Nonlinear passive control of a class of coupled partial differential equation models

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Abstract: In this work, the stabilization problem of a possible open-loop unstable steady-state for a class of semilinear parabolic partial differential equation models with an averaged measurement and homogeneously distributed control action is addressed. Following notions of passivity-based control for finite-dimensional systems, a feedback passive control is constructed. The combination of Lyapunov and modal techniques gives sufficient conditions to ensure the stability of the closed-loop system by characterizing the zero dynamics behavior in terms of the sensor location and the controller gain. For implementation purposes, an estimator with a pointwise innovation scheme is considered. The performance of the designed controller is shown by numerical simulations.

Keywords: distributed parameter systems, passivity based control, sensor and actuator placement

1. INTRODUCTION

Passivity based control for nonlinear finite-dimensional systems has proven to be a useful tool for control synthesis, it has been applied to stable and unstable plants, from mechanical and electrical systems, see, e.g., [Ortega et al., 2013], to chemical processes, see, e.g., [Doerfler et al., 2009, Sira-Ramirez and Angulo-Nunez, 1997]. The extension to infinite dimensional systems has been done following the *early-* and *late-lumping* approaches. In the *early-lumping* framework, the partial differential equation (PDE) model is approximated using a finite-dimensional model and then existing results on passivity based control has been applied. On the other hand, in the *late-lumping* approach the extension of passivity concepts has been performed for some types of PDE models exploiting its distributed structure and considering different input and output configurations.

In the context of *early-lumping* approaches, in [Franco-de los Reyes and Álvarez, 2017, Nájera et al., 2015] finite differences are used to obtain a finite-dimensional model of a tubular reactor and then a feedback passive controller is designed. In [Christofides, 2012], Galerkin and approximated inertial manifolds methods are used to approximate parabolic PDE models and then geometric control tools are used. Regarding *late-lumping* techniques, in [Christofides, 2012] geometric control is used for the stabilization of a plug flow tubular reactor with collocated sensor and actuator setup. Passivity for linear PDE models with collocated setup is analyzed in [Bondarko and Fradkov, 2002], while semilinear parabolic systems are considered in [Wang and Wu, 2014] where a feedback passivity-based controller is built exploiting Lyapunov theory. In the framework of thermodynamics in [Alonso et al., 2000, Ruszkowski et al., 2005] passivity-based control is studied for semilinear PDEs models. In the more general context of dissipativity-based control design, in [Schaum and Meurer, 2019] stabilization through linear output-feedback control of a semilinear heat equation with collocated sensor-actuator setup and an output dependent nonlinearity is considered.

In the present work, the ideas exploited in [Franco-de los Reyes and Álvarez, 2017, Nájera et al., 2015] following the *early-lumping* approach to design a passive controller for a tubular reactor model are extended to a class of infinite-dimensional systems. A passive controller is introduced for a class of diffusion-convection-reaction semilinear systems with regional sensor and homogeneous actuator. Conditions for the stability of the closed-loop system are established based on the stability of the origin of the related zero dynamics, which can be analyzed as a Lur'e system, i.e., an interconnection of a linear dynamic system and an static nonlinearity and by applying modal analysis. Stability conditions are established in terms of the sensor location and the controller gain. In comparison to [Franco-de los Reyes et al., 2019a,b], where a similar approach has been followed, here different system structure, controller and sensor-actuator setup are considered.

The rest of the paper is organized as follows, in Section 2 the control problem is introduced, in Section 3 the passive controller is built and closed-loop stability ensured. In Section 4 the output-feedback version of the proposed controller is presented. Simulation results are shown in Section 5 and final conclusions are given in Section 6.

2. PROBLEM FORMULATION

Consider the diffusion-convection-reaction system with multiple steady-states

$$\partial_t \bar{x}_1 = d\partial_z^2 \bar{x}_1 - \partial_z \bar{x}_1 + \phi(\bar{x}_1) + u, \quad \bar{x}_1(0) = \bar{x}_{10} \tag{1a}$$

$$\partial_t \bar{x}_2 = d\partial_z^2 \bar{x}_2 - \partial_z \bar{x}_2 + a_{21}\bar{x}_1, \quad \bar{x}_2(0) = \bar{x}_{20} \tag{1b}$$

$$\bar{y} = \textstyle\int_0^1 \gamma \bar{x}_1 \mathrm{d}z \tag{1c}$$

subject to the boundary conditions

$$\delta\partial_z \bar{x}_i(0,t) - \bar{x}_i(0,t) = 0,\ \delta\partial_z \bar{x}_i(1,t) = 0,\ i = 1,2, \tag{1d}$$

Herein $t \in \mathbb{R}_+$ is time, $z \in (0,1)$ the spatial domain, $x_i(z,t)$, $i = 1,2$ are the state variables, $u(t) \in \mathbb{R}$ is the (homogeneously distributed) control input and $y(t) \in \mathbb{R}$ the control output. The function $\phi(x_1(z,t))$ is a smooth nonlinearity, $a_{21} > 0$ is an interconnection gain, d is the diffusion coefficient and $\gamma(z)$ is the sensor function that defines the output as an average of the first state in a small region of space, and is given by

$$\gamma(z) = \begin{cases} \frac{1}{2\epsilon} & \text{if } z \in [z-\epsilon, z+\epsilon] \\ 0 & \text{else} \end{cases},$$

where ζ is the sensor location and $\epsilon > 0$ is a small fixed constant. In the following a transformation is used to obtain an equivalent system to (1) with self-adjoint operators. For this aim consider the change of variable given by

$$x_i = \beta(z)\bar{x}_i,\ i = 1,2, \qquad \beta(z) = \mathrm{e}^{-\frac{z}{2d}}, \tag{2}$$

where $\beta(z)$ is an integrating factor [Meurer and Andrej, 2018]. In this coordinates the system dynamics in abstract form are given by

$$\partial_t x_1 = \mathcal{A}_1 x_1 + \varphi(x_1) + \mathcal{B}u, \qquad x_1(0) = x_{10}, \tag{3a}$$

$$\partial_t x_2 = \mathcal{A}_2 x_2 + a_{21} x_1, \qquad x_2(0) = x_{20}, \tag{3b}$$

$$y = \mathcal{C}x_1 \tag{3c}$$

defined on the Hilbert space $\mathcal{H}^2$, where $\mathcal{H} = L^2(0,1)$ (with L^2-norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ where $\langle\cdot,\cdot\rangle$ denotes the standard inner product). $\mathcal{A}_i : \mathcal{D}(\mathcal{A}_i) \subset \mathcal{H} \to \mathcal{H}$, $i = 1,2$ are linear differential operators, $\mathcal{B}$ is the input operator and $\mathcal{C}$ the output operator, which are defined as

$$\begin{aligned} \mathcal{A}_i x_i &= d\partial_z^2 x_i - \tfrac{1}{4d} x_i, \qquad i = 1,2, \\ \mathcal{D}(\mathcal{A}_i) &= \big\{ v \in \mathcal{H} \,|\, d\partial_z v(0) - \tfrac{1}{2} v(0) = 0, \\ &\qquad d\partial_z v(1) + \tfrac{1}{2} v(1) = 0 \big\}, \\ \mathcal{B} &= \beta, \quad \mathcal{C}(\cdot) = \langle \bar{\gamma}, (\cdot) \rangle, \quad \bar{\gamma} = \bar{\beta}\gamma \\ \varphi(x_1) &= \beta\phi(\bar{\beta}x_1), \quad \bar{\beta} = \mathrm{e}^{\frac{z}{2d}}. \end{aligned}$$

The existence of a local unique strong solution of (3) is ensured for each $x_0 \in H^1([0,1])$, where $H^1([0,1])$ is the Sobolev space of function with first derivative (see e.g. [Fridman and Orlov, 2009, Schaum et al., 2014]).

Motivated by practical situations, e.g. from chemical engineering, where $\mathcal{A}_i$, $i = 1,2$ stand for diffusion-convection operators and φ is a bounded, potentially destabilizing term, the following assumptions are in order:

(A1) Let $\mathcal{A}_1$ be a self-adjoint Riesz spectral operator with real eigenvalues λ_n, $n \in \mathbb{N}$ fulfilling $0 > \lambda_1 \geq \lambda_2 \geq \ldots$ for which the algebraic and geometric multiplicities are the same, and whose eigenfunctions ϕ_n, given by the solution to the eigenvalue problem

$$\mathcal{A}\phi_n(z) - \lambda_n \psi_n(z) = 0, \quad n \in \mathbb{N}, \tag{4}$$

form a Riesz basis, i.e., $\langle \psi_n, \psi_k \rangle = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta.

(A2) The operator $\mathcal{A}_2$ generates a C_0-semigroup of contractions $S_2(t) = \mathrm{e}^{\mathcal{A}_2 t}$ which satisfy $\|S_2(t)\| \leq \mathrm{e}^{-\nu_2 t}$ where ν_2 is its growth bound [Curtain and Zwart, 2012].

(A3) The source term φ belongs to the sector $[-\kappa, \kappa]$ and satisfies $\|\varphi(z, x_1)\| \leq \kappa \|x_1\|\, \forall x_1 \in \mathcal{H}$ uniformly in z and $\varphi(z, 0) = 0$.

The control problem consists in selecting the sensor location ζ and design an output feedback controller such that the corresponding closed-loop system has the zero profile as unique and exponentially stable steady-state.

3. FEEDBACK-PASSIVE CONTROL

Here, for system (3) a passive controller is constructed, the closed-loop stability is ensured by means of the stability of the zero dynamics and its dependency on the sensor location.

3.1 Controller construction

Consider the output (3c) and take its first time derivative

$$\dot{y} = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}x_1 = \mathcal{C}\partial_t x_1 = \mathcal{C}\left(\mathcal{A}_1 x_1 + \varphi(x_1)\right) + \mathcal{C}\mathcal{B}u.$$

If $\mathcal{CB} \neq 0$ then the characteristic order from y to u is equal to one [Christofides, 2012]. Note that this concept is the extension of the relative degree property for finite-dimensional systems. Taking into account the considered input and output operators it follows that

$$\mathcal{CB} = \langle \bar{\gamma}, \beta \rangle = \tfrac{1}{2\epsilon} \neq 0,\ \forall z \in (0,1),\ \forall t \in \mathbb{R}_+. \tag{5}$$

Consequently the characteristic index is always one and the existence of the state-feedback controller

$$u = 2\epsilon\left[v - \mathcal{C}\left(\mathcal{A}_1 x_1 + \varphi(x_1)\right)\right], \tag{6}$$

is ensured and it can be used to stabilize the (possibly open-loop unstable) zero profile of system (3), as it will be established later.

Remark 1. Using $S(x) = \frac{1}{2}y^2 \geq 0$ as storage function it follows that $\frac{\mathrm{d}S(x)}{\mathrm{d}t} = yv$, which shows the passivity property introduced by the controller (6). In particular, using $v = -ky$ the exponentially stable output dynamics

$$\dot{y} = -ky, \quad y(0) = y_0$$

is enforced.

The corresponding closed-loop dynamics are given by

$$\partial_t x_1 = \mathcal{A}_1^c x_1 + \Delta\varphi(x_1) - 2\epsilon k \mathcal{B} y, \quad x_1^z(0) = x_{10}^z \tag{7a}$$
$$\partial_t x_2 = \mathcal{A}_2 x_2 + a_{21} x_1, \quad x_2^z(0) = x_{20}^z \tag{7b}$$
$$y = \mathcal{C} x_1 \tag{7c}$$

where $\mathcal{A}_1^c$, with domain $\mathcal{D}(\mathcal{A}_1^c) = \mathcal{D}(\mathcal{A}_1)$, is the closed-loop operator and $\Delta\varphi$ a modified nonlinear term, which are defined as

$$\mathcal{A}_1^c x_1 = \mathcal{A}_1 x_1 - 2\epsilon \mathcal{B}\mathcal{C}\mathcal{A}_1 x_1, \tag{7d}$$
$$\Delta\varphi(x_1) = \varphi(x_1) - 2\epsilon \mathcal{B}\mathcal{C}\varphi(x_1). \tag{7e}$$

Note that in (7e) the nonlinear component of the controller (6) takes the weighted (by $2\epsilon\mathcal{B}$) and averaged (by $\mathcal{C}$) nonlinear term φ and subtract it from the original nonlinearity, potentially mitigating its destabilizing effect. By assumption (A3) it follows that

$$\|\Delta\varphi(x_1)\| \leq \bar{\kappa}\,\|x_1\|\,,$$

where $\bar{\kappa}(\zeta)$ is sensor location dependent highlighting that ζ is a key design degree of freedom in the stabilization task. For the stability assessment the characterization of the zero dynamics is addressed next.

3.2 Zero dynamics

The related zero dynamics associated to (7) are given by

$$\partial_t x_1^z = \mathcal{A}_1^z x_1^z + \Delta\varphi(x_1^z), \quad x_1^z(0) = x_{10}^z, \tag{8a}$$
$$\partial_t x_2 = \mathcal{A}_2 x_2 + a_{21} x_1^z, \quad x_2(0) = x_{20}, \tag{8b}$$
$$y = \mathcal{C} x_1^z = 0, \tag{8c}$$

with the zero dynamics operator $\mathcal{A}_1^z$ defined as

$$\mathcal{A}_1^z x_1 = \mathcal{A}_1^c x_1, \quad \mathcal{D}(\mathcal{A}_1^z) = \{x \in \mathcal{D}(\mathcal{A}_1)\,|\,\mathcal{C}x = 0\}\,. \tag{8d}$$

In the following Lemma sufficient conditions for the stability of the solution $(x_1^z, x_2) = (0, 0)$ are established.

