Continuous-discrete observer for nonlinear systems with sampled measurements: application to a 1-DOF helicopter

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Abstract: This work presents a new approach for observer design for a general class of state affine nonlinear systems in the presence of uncertainties in the state equations and the sampled output measurements. A new high-gain observer design is developed and analyzed under insightful conditions. This result is achieved by considering a persistent excitation condition that can be validated on-line. The algorithm is applied to a 1-D helicopter and validated with simulations

Keywords: Nonlinear systems, sampled measurement and continuous-discrete observer

1. INTRODUCTION

Over the last decades, the observer design has been investigated by numerous researchers, but is still an open problem. Many different class of systems have been considered, and mainly for continuously available measurements, Alessandri and Rossi [2015], Besançon [1999]. A strong property of linear systems is the fact that the observability does not depend on the input. This idea has been transposed to nonlinear systems giving birth to the general high-gain observer design in the seminal paper. Contrarily to the linear case, the observability of a nonlinear system can be lost depending on the input. Several different general forms have been proposed for the observer design of non-uniformly observable systems. Such design usually requires additional assumptions on the input, called persistent excitation condition.

A typical difficulty encountered for observer design is that the output of the systems is assumed continuous. However, in many practical cases the output is available at discrete time. Several results have already been proposed for this challenging problem, e.g. Folin et al. [2016]. The first observer considering this constraint has been presented in Deza et al. [1992] and it is based on a Kalman filter. In the continuous case, a continuous estimation of the output has been used in Karafyllis and Kravaris [2009] and reconsidered in Ahmed-Ali and Lamnabhi-Lagarrique [2012]. Following the same idea, an observer for a class of multi-output systems has been proposed in Farza et al. [2014b] and the effect of uncertainty on the model has been further studied in Farza et al. [2014a]. A similar design based on the continuous case has been proposed in Raff et al. [2008] but the correction term is kept constant between two measures. The case of systems whose observability depends on the input has been considered in Nadri et al. [2004], it is based on a state estimator which is reinitialized when a new measurement is available. Several publications have appeared in recent years documenting the observer design problem for nonlinear systems with sampled output, Hammouri et al. [2006], Nadri et al. [2013], here, the dynamical systems is used to provide a state prediction over the sampling intervals. The state prediction is updated by the output measurements, which are sampled at instant t_k. It has been proposed a correction term, that is shaped by a gain which is computed through the resolution of LMI (Linear Matrix Inequality). To the authors best knowledge, no observer has been proposed for the class of nonlinear state affine

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systems considered here when the output is only available at discrete times.

In this paper, we focus on the design of a class state-affine uncertain nonlinear system with sampled output measurements, i.e., its measurements are available at sampling instants t_k. When, the uncertainties are not present, the nonlinear system is observable for any persistent input. An observer for these systems is considered in Besancon [1999]. The designed continuous observer with the sampled output measurements is achieved from a re-designed version of the continuous-time observer proposed by Besancon [1999]. There are two main contributions of the proposed observer with the sampled output measurements. The first contribution concerns the design of a continuous observer. The convergence analysis is studied by considering uncertainties and measurement noise. The observation error lies in a region centered at the origin whose radio depending on the bounds of the uncertainties, the noise measurements and the maximum sampling. The second contribution is the easiness to compute the correction term of observer which is updated at sampling instants.

The paper is organized has follows. The class of considered systems is described in the next section with some assumptions required for the continuous observer design. In section 3, a simple observer based on high-gain design is provided for the case of continuously available measurements. The continuous observer is redesigned in section where it is considered the sampled output. The proposed observer is applied to a 1-DOF helicopter and simulations are given in section 5. Finally, section 6 concludes the paper.

2. PROBLEM STATEMENT

Consider the following class of multi-variable state-affine nonlinear system

$$\begin{cases} \dot{x}(t) = A\left(u(t), x(t)\right) x(t) + \varphi\left(x(t), u(t)\right) + B\varepsilon(t) \\ y(t_k) = Cx\left(t_k\right) = x^1(t_k) \end{cases} \tag{1}$$

with

$$A(u(t), x(t)) = \begin{bmatrix} 0 & A_1(u(t), x^1(t)) & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & & & A_{q-1}(u(t), x^1(t), \cdots, x^{q-1}(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\varphi(u(t), x(t)) = \begin{pmatrix} \varphi_1(u(t), x^1(t)) \\ \varphi_2(u(t), x^1(t), x^2(t)) \\ \vdots \\ \varphi_{n-1}(u(t), x^1(t), \cdots, x^{q-1}(t)) \\ \varphi_n(u(t), x(t)) \end{pmatrix}$$

$$C = \left[\, I_{n_1 \times n_1} \;\; 0_{n_1 \times n_2} \;\; \cdots \;\; 0_{n_1 \times n_q} \,\right]$$

and $B = \left[\, 0_{n_1} \; 0_{n_1} \; \cdots \; I_{n_1} \,\right]^T$ where the state $x(t) = (x^1 ... x^q)^T \in \mathbb{R}^n$, $x^k \in \mathbb{R}^{n_k}$, $k = 1, ..., q$ with $n_1 = p$ and $\sum_{k=1}^{q} n_k = n$ and each $A_k(u, x)$ is a $n_k \times n_{k+1}$ matrix which is triangular w.r.t. x i.e. $A_k(u, x) = A(u, x^1, ..., x^k)$, $k = 1, ..., q-1$; $\varphi(x(t), u(t))$ is a nonlinear vector function that has a triangular structure w.r.t. x; $u \in \mathbb{R}^s$ denotes the system input; $\epsilon(t)$ is an unknown function describing the system uncertainties and may depend on the state, $\epsilon : \mathbb{R}^+ \mapsto \mathbb{R}^p$ and $y(t_k) \in \mathbb{R}^p$ is the discrete-time output. Furthermore $0 \leq t_0 < \cdots < t_k < \ldots$, $\Delta_k = t_{k+1} - t_k$ and $\lim_{k \to +\infty} t_k = +\infty$, we assume that there exists $\Delta_M > 0$ such that $0 < \Delta_k < \Delta_M$, $\forall k \geq 0$.

Now, some assumptions are provided (see Besançon et al. [1996] and Farza et al. [2015]):

A1 The state $x(t)$ and the control $u(t)$ are bounded, i.e., $x(t) \in X$ and $u(t) \in U$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compacts sets.
A2 The functions $A(u(t), x(t))$ and $\varphi\left(x(t), u(t)\right)$ are Lipschitz w.r.t. x uniformly w.r.t. u where $(u, x) \in U \times X$. