***] *Engineering and Technology Institute Groningen (ENTEG), University of Groningen, Nijenborgh 4, 9747AG, The Netherlands ({l.p.borja.rosales,b.jayawardhana}@rug.nl)*

Abstract: In this work, a class of virtual control systems associated to mechanical systems in the Euler-Lagrange (EL) and port-Hamiltonian (pH) energy-based frameworks is introduced, where the behavior of the original system is embedded into the dynamics of the virtual one. The construction of the virtual mechanical systems is based on the notion of *virtual forces* which are mathematical objects that behave like *true forces*. Remarkably, the virtual mechanical systems preserve the energy conservation properties of the original mechanical systems, e.g., passivity. Moreover, the aforementioned virtual forces exhibit coordinate-free properties.

Keywords: passivity, virtual systems, Euler-Lagrange, port-Hamiltonian, nonlinear systems

1. INTRODUCTION

In this work, motivated by the recent applications of virtual systems in analysis and control, a unified approach for the construction of *virtual control systems* associated to mechanical systems is presented. A virtual control systems can produce all the solutions of the original one. Virtual systems have been extensively used in the field of systems and control. These are commonly found in state estimation and tracking problems. For instance, in state estimation, the original system is the reference system and the virtual system is the observer itself.

The virtual systems point of view in mechanical EL systems is introduced in the celebrated work of Slotine and Li (1987) within the context of *sliding control* (SC), where a structure preserving *virtual mechanical system* was employed to design a trajectory tracking controller. This idea is later revisited in Ortega et al. (2013) from a passivity-based control (PBC) perspective, where the virtual system is interpreted as the target closed-loop dynamics in the design process. A free-coordinate interpretation of this scheme without gravity effects is presented in van der Schaft (2017). In the work Reyes-Báez et al. (2018b), virtual systems in the EL framework are used for distributed PBC design of mechanical network dynamics. Another control design method for mechanical EL systems also using virtual systems is briefly discussed in Jouffroy and Fossen (2010); Manchester et al. (2018), where the controller synthesis is performed on directly on the virtual system. Such controller is then used to close the loop of the original mechanical system.

Similar to the EL framework, a class of virtual systems have been used in control design of mechanical pH systems. Specifically, the structure preserving control techniques propose virtual systems as target behaviors. For instance, partial linearizion Venkatraman et al. (2010); Dirksz and Scherpen (2010), interconnection and Damping assignment passivity-Based Control (IDA-PBC) Ortega et al. (2002), control by interconnection (CbI) method Ortega et al. (2008), among others. When these methods are applied to mechanical systems, it is a common practice to construct the virtual systems after an intermediate *canonical generalized transformation* that lets them to rewrite the system as a system whose inertia matrix is constant, see Fujimoto et al. (2003); Venkatraman et al. (2010). In the recent work of Reyes-Báez et al. (2017), a *passivity-based sliding control* is developed [1] for mechanical pH systems also using the virtual systems.
A different point of attack for the control of mechanical pH using virtual systems is adopted also by the authors in Reyes-Báez et al. (2018c, 2019); Reyes Báez (2019),

[1] This can be seen as a Hamiltonian counterpart of the one in Slotine and Li (1987) in the case where all parameters are known.

where the control design goal is to impose contractive behavior for the closed-loop virtual system, see Lohmiller and Slotine (1998).

The paper is organized as follows: In Section 2 the notion of virtual (control) systems is introduced. In Section 3 a class of virtual mechanical systems in the EL framework is presented, together with a energy-based interpretation. The pH counterpart of for mechanical systems and properties is is presented in Section 4. But due to space limitations, the free-coordinate interpretation of pH system is not worked in this work.

2. PRELIMINARIES ON VIRTUAL SYSTEMS

Let Σ_u be a nonlinear control system, affine in the input u, with state space manifold $\mathcal{X}$ of dimension N, which in local coordinates $(x_1, \dots, x_N)$ is given by

$$\Sigma_u : \begin{cases} \dot{x} = f(x,t) + \displaystyle\sum_{i=1}^{n} g_i(x,t)u_i, \\ y = h(x,t), \end{cases} \tag{1}$$

where $x \in \mathcal{X}$ is the state, input $u \in \mathcal{U} \subset \mathbb{R}^n$, output $y \in \mathcal{Y} \subset \mathbb{R}^n$, time-dependent vector fields $f, g_i \in \mathfrak{X}^\infty(\mathcal{X} \times \mathbb{R})$ and the scalar function $h \in \mathcal{C}^\infty(\mathcal{X} \times \mathbb{R})$. The input space $\mathcal{U}$ and output space $\mathcal{Y}$ are open subsets of $\mathbb{R}^n$. Solutions to Σ_u are given by trajectories $t \in [t_0, T] \mapsto x(t) = \psi^u_{t_0}(x_0, t)$ resulting from the initial condition $x_0 \in \mathcal{X}$, for a fixed input function $u : [t_0, T] \to \mathcal{U}$, with $\psi^{u_0}_{t_0}(x_0, t_0) = x_0$. Consider a forward invariant and connected neighborhood $\mathcal{C}$ of $\mathcal{X}$ such that $\psi^u_{t_0}(t, x_0)$ is forward complete for every $x_0 \in \mathcal{C}$, i.e., $\psi^u_{t_0}(x_0, t) \in \mathcal{C}$ for each t_0, each function u and each $t \geq t_0$. By connectedness, any pair of points in $\mathcal{C}$ can be connected by a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{C}$.

In the following definition the different notions of virtual (control) system introduced in Wang and Slotine (2005); Jouffroy and Fossen (2010); Forni and Sepulchre (2014) are unified and generalized.

Definition 1. Consider systems Σ_u in (1). Suppose that $\mathcal{C}_v \subseteq \mathcal{X}$ and $\mathcal{C}_x \subseteq \mathcal{X}$ are connected and forward invariant for these systems. A *virtual control system* associated to Σ_u is defined as

$$\Sigma^v_u : \begin{cases} \dot{x}_v = \Gamma_v(x_v, x, u_v, t), \\ y_v = h_v(x_v, x, t), \qquad \forall t \geq t_0, \end{cases} \tag{2}$$

with state $x_v \in \mathcal{X}$ and is parametrized by the trajectory $x \in \mathcal{X}$ of Σ_u, where $h_v : \mathcal{C}_v \times \mathcal{C}_x \times \mathbb{R}_{\geq 0} \to \mathcal{Y}$ and $\Gamma_v : \mathcal{C}_v \times \mathcal{C}_x \times \mathcal{U} \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ are such that

$$\begin{aligned} \Gamma(x, x, u, t) &= f(x,t) + \sum_{i=1}^{n} g_i(x,t)u_i, \\ h_v(x, x, t) &= h(x,t), \quad \forall x, \forall u, \forall t \geq t_0. \end{aligned} \tag{3}$$

It follows that any solution $x(t) = \psi_{t_0}(t, x_o)$ of the *original control system* Σ_u in (1), starting at $x_0 \in \mathcal{C}_x$ for a certain input u, generates the solution $x_v(t) = \psi_{t_0}(t, x_0)$ to the virtual system Σ^v_u in (2), starting at $x_{v0} = x_0 \in \mathcal{C}_v$ with $u_v = u$, for all $t > t_0$. However, *not* every virtual system's solution $x_v(t)$ corresponds to an original system's solution. Thus, *for* any *trajectory* $x(t)$ *of the original system, we may consider* (2) *as a time-varying system with state* x_v.

3. VIRTUAL MECHANICAL SYSTEMS IN THE EULER-LAGRANGE FRAMEWORK

Let $\mathcal{Q}$ be the configuration manifold of a mechanical system of n degrees of freedom (dof) with a local coordinates $q = (q_1, \dots, q_n)$ at q. Consider function $L : T\mathcal{Q} \to \mathbb{R}$ called the *Lagrangian* with coordinates (q, v) of $T\mathcal{Q}$.

In this work only *simple* mechanical systems are considered. For this class of systems simply the Lagrangian function is the difference between the kinetic (co)-energy and the potential energy, in local coordinates, given as

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q} - P(q), \tag{4}$$

and the Euler-Lagrange equations are given by

$$M(q)\dot{q} + C(q, \dot{q})\dot{q} + g(q) = B(q)\tau, \tag{5}$$

with $g(q)$ the differential of $P(q)$ and $C(q, \dot{q})$ any matrix satisfying the relation (see relation

$$C(q, \dot{q})\dot{q} = \dot{M}(q)\dot{q} - \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^\top M(q)\dot{q}\right). \tag{6}$$

The forces $C(q, \dot{q})\dot{q}$ correspond to the centrifugal $(i = j)$ and Coriolis $(i \neq j)$ effects, respectively.

The covector $B(q)\tau$, with inputs $\tau \in \mathcal{U}$, represents the vector of external forces. Matrix $B(q)$ indicates how the inputs τ influences the system. If rank $B(q) = m < n$ then we say that system (5) is *underactuated*.

For system (5) the $n \times n$ matrix with (i, j)-th element $\frac{\partial^2 L}{\partial \dot{q}_i \dot{q}_j}(q, \dot{q})$ is equal to $M(q)$ and thus nonsingular. Hence equation (5) define the affine system

$$\frac{d}{dt}\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} v \\ -M^{-1}\left(C(q, v)v + g(q)\right) \end{bmatrix} + \begin{bmatrix} 0_n \\ M^{-1}B \end{bmatrix}\tau. \tag{7}$$

of the form (1) with state space $\mathcal{X} = T\mathcal{Q}$.

It is well known that the EL equations (5) exhibit several important dynamic properties; see Ortega et al. (2013) and references therein. Among those properties, the skew-symmetry of the matrix $N(q, \dot{q}) := \dot{M}(q) - 2C(q, \dot{q})$ receives special attention since it is close related to the energy conservation of the EL system (5). To see this consider the total (co-)energy

$$\mathcal{E}(q, \dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q} + P(q), \tag{8}$$

Then, the (co-)energy balance reads as follows

$$\dot{\mathcal{E}}(q, \dot{q}) = \dot{q}^\top B(q)\tau + \frac{1}{2}\dot{q}^\top N(q, \dot{q})\dot{q} = \dot{q}^\top B(q)\tau. \tag{9}$$

This shows that the energy is conserved. In the dissipativity theory Willems (1972); van der Schaft (2017), the system (5) is called *lossless* if condition (9) is satisfied. The scalar function (8) is called a *storage function*.

From a Riemannian geometry point of view, the skew-symmetry of $N(q,\dot q)$ is a clear expression in *local coordinates* of the torsion-free property and compatibility condition of the *Levi-Civita affine connection* $\overset{M}{\nabla}$ with the metric $M\langle v,v\rangle := v^\top M(q)v$ (see Appendix A for more details). This is shown in the following corollary.

Corollary 1. Consider the Levi-Civita connection $\overset{M}{\nabla}$ associated to the inertia matrix $M(q)$, and let $\dot q = X(q)$ and $\dot q_v = Y(q)$ in $T_q\mathcal{Q}$. Then, the compatibility condition [2]

$$L_X\left(M\langle Y,Y\rangle\right) = M\langle \overset{M}{\nabla}_X Y, Y\rangle + M\langle Y, \overset{M}{\nabla}_X Y\rangle \tag{10}$$

is expressed in local coordinates as

$$Y^\top N(q,X)Y = Y^\top\left[\dot M(q) - 2C(q,X)\right]Y = 0. \tag{11}$$

where $\overset{M}{\nabla}_X Y$ is covariant derivative of $Y(q)$ along $X(q)$.

Remark 1. The energy conservation condition (9) requires that (10) (or in coordinates (11)) to hold only along $\dot q_v = \dot q$, instead of for every tangent vector $\dot q_v \in T_q\mathcal{Q}$.

Notice that the induced maps by $N(q,\dot q)$ defined as

$$F_N(q,\dot q) := N(q,\dot q)\dot q, \quad F_{N_v}(q,\dot q,\dot q_v) := N(q,\dot q)\dot q_v \tag{12}$$

have units of force, and their corresponding "power" is given by $\dot q^\top F(q,\dot q) = 0$ and $\dot q_v^\top F_{N_v}(q,\dot q,\dot q_v) = 0$. The first is a consequence of the energy conservation (see (9)) and the later is a consequence of the compatibility condition (see (11)). Nevertheless, $F_{N_v}(q,\dot q,\dot q_v)$ not necessarily defines a *true force* since the tangent vector $\dot q_v \in T_q\mathcal{Q}$ may not correspond to the velocity of q. This would be the case only if $\dot q_v = \dot q$, implying that the following holds

$$F_{N_v}(q,\dot q,\dot q) = F_N(q,\dot q) \tag{13}$$

Such $F_{N_v}(q,\dot q,\dot q_v)$ is referred as a *virtual force.*

Exploiting the above introduced notion of virtual forces, in the next proposition a class of virtual control system associated to the (original) EL system (5) are introduced.

Proposition 1. Consider the EL system in(5). Consider also the system defined by

$$\begin{aligned} \dot q_v &= v_v, \\ M(q)\dot v_v + C(q,v)v_v + g_v(q_v) &= B(q)\tau_v, \\ y_v &= B^\top(q)v_v, \end{aligned} \tag{14}$$

with state $(q_v,v_v) \in \mathcal{Q}\times\mathbb{R}^n$ and parametrized by $(q,\dot q) \in T\mathcal{Q}$, where $g_v(q_v)$ is such that $g_v(q) = g(q)$, $B(q)\tau_v \in T^*\mathcal{Q}$ is a co-vector with inputs τ_v, and y_v is an output. Then, system (14) defines a virtual control system for the original system (5).

Remark 2. The virtual system (14) can be seen as the second-order version of the one introduced in (van der Schaft, 2017, Definition 4.6.2). Here we have also considered the virtual potential energy function $P_v(q_v)$.

3.1 Losslessness property preserving

Remarkably, not only the EL system (5) is lossless (i.e., preserves the energy) with respect to the output $y = B^\top(q)v$, but also the virtual system (14) turns out to be lossless with respect to the output $y_v = B^\top(q)v_v$, *for every time-functions* $(q(\cdot),v(\cdot))$. As we will see in the following proposition, the skews-symmetry of $N(q,\dot q)$ is crucially used in the proof's computations. To this end, consider the storage function of the virtual system (14) as the function of (q_v,v_v) given by

$$\mathcal{E}_v(q_v,v_v,q,v) := \frac{1}{2}v_v^\top M(q)v_v + P_v(q_v), \tag{15}$$

parametrized by (q,v), where $P_v(q_v)$ is such that $g_v(q_v) = \partial P_v/\partial q_v(q_v)$.

Proposition 2. For any curve $(q(\cdot),v(\cdot))$, the virtual system (14) with input τ_v and output y_v is lossless, with the (q,v)-parametrized storage function (15).

Proposition 2 can be easily extended to the case when the original EL system (5) contains dissipative forces. In particular, if the dissipation is modeled by Rayleigh function $R(\dot q)$ satisfying $\dot q^\top \frac{\partial R}{\partial \dot q}(\dot q) \geq 0$, then the virtual system (14) is passive with input-output pair (τ_v,y_v) and storage function (15). Indeed,

$$\dot{\mathcal{E}}_v(q_v,v_v,q,v) \leq -\dot q_v^\top \frac{\partial R}{\partial \dot q}(\dot q_v)\dot q_v + y_v^\top\tau_v. \tag{16}$$

Moreover, with Proposition 2 the standard lossless (or passivity) preserving interconnection properties of the EL system (5) can be extended to the virtual system (14).

Remark 3. The virtual system (14) possesses the contractivity property Lohmiller and Slotine (1998); Jouffroy and Fossen (2010). However, this is not treated in this work.

3.2 Coordinate-free description

For $\dot q = X(q)$, $\overset{M}{\nabla}_X X$ is locally given by

$$\overset{M}{\nabla}_{\dot q}\dot q = \ddot q + M^{-1}(q)C(q,\dot q)\dot q(t). \tag{17}$$

It follows that (5) is expressed in a free-coordinate manner in the context of Riemannian geometry as follows:

$$\begin{aligned} \dot q &= X(q), \\ \overset{M}{\nabla}_{X(q)}X(q) + \operatorname{grad}(P(q)) &= M^{-1}(q)B(q)\tau, \end{aligned} \tag{18}$$

where $\operatorname{grad}(P(q)) \in T_q\mathcal{Q}$ is the gradient of the potential energy function $P(q)$ which in local coordinates is given by $\operatorname{grad}(P(q)) = M^{-1}(q)\frac{\partial P}{\partial q}$. For the external input term $M^{-1}(q)B(q)\tau$, from a geometric point of view, the force $B(q)\tau$ is an element of the cotangent space $T_q^*\mathcal{Q}$. In this case, $M^{-1}(q)$ defines a map from the cotangent space and to tangent space; this yields $M^{-1}(q)B(q)\tau \in T_q\mathcal{Q}$.

Remark 4. If $\tau = 0$, then the (18) reduces to the *geodesic equation* $\overset{M}{\nabla}_{\dot q}\dot q = 0_n$ and q is the geodesic curve, see (A.7).

Proposition 3. The coordinate-free description of the virtual system (14) is given by

$$\begin{aligned} \dot q_v(t) &= Y(q_v), \\ \overset{M}{\nabla}_{X(q)}Y(q_v) + \operatorname{grad}(P(q)) &= M^{-1}(q)B(q)\tau_v. \end{aligned} \tag{19}$$

[2] $L_X(\cdot)$ denotes the operator Lie derivative along the vector $X(q)$.

4. VIRTUAL MECHANICAL SYSTEMS IN THE PORT-HAMILTONIAN FRAMEWORK

As an alternative to the Euler-Lagrange framework for mechanical systems, the (port-)Hamiltonian formulation can be adopted van der Schaft and Maschke (1995).

In this setting the state space is given by the cotangent bundle $\mathcal{X} = T^*\mathcal{Q}$ with natural coordinates $x = (q, p)$. For simple mechanical systems the Hamiltonian function corresponds to the total energy defined by

$$H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + P(q), \tag{20}$$

where $p := M(q)\dot{q}$ is the generalized momentum, and the port-Hamiltonian dynamics is

$$\begin{aligned}\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} &= (J(q,p) - R(q,p))\begin{bmatrix}\dfrac{\partial H}{\partial q}(q,p)\\ \dfrac{\partial H}{\partial p}(q,p)\end{bmatrix} + \begin{bmatrix}0_n\\ B(q)\end{bmatrix}\tau,\\ y &= B^\top(q)\frac{\partial H}{\partial p}(q,p),\end{aligned} \tag{21}$$

The interconnection and dissipation matrices given by

$$J(x) = \begin{bmatrix}0_n & I_n\\ -I_n & 0_n\end{bmatrix}, \quad R(x) = \begin{bmatrix}0_n & 0_n\\ 0_n & D(q)\end{bmatrix} \tag{22}$$

respectively. The $n \times n$ matrix $D(q) = D^\top(q) \geq 0_n$ is a dissipation term. Similar to the EL framework, the energy balance for system (21) is given by the time derivative of the Hamiltonian (20) along the system's trajectories, i.e.,

$$\dot{H}(q,p) = -\frac{\partial H^\top}{\partial p}(q,p)D(q)\frac{\partial H}{\partial p}(q,p) + \tau \leq y^\top\tau. \tag{23}$$