Lemma 1. Let assumptions (A1)-(A3) hold and additionally assume that the origin is the unique steady-state for the zero dynamics. If the operator $\mathcal{A}_1^z$ defined in (8d) generates a C_0-semigroup of contractions $S_1^z(t) = \mathrm{e}^{\mathcal{A}_1^z t}$ with growth bound ν_z which satisfies

$$\nu_z - \bar{\kappa} := \upsilon_z > 0, \tag{9}$$

then $(x_1^z, x_2) = (0, 0)$ is globally exponentially stable in the L^2-norm and input-to-state stable with respect to additive disturbances.

Proof: Considering a bounded additive disturbance $\varsigma(t) \in L^2$ in the x_1^z dynamics (8a), the formal solutions for (8) are given by

$$x_1^z(t) = x_{10}^z S_1^z(t) + \textstyle\int_0^t S_1^z(t-\tau)\,[\Delta\varphi(x_1^z(\tau)) + \varsigma(\tau)]\,\mathrm{d}\tau,$$
$$x_2(t) = x_{20} S_2(t) + \textstyle\int_0^t a_{21} S_2(t-\tau) x_1^z(\tau)\mathrm{d}\tau,$$

taking norms, using Assumption (3) and applying the triangle inequality the following is obtained

$$\|x_1^z(t)\| \leq \|x_{10}^z\|\mathrm{e}^{-\nu_z t} + \textstyle\int_0^t \mathrm{e}^{-\nu_z(t-\tau)}\Big(\bar{\kappa}\|x_1^z(\tau)\| + \|\varsigma(\tau)\|\Big)\mathrm{d}\tau,$$
$$\|x_2(t)\| \leq \|x_{20}\|\mathrm{e}^{-\nu_2 t} + |a_{21}|\textstyle\int_0^t \mathrm{e}^{-\nu_2(t-\tau)}\|x_1^z(\tau)\|\mathrm{d}\tau.$$

Denote the right-hand-side of the above inequalities as ξ_i, $i = 1, 2$. It holds that $\xi_i(0) = \|x_i(0)\|$, $i = 1, 2$, $\|x_1^z(t)\| \leq \xi_1(t)$, $\|x_2(t)\| \leq \xi_2(t)$ for al $t \geq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} \leq \begin{bmatrix}-\upsilon_z & 0\\ |a_{21}| & -\nu_2\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}\|\varsigma\|.$$

Due to the triangular matrix in the above and that by Assumption (A2) $\nu_2 > 0$, the following holds: (i) $\varsigma(t) = 0$ implies that the zero solution $\xi_i = 0$, $i = 1, 2$ is exponentially stable if (9) is fulfilled and as a consequence the zero solution of the zero dynamics is exponentially stable in the L^2-norm, and (ii) when the disturbance $\varsigma(t)$ is present the following is satisfied

$$\|x_1^z(t)\| \leq \|x_{10}^z\|\mathrm{e}^{-\upsilon_z t} + \tfrac{1}{\upsilon_z}|\varsigma(\tau)|_\infty,$$
$$\|x_2(t)\| \leq \|x_{20}\|\mathrm{e}^{-\nu_2 t} + \tfrac{|a_{21}|}{\nu_2}\|x_1^z(t)\|.$$

where $|\cdot|_\infty$ denotes the supremum-norm. This implies input-to-state-stability [Karafyllis and Krstic, 2019, Sontag, 1995]. $\square$

3.3 Closed-loop stability

The closed-loop dynamics (7) can be rewritten as the following cascaded interconnection

$$\dot{y} = -ky, \quad y(0) = y_0 \tag{10a}$$
$$\partial_t x_1 = \mathcal{A}_1^c x_1 + \Delta\varphi(x_1) - 2\epsilon k \mathcal{B} y, \quad x_1(0) = x_{10} \tag{10b}$$
$$\partial_t x_2 = \mathcal{A}_2 x_2 + a_{21} x_1, \quad x_2(0) = x_{20}. \tag{10c}$$

Since the domain of the closed-loop dynamics differs from the one of the zero dynamics, write

$$x_1 = x_1^z + \tilde{x}_1, \quad \tilde{x}_1 = x_1 - x_1^z \in \mathcal{H}.$$

Consequently, it holds that the output y converge to zero together with $\tilde{x}_1$ and there exists a positive constant M such that

$$|y| \leq |y_0|\mathrm{e}^{-kt} \Rightarrow \|\tilde{x}_1\| \leq M\|\tilde{x}_{10}\|\mathrm{e}^{-kt}.$$

Considering the formal solutions of the closed-loop system with $\|x_1\| \leq \|x_1^z\| + \|\tilde{x}_1\|$, applying the triangle inequality and using the input-to-state stability property of the zero dynamics with $\varsigma(t) = -2\epsilon k \mathcal{B} y$, the following holds

$$\|x_1(t)\| \leq (\|x_{10}^z\| - \eta|y_0|)\,\mathrm{e}^{-\upsilon_z t} + (M\|\tilde{x}_{10}\| + \eta|y_0|)\,e^{-kt}$$
$$\|x_2(t)\| \leq \|x_{20}^z\|e^{-\nu_2 t} + \tfrac{a_{21}}{\nu_2}\|x_1(t)\|,$$

where $\eta = \frac{2\epsilon\|\mathcal{B}\|k}{\upsilon_z - k}$. Consequently (x_1^z, x_2) converge exponentially to zero, in the L^2-norm, with rate $\min\{\upsilon_z, k\}$. This result is stated in the following proposition.

Proposition 1. Let the assumptions of Lemma 1 hold. Then, the controller (6) exponentially stabilizes the origin of the system (3) in the L^2-norm with convergence rate given by $\min\{\upsilon_z, k\}$.

3.4 Sensor placement

According to Proposition 1 and Lemma 1, the key property for the functioning of the proposed control scheme is the exponential stability of the origin of the zero dynamics in the L^2-norm. In the following this property is characterized in terms of the sensor location using Lyapunov techniques and modal representation. For this aim, consider the following Lyapunov functional

$$V = \tfrac{1}{2}\langle x_1^z, x_1^z\rangle > 0. \tag{11}$$

Its time derivative along the trajectories of (8a) reads

$$\begin{aligned}\tfrac{\mathrm{d}V}{\mathrm{d}t} &= \tfrac{1}{2}\langle \partial_t x_1^z, x_1^z\rangle + \tfrac{1}{2}\langle x_1^z, \partial_t x_1^z\rangle \\ &= \tfrac{1}{2}\langle \mathcal{A}_1^z x_1^z + \Delta\varphi(x_1^z), x_1^z\rangle + \tfrac{1}{2}\langle x_1^z, \mathcal{A}_1^z x_1^z + \Delta\varphi(x_1^z)\rangle \\ &= \langle \mathcal{A}_1 x_1^z, x_1^z\rangle - 2\epsilon\mathcal{C}\mathcal{A}_1 x_1^z\langle \mathcal{B}, x_1^z\rangle + \langle \Delta\varphi(x_1^z), x_1^z\rangle\end{aligned}$$

In the following the notation $\sum = \sum_{n=1}^{\infty}$ and $\sum' = \sum_{n=2}^{\infty}$ is used. From Assumption (A1), the first state has the modal representation $x_1 = \sum a_n\psi_n$ and the output can be written as $y = \sum a_n c_n$, where $a_n = \langle x_1, \psi_n\rangle$ and $c_n = \langle \bar{\gamma}, \psi_n\rangle$. Furthermore, the action of the operator $\mathcal{A}_1^z$ can be expressed as $\mathcal{A}_1^z x_1 = \sum\lambda_n a_n\psi_n$, thus it follows that $\mathcal{C}\mathcal{A}_1 x_1^z = \sum\lambda_n a_n c_n$ and $\langle \mathcal{B}, x_1\rangle = \sum a_n b_n$ where $b_n = \langle \beta, \psi_n\rangle$. Considering all the introduced modal representations, the time derivative of V reads

$$\begin{aligned}\tfrac{\mathrm{d}V}{\mathrm{d}t} &= \textstyle\sum\lambda_n a_n \langle \psi_n, \sum a_j\psi_j\rangle - 2\epsilon\sum\lambda_n a_n c_n \sum a_j b_j \\ &\quad + \langle \Delta\varphi(x_1^z), x_1^z\rangle, \\ &= \textstyle\sum\lambda_n a_n^2 - 2\epsilon\sum\lambda_n a_n c_n\sum a_j b_j + \langle \Delta\varphi(x_1^z), x_1^z\rangle, \\ &= \textstyle\lambda_1 a_1^2 + \sum'\lambda_n a_n^2 + \langle \Delta\varphi(x_1^z), x_1^z\rangle - \\ &\quad \textstyle - 2\epsilon\left(\lambda_1 a_1 c_1 + \sum'\lambda_n a_n c_n\right)\left(a_1 b_1 + \sum' a_j b_j\right).\end{aligned}$$