It follows that the map $\tau \mapsto y$ is *passive* with storage function (20). Furthermore, the system is *lossless* for if $D(q) = 0_n$ (energy conservation).
System in (21) can be equivalently rewritten as follows (see Reyes-Báez et al. (2018a) for details)

$$\begin{aligned}\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} &= \begin{bmatrix}0_n & I_n\\ -I_n & -(E(q,p)+D(q))\end{bmatrix}\begin{bmatrix}\dfrac{\partial P}{\partial q}\\ \dfrac{\partial H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ B\end{bmatrix}\tau,\\ y_E &= \begin{bmatrix}0_n & B^\top(q)\end{bmatrix}\begin{bmatrix}\dfrac{\partial P}{\partial q}(q)\\ \dfrac{\partial H}{\partial p}(q,p)\end{bmatrix},\end{aligned} \tag{24}$$

where $E(q,p) := S_H(q,p) - \frac{1}{2}\dot{M}(q)$, with $S_H(q,p)$ a skew-symmetric matrix[3], and the output satisfies $y_E = y$. The main characteristic of this alternative form is that the forces associated to the inertia matrix of (21), i.e. $\frac{\partial}{\partial q}\left(\frac{1}{2}p^\top M^{-1}(q)p\right)$, are decoupled from the force $\frac{\partial H}{\partial q}(q,p)$ by means of the physical identity

$$\frac{\partial}{\partial q}\left(\tfrac{1}{2}p^\top M^{-1}(q)p\right) = E(q,p)M^{-1}(q)p. \tag{25}$$

Notice that this is possible without any change of coordinates or feedback, which is a common practice in the literature; e.g. Venkatraman et al. (2010) and Romero et al. (2015). Moreover, it can be shown that the force $E(q,p)M^{-1}(q)p$ is the Hamiltonian counterpart[4] of the force $C(q,\dot{q})\dot{q}$ in the Lagrangian framework.

As expected, the input-output pair (u, y_E) defines a passive map with the same storage function (20) as the storage function as well. Indeed, the time derivative of $H(q,p)$ a long the trajectories of (24) is given by

$$\dot{H} = \begin{bmatrix}\dfrac{\partial H}{\partial q}\\ \dfrac{\partial H}{\partial p}\end{bmatrix}^\top\left(\begin{bmatrix}0_n & I_n\\ -I_n & -(E+D)\end{bmatrix}\begin{bmatrix}\dfrac{\partial P}{\partial q}\\ \dfrac{\partial H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ B\end{bmatrix}\tau\right) \leq y_E^\top\tau. \tag{26}$$

Notice that the first term inside the bracket can be conveniently rewritten as

$$\begin{bmatrix}\dfrac{\partial H}{\partial q}\\ \dfrac{\partial H}{\partial p}\end{bmatrix}^\top \mathcal{F}\begin{bmatrix}\dfrac{\partial P}{\partial q}\\ \dfrac{\partial H}{\partial p}\end{bmatrix} = \begin{bmatrix}\dfrac{\partial P}{\partial q}\\ \dfrac{\partial H}{\partial p}\\ \dfrac{\partial}{\partial q}\left(\frac{1}{2}p^\top M^{-1}p\right)\end{bmatrix}^\top \overline{\mathcal{F}}\begin{bmatrix}\dfrac{\partial P}{\partial q}\\ \dfrac{\partial H}{\partial p}\\ M^{-1}p\end{bmatrix} = 0. \tag{27}$$

where $\mathcal{F}(q,p)$ and $\overline{\mathcal{F}}(q,p)$ are respectively given by

$$\mathcal{F}(q,p) := \begin{bmatrix}0_n & I_n\\ -I_n & -E\end{bmatrix};\ \overline{\mathcal{F}}(q,p) = \begin{bmatrix}0_n & I_n & 0_n\\ -I_n & -E & 0_n\\ 0_n & 0_n & I_n\end{bmatrix}. \tag{28}$$

With the above observations let us define the map $\mathbb{F}(q,p) : C^\infty(T^*\mathcal{Q}) \to \mathbb{R}^2$ in coordinates as

$$\mathbb{F}(q,p)[H(q,p)] := \mathcal{F}(q,p)\begin{bmatrix}\dfrac{\partial P}{\partial q}(q)\\ \dfrac{\partial H}{\partial p}(q,p)\end{bmatrix}, \tag{29}$$

with[5] $H(q,p) \in C^\infty(T^*\mathcal{Q})$ given in (20). From the energy balance identity in (27), it follows that

$$\frac{\partial H^\top}{\partial x}(q,p)\mathbb{F}(q,p)[H(q,p)] = 0. \tag{30}$$

This suggests that the quantity $\mathbb{F}(q,p)[H(q,p)]$ can be interpreted as the Hamiltonian counterpart of the workless force $F_N(q,\dot{q})$ in (12). Similarly, consider a smooth function $H_v(q_v,p_v,q) \in C^\infty(T^*\mathcal{Q} \times \mathcal{Q})$ of the form

$$H_v(q_v,p_v,q) = \frac{1}{2}p_v^\top M^{-1}(q)p_v + P_v(q_v), \tag{31}$$

and parametrized by $q(\cdot)$ from (24), where function $P_v(q_v)$ is such that $P_v(q) = P(q)$. Consider also the map defined as $\overline{\mathbb{F}}(q,p) : C^\infty(T^*\mathcal{Q} \times \mathcal{Q}) \to \mathbb{R}^3$ with

$$\overline{\mathbb{F}}(q,p)[H_v(q_v,p_v,q)] := \overline{\mathcal{F}}(q,p)\begin{bmatrix}\dfrac{\partial P_v}{\partial q_v}(q_v)\\ \dfrac{\partial H_v}{\partial p_v}(q_v,p_v,q)\\ M^{-1}(q)p\end{bmatrix}. \tag{32}$$