Since $y = 0$ it holds that $a_1 = -\frac{1}{c_1}\sum' a_n c_n$ and thus

$$\begin{aligned}\tfrac{\mathrm{d}V}{\mathrm{d}t} &= \textstyle\sum'\left(\lambda_n + \frac{\lambda_1}{c_1^2}c_n\right)a_n^2 + \frac{\lambda_1}{c_1^2}\sum' a_n c_n \sum'_{n\neq j} a_j c_j \\ &\quad \textstyle - 2\epsilon\sum'(\lambda_n - \lambda_1)a_n c_n \sum'\left(b_j - \frac{b_1}{c_1}c_j\right)a_j + \\ &\quad + \langle \Delta\varphi(x_1^z), x_1^z\rangle.\end{aligned}$$

Writing the summations above in a quadratic form yields

$$\tfrac{\mathrm{d}V}{\mathrm{d}t} = \boldsymbol{a}^T\boldsymbol{Z}\boldsymbol{a} + \langle \Delta\varphi(x_1^z), x_1^z\rangle$$

where $\boldsymbol{a} = [a_n]_{n=2,\ldots}$ is an infinite-dimensional vector and $\boldsymbol{Z} = [z_{n,j}]$, $n, j = 2, \ldots$ an infinite-dimensional matrix with entries defined as follows

$$z_{n,j} = \begin{cases} \lambda_n + \frac{\lambda_1}{c_1^2}c_n^2 - 2\epsilon c_n(\lambda_n - \lambda_1)\left(b_n - \frac{b_1}{\bar{c}_1}c_n\right) & n = j \\ \frac{\lambda_1}{c_1^2}c_n c_j - 2\epsilon c_n(\lambda_n - \lambda_1)\left(b_j - \frac{b_1}{c_1}c_j\right) & n \neq j \end{cases}. \tag{12}$$

Thus it holds that

$$\begin{aligned}\tfrac{\mathrm{d}V}{\mathrm{d}t} &\leq -\nu_z\|x_1^z\|^2 + \langle \bar{\kappa}\|x_1^z\|, x_1^z\rangle \leq -(\nu_z - \bar{\kappa})\|x_1^z\|^2, \\ &\leq -(\nu_z - \bar{\kappa})V(x_z),\end{aligned}$$

where $\nu_z = \sup_{n\geq 2}\lambda(\boldsymbol{Z})$. To ensure that the time derivative of $V(x_1^z)$ is negative definite, the matrix $\boldsymbol{Z}$ must be Hurwitz. This can be established by applying the Geršgorin theorem for infinite-dimensional matrices (see [Theorem (16c) in Aleksić et al. [2014]], c.p. [Franco-de los Reyes et al., 2019a,b]). Adapting this result to the matrix $\boldsymbol{Z}$ its maximum eigenvalue can be estimated as

$$\nu_z = \mathrm{Re}(\sup_{n\geq 2}\{\lambda(\boldsymbol{Z})\}) \approx \sup_{n\geq 2}\{z_{n,n}\} \tag{13}$$

where $z_{n,n}$ is given in (12). The sensor location ζ should be selected such that $c_1 \neq 0$ and ν_z is large enough to satisfy (9). This is summarized in the following lemma.

Lemma 2. Consider the zero dynamics (8). Let the sensor location ζ be such that $c_1 \neq 0$ and the matrix $\boldsymbol{Z}$ defined in (12) is Hurwitz and ν_z in (13) is such that (9) holds true. Then the solution $(x_1^z, x_2) = 0$ of the zero dynamics is exponentially stable in the L^2-norm.

Remark 2. In order to ensure that the zero solution is the unique steady-state profile for the zero dynamics a bifurcation analysis with respect to the sensor location ζ can be carried out, which in combination with the previous Lemma can be used as sensor location criterion.

4. OUTPUT FEEDBACK CONTROLLER

The controller (6) requires knowledge of the first state but since it is not available for implemmaentation purposes the controller is build on the basis of a reduced order point injection estimator [Schaum et al., 2017] as follows

$$\partial_t\hat{x}_1 = \mathcal{A}_1\hat{x}_1 + \varphi(\hat{x}_1) + \mathcal{B}u, \quad \hat{x}_1(0) = \hat{x}_{10} \tag{14a}$$

$$\hat{x}_1(\zeta, t) = y(t) \tag{14b}$$

$$u = -2\epsilon\left[ky + \mathcal{C}\left(\mathcal{A}_1 x_1 + \varphi(x_1)\right)\right]. \tag{14c}$$

With closed-loop dynamics

$$\partial_t e = \mathcal{A}_1 e + \vartheta(e), \qquad e(0) = e_0 \tag{15a}$$

$$e(\zeta, t) = 0 \tag{15b}$$

$$\partial_t x_1 = \mathcal{A}_1^c x_1 + \Delta\varphi(x_1) + \theta(e), \quad x_1(0) = x_{10} \tag{15c}$$

$$\partial_t x_2 = \mathcal{A}_2 x_2 + a_{21}x_1, \qquad x_2(0) = x_{20} \tag{15d}$$

where $e = \hat{x} - x$ is the estimation error, $\vartheta(e) = \varphi(x + e) - \varphi(x)$ and $\theta(e)$ is a Lipschitz bounded interconnection term which satisfies $\theta(0) = 0$. Due to the cascaded structure of the resulting closed-loop dynamics and its input-to-state stability property, the exponential stability of the origin $(e, x_i) = 0$, $i = 1, 2$ follows if Proposition 1 is fulfilled and the estimation error converge exponentially. According to [Schaum et al., 2017] exponential stability of $e = 0$ is ensured if the sensor location is selected appropriately. In particular, using the same sensor location as in the controller the stability is ensured.

5. SIMULATION STUDY

Consider a system with given by the enthalpy balance introduced in [Raymond and Amundson, 1964] coupled with a linear convection-diffusion equation, which n the form (1) is described by following parameters and nonlinearity

$$d = 0.2, \quad a_{21} = 0.5, \quad \phi(\bar{x}_1) = 0.5 \times 10^7(1 - \bar{x}_1)\mathrm{e}^{-\frac{20}{1+\bar{x}_1}}.$$

In Fig. 1 (left) the three open-loop steady-state profile pairs of the system are shown, note that two of them are stable and the one in the middle, denoted as $\bar{x}_1^*, \bar{x}_2^*$, is unstable and selected for closed-loop operation. The open-loop dynamic response of the system initialized near the unstable profile pair is shown in Fig. 1 (right) and confirms its instability.

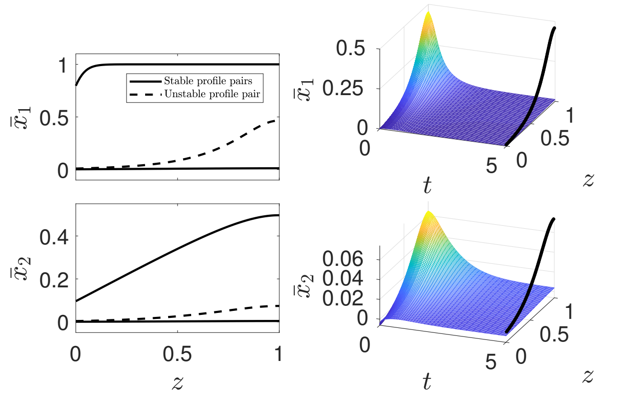

Fig. 1. Open-loop system steady-states (left) and profile evolution (instability of the target steady-state).

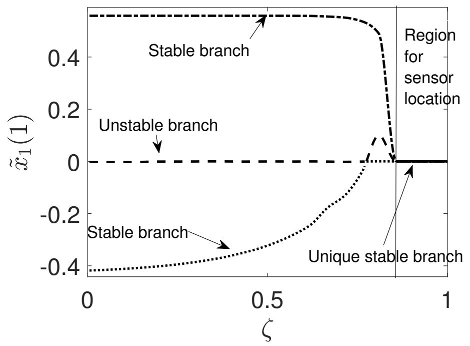

Fig. 2. Steady-states dependency on the sensor location: triplicity for $\zeta \in [0, 0.87)$, uniqueness for $\zeta \in [0.87, 1]$.

The eigenvalues and eigenfunctions are given by

$$\lambda_n = -\left(\tfrac{1}{2d}\right)^2 - \omega_n^2,\ \tan(\omega_n) = \tfrac{\omega_n}{\omega_n^2 - (\frac{1}{2d})^2},\ \omega_n \neq 0,$$
$$\psi_n = B_n \left(2d\omega_n \cos(\omega_n z) + \sin(\omega_n z)\right),$$

where B_n are normalization constants. The series b_n and c_n are given by

$$b_n = \tfrac{2B_n}{\omega_n^2 + \left(\frac{1}{2d}\right)^2}\left(\omega_n + e^{-\frac{\zeta}{2d}}\left(d\left(\omega_n^2 - \tfrac{1}{2d}\right)\sin(\omega_n) + \right.\right.$$
$$\left.\left. +\ \omega_n \cos(\omega_n)\right)\right)$$
$$c_n = B_n e^{\frac{\zeta}{2d}}\left(2d\omega_n \cos(\omega_n \zeta) + \sin(\omega_n \zeta)\right),$$