[3] This matrix is related to the compatibility condition (11) as follows: $-2S_H(q,p)M^{-1}(q)p = N(q, M^{-1}(q)p)M^{-1}(q)p$.

[4] Similar results were obtained the works of Sarras et al. (2012) and Stadlmayr and Schlacher (2008).

[5] $C^\infty(T^*\mathcal{Q})$ is the set of smooth scalar functions $H : T^*\mathcal{Q} \to \mathbb{R}$.

By construction, it straightforward to verify that

$$\frac{\partial H_v^\top}{\partial \overline{x}_v}(q_v, p_v, q)\overline{\mathbb{F}}(q,p)[H(q_v,p_v,q)] = 0, \qquad (33)$$

where $\overline{x}_v = (x_v, q)$, $x_v = (q_v, p_v)$ and q is the solution to $\dot{q} = M^{-1}(q)p$. This in turn implies that whenever $(q_v, p_v) = (q, p)$ we have that

$$\frac{\partial H_v^\top}{\partial \overline{x}_v}(q,p,q)\overline{\mathbb{F}}(q,p)[H(q,p,q)] = \frac{\partial H^\top}{\partial x}(q,p)\mathbb{F}(q,p)[H(q,p)]. \qquad (34)$$

Hence, similar (29), $\overline{\mathbb{F}}(q,p)[H_v(q_v,p_v,q)]$ can be understood as the Hamiltonian counterpart of the workless virtual force $F_{N_v}(q,\dot{q},q_v)$ in (12).

Proposition 4. Consider system (21) and its alternative coordinate formulation (24). Consider also the input-state-output system given by

$$\begin{bmatrix} \dot{q}_v \\ \dot{p}_v \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(q,p)+D) \end{bmatrix} \begin{bmatrix} \dfrac{\partial H_v}{\partial q_v} \\ \dfrac{\partial H_v}{\partial p_v}1 \end{bmatrix} + \begin{bmatrix} 0_n \\ B \end{bmatrix}\tau_v$$
$$y_v = \begin{bmatrix} 0_n & B^\top(q) \end{bmatrix} \begin{bmatrix} \dfrac{\partial H_v}{\partial q_v}(q_v,p_v,q) \\ \dfrac{\partial H_v}{\partial p_v}(q_v,p_v,q) \end{bmatrix}, \qquad (35)$$

in the state $(q_v, p_v) \in \mathcal{Q} \times \mathbb{R}^n$ and parametrized by $(q,p) \in T^*\mathcal{Q}$, where $H_v(q_v,p_v,q)$ is given as in (31), $B(q)\tau_v \in T^*\mathcal{Q}$ is a covector with inputs τ_v, and y_v is an output. Then, system (35) defines a virtual control system for the original system (21).

4.1 Structure preserving property

As its EL counterpart, not only the pH system (21) (respectively the alternative form (24)) is passive with input-output pair (τ, y) (respectively (τ, y_E)) and storage function (20), but also the virtual system (35) with input-output pair (τ_v, y_v) and storage function given by (31). In the following proposition it is shown that the passivity preserving property relies explicitly in the "workless" property of the map $\overline{\mathbb{F}}(q,p)$ in (33).

Proposition 5. For any curve $(q(\cdot), p(\cdot))$ the virtual system (35) with input τ_v and output y_v is passive, with q-parametrized storage function (31).

5. CONCLUSION

In this work, virtual systems associated to simple mechanical systems have been presented. It was shown how the concept of virtual forces can be used to construct such virtual mechanical systems in both, the EL and pH frameworks. These virtual systems, besides having all the original system's solution embedded, preserve some structural properties such as the lossless property, which is associated to energy conservation. Moreover, using the geometry structure of the original system's state space, it was shown that the virtual systems preserve also some coordinate-free properties.

Appendix A. THE LEVI-CIVITA CONNECTION AND COVARIANT DERIVATIVE IN COORDINATES

The matrix $M(q)$ defines a *Riemannian metric* given by

$$M\langle v, w\rangle := v^\top M(q) w, \quad \text{for} \quad u, w \in T_q\mathcal{Q}. \qquad (A.1)$$

Thus, a geometric interpretation of the EL equations (5) can be given within the context of *Riemannian manifolds* (van der Schaft, 2017, Section 4.6).

Let $\Gamma(q_a, q_b) = \{\gamma : C^2[0,1] \to \mathcal{Q} | \gamma(0) = q_a, \gamma(1) = q_b\}$ be the collection of twice continuously differentiable curves on $[0,1]$ connecting $q_a \in \mathcal{Q}$ and $q_b \in \mathcal{Q}$, with local representative given by $t \mapsto q(t) = [q_1(t), \ldots, q_n(t)]^\top$.

Definition 2. For any vector fields $X, Y \in \mathfrak{X}^\infty(T\mathcal{Q})$ and any real function $f \in \mathcal{C}^\infty(\mathcal{Q})$, an *affine connection* ∇ is a map $(X,Y) \mapsto \nabla_X Y \in \mathfrak{X}^\infty(T\mathcal{Q})$ such that

(a) $\nabla_X Y$ is bilinear in X and Y,
(b) $\nabla_{fX} Y = f\nabla_X Y$,
(c) $\nabla_X fY = f\nabla_X Y + (L_X f)Y$.

The vector $\nabla_X Y$ is called the *covariant derivative* of Y with respect to X. Property (b) implies that $\nabla_X Y$ at $q \in \mathcal{Q}$ depends on X only through its value $X(q)$.

The *torsion* of an affine connection ∇ is defined as

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]. \qquad (A.2)$$

with $[X,Y]$ the Lie bracket of X, Y. If $T(X,Y) = 0$, we say the connection is *torsion-free*. An affine connection ∇ on $\mathcal{Q}$ is said *metric* or *compatible* with the Riemannian metric in (A.1) if

$$L_X\left(M\langle Y,Z\rangle\right) = M\langle \nabla_X Y, Z\rangle + M\langle X, \nabla_X Z\rangle \qquad (A.3)$$

for all vector fields $X, Y, Z \in \mathfrak{X}^\infty(\mathcal{Q})$. In a chart $(\mathcal{Q}, \varphi)$ at q and corresponding basis $\{\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}\}$ of $T_q\mathcal{Q}$, the ℓ-th component of $\nabla_{\frac{\partial}{\partial q_i}} \frac{\partial}{\partial q_j} \in \mathfrak{X}^\infty(\mathcal{W})$ is locally written as

$$\left(\nabla_{\frac{\partial}{\partial q_i}} \frac{\partial}{\partial q_j}\right)_\ell = \sum_{i,j=1}^n \Gamma_{ij}^\ell(q)\frac{\partial}{\partial x_\ell}, \quad \ell \in \{1, \ldots, n\}, \qquad (A.4)$$

where the n^3 smooth functions $\Gamma_{ij}^\ell(q)$ are uniquely defined. With these functions, called the *Christoffel symbols of second kind*, the ℓ-th component of $\nabla_X Y$ is

$$(\nabla_X Y)_\ell = \sum_{j=1}^n \frac{\partial Y_\ell}{\partial q_j}X_j + \sum_{i,j=1}^n \Gamma_{ij}^\ell X_i Y_j, \qquad (A.5)$$

for all $\ell \in \{1, \ldots, n\}$. This in turn implies that the ℓ-th component of the covariant derivative of $Y \in \mathfrak{X}^\infty(\mathcal{Q})$ along a curve $\gamma \in \Gamma(q_a, q_b)$ is given by

$$(\nabla_{\gamma'(t)} Y(\gamma(t)))_\ell = \dot{Y}_\ell(\gamma(t)) + \sum_{i,j=1}^n \Gamma_{ij}^\ell(\gamma(t))\gamma_i'(t)Y_j(\gamma(t)), \qquad (A.6)$$

A curve γ is called a *geodesic* of the affine connection if

$$\nabla_{\gamma'(t)}\gamma'(t) = 0_n. \qquad (A.7)$$

The metric (A.1) defines a unique affine connection $\overset{M}{\nabla}$ on $\mathcal{Q}$, called the *Levi-Civita connection*, which is torsion-free and compatible, see Bullo and Lewis (2004) for details. In this case the so-called Christoffel symbols are given by

$$\Gamma_{ij}^{\ell}(q) := \sum_{k=1}^{n} M_{\ell k}^{-1}(q) c_{ijk}(q), \tag{A.8}$$

where $M_{\ell k}^{-1}(q)$ is the (ℓ, k)-th element of matrix $M^{-1}(q)$, and the functions $c_{ijk}(q)$ are the Christoffel symbols of *first kind* defined as

$$c_{ijk}(q) := \frac{1}{2}\left[\frac{\partial M_{kj}}{\partial q_i}(q) + \frac{\partial M_{ki}}{\partial q_j}(q) - \frac{\partial M_{ij}}{\partial q_k}(q)\right]. \tag{A.9}$$

In this setting the forces to be functions from $T\mathcal{Q}$ to $T^*\mathcal{Q}$ (Bullo and Lewis, 2004, Section 4.4). Then, the map is $X \in T_q\mathcal{Q} \mapsto M(q)X \in T_q^*\mathcal{Q}$, then $M^{-1}(q)g(q), M^{-1}(q)B(q)\tau \in T_q\mathcal{Q}$. Recall from (4) that $g(q)$ is a potential force, and $g(q) = dP(q)$ (differential of $P(q)$); when passing the differential of $P(q)$ through the mapping $M^{-1}(q)$, we get $M^{-1}(q)(dP(q)) = \mathrm{grad}(P(q))$, the gradient of $P(q)$. Therefore, equation (5) can be rewritten in a coordinate-free manner as (19).