Note that b_n are constants, and that c_n depends on the sensor location ζ. According to the previous sections, ζ must be selected to satisfy the conditions of Lemma 1 and Lemma 2. The first assumption is the uniqueness of the steady-state solution for the zero dynamics, i.e., the uniqueness of the zero profile pair for the boundary value problem

$$\begin{aligned} 0 &= \mathcal{A}_1^z x_1 + \varphi(x_1^z), && x_1 \in \mathcal{D}(\mathcal{A}_1^z) \\ 0 &= \mathcal{A}_2 x_2 + a_{21} x_1^z, && x_2 \in \mathcal{D}(\mathcal{A}_2) \\ 0 &= \mathcal{C} x_1 = y, && \end{aligned}$$

where the restriction $y = x_1^z(\zeta) = 0$ for $\zeta \in [0, 1]$ is the unique degree of freedom. The uniqueness of the solution $(x_1^z, x_2) = 0$ requires that for the first state $x_1 = 0$ must be a unique solution. The analysis of the above boundary

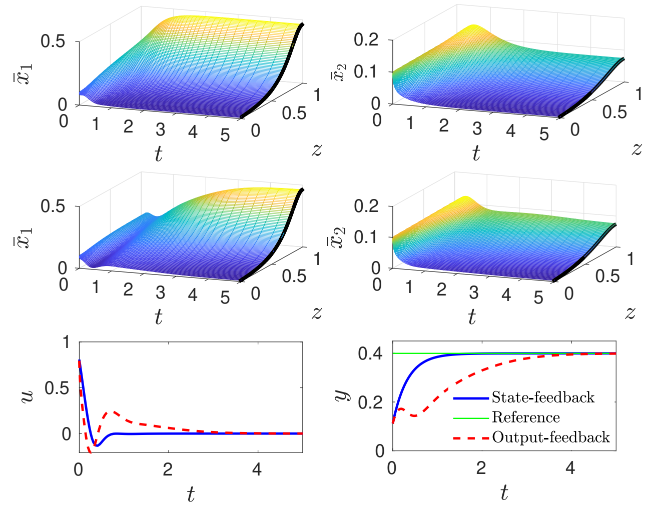

Fig. 3. Closed-loop behavior. State profile evolution: state (left) and output-feedback (right), control effort and output responses (bottom).

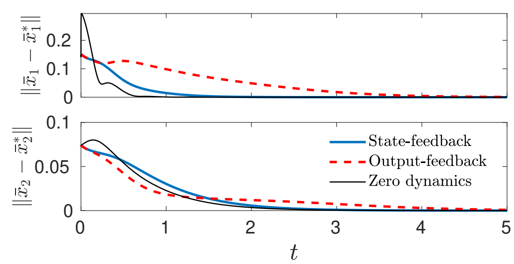

Fig. 4. Closed-loop behavior, L^2-approximated norms of the states for state and output-feedback cases and the zero dynamics.

value problem is done by constructing a bifurcation diagram (based on a finite differences approximation) using the `Matcont` software package [Dhooge et al., 2003]. The obtained result is shown in Fig. 2. It can be seen that the zero solution is unique if $\zeta \in \mathcal{I}_z = (0.87, 1)$. Accordingly $\zeta = 0.9$ is selected so that the conditions of Lemma 2 are fulfilled, i.e., $c_1(\zeta) \neq 0$ and $\nu_z \approx -4.13$. The numerical computation of κ can be done using a Lipschitz constant that for this case gives $\kappa = 3$. Thus condition (9) is fulfilled with $\upsilon_z = -1.13$. This ensures the convergence to zero of the zero dynamics (that determines the rate of convergence of the closed-loop states) and for the closes-loop system (15). The gain k is used to accelerate the rate of convergence of the output.

In Fig. 3 the closed-loop behavior with controllers (6) and (14) with $k = 3$ is shown in original coordinates. The system is initialized at the lower steady-state. It can be seen that for the state feedback case the output goes to zero in about 1 time unit so the system in the zero dynamics converge to the desired steady-state in 2 time units. For the output-feedback case the convergence is slower due to the estimation convergence time. In Fig. 4 the approximated L^2-norms of the corresponding state profiles are shown. It can be seen that the zero dynamics response is the best attainable behavior.

All simulation were carried out using finite differences with 100 collocation points, approximating the integral with the trapezoidal rule using the Matlab command `trapz` and solving the obtained system of ordinary differential equations with the method `ode15s` in Matlab.

6. CONCLUSIONS

The output-feedback stabilization problem of an unstable profile of a class of semilinear PDE models with averaged measurement and homogeneous control action has been addressed. The controller scheme is constructed following a similar notion to feedback passivity for nonlinear finite-dimensional systems. The close-loop stability is ensured by the characterization of the zero dynamics in terms of the sensor location by using a combined approach with Lyapunov and modal techniques. For implementation purposes an observer scheme is added to the design. Numerical simulations show the satisfactory performance of the proposed approach.

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