REFERENCES

Bullo, F. and Lewis, A.D. (2004). *Geometric control of mechanical systems: modeling, analysis, and design for simple mechanical control systems*, volume 49. Springer Science & Business Media.

Dirksz, D.A. and Scherpen, J.M. (2010). Adaptive tracking control of fully actuated port-Hamiltonian mechanical systems. In *CCA*, 1678–1683.

Forni, F. and Sepulchre, R. (2014). A differential Lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*.

Fujimoto, K., Sakurama, K., and Sugie, T. (2003). Trajectory tracking control of port-controlled Hamiltonian systems via generalized canonical transformations. *Automatica*, 39(12), 2059–2069.

Jouffroy, J. and Fossen, T.I. (2010). A tutorial on incremental stability analysis using contraction theory. *Modeling, Identification and control*, 31(3), 93–106.

Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for non-linear systems. *Automatica*.

Manchester, I.R., Tang, J.Z., and Slotine, J.J.E. (2018). Unifying robot trajectory tracking with control contraction metrics. In *Robotics Research*, 403–418. Springer.

Ortega, R., Perez, J.A.L., Nicklasson, P.J., and Sira-Ramirez, H. (2013). *Passivity-based control of Euler-Lagrange systems*. Springer Science & Business Media.

Ortega, R., van der Schaft, A., Castaños, F., and Astolfi, A. (2008). Control by interconnection and standard passivity-based control of port-Hamiltonian systems. *IEEE Transactions on Automatic Control*, 53.

Ortega, R., van der Schaft, A., Maschke, B., and Escobar, G. (2002). Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, 38(4), 585–596.

Reyes-Báez, R., van der Schaft, A.J., and Jayawardhana, B. (2018a). Virtual differential passivity based control for a class of mechanical systems in the port-hamiltonian framework. *Submitted*.

Reyes Báez, R. (2019). *Virtual Contraction and Passivity based Control of Nonlinear Mechanical Systems*. Ph.D. thesis, Rijksuniversiteit Groningen.

Reyes-Báez, R., Donaire, A., van der Schaft, A., Jayawardhana, B., and Perez, T. (2019). Tracking control of marine craft in the port-Hamiltonian framework: A virtual differential passivity approach. In *Proceedings European Control Conference (ECC)*.

Reyes-Báez, R., van der Schaft, A., and Jayawardhana, B. (2017). Tracking control of fully-actuated port-Hamiltonian mechanical systems via sliding manifolds and contraction analysis. *IFAC-PapersOnLine*, 50(1), 8256 – 8261. 20th IFAC World Congress.

Reyes-Báez, R., van der Schaft, A., and Jayawardhana, B. (2018b). Passivity based distributed tracking control of networked euler-lagrange systems. *IFAC-PapersOnLine*, 51(23), 136 – 141. doi: https://doi.org/10.1016/j.ifacol.2018.12.024. 7th IFAC Workshop on Distributed Estimation and Control in Networked Systems NECSYS 2018.

Reyes-Báez, R., van der Schaft, A., and Jayawardhana, B. (2018c). Virtual differential passivity based control for tracking of flexible-joints robots. volume 51, 169 – 174. 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2018.

Romero, J.G., Ortega, R., and Sarras, I. (2015). A globally exponentially stable tracking controller for mechanical systems using position feedback. *IEEE Transactions on Automatic Control*, 60(3), 818–823.

Sarras, I., Ortega, R., and Van Der Schaft, A. (2012). On the modeling, linearization and energy shaping control of mechanical systems. In *LHMNLC 2012*, volume 4, 161–166.

Slotine, J.J.E. and Li, W. (1987). On the adaptive control of robot manipulators. *The international journal of robotics research*, 6(3), 49–59.

Stadlmayr, R. and Schlacher, K. (2008). Tracking control for port-Hamiltonian systems using feedforward and feedback control and a state observer. *IFAC Proceedings Volumes*, 41(2), 1833–1838.

van der Schaft, A.J. (2017). *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer International Publishing.

van der Schaft, A.J. and Maschke, B.M. (1995). The hamiltonian formulation of energy conserving physical systems with external ports. *Archiv fur Elektronik und Ubertragungstechnik*, 49(5), 362–371.

Venkatraman, A., Ortega, R., Sarras, I., and van der Schaft, A. (2010). Speed observation and position feedback stabilization of partially linearizable mechanical systems. *IEEE Transactions on Automatic Control*.

Wang, W. and Slotine, J.J.E. (2005). On partial contraction analysis for coupled nonlinear oscillators. *Biological cybernetics*, 92(1).

Willems, J.C. (1972). Dissipative dynamical systems, part i: General theory. *Arch. Rational Mech. Anal.*, 45.

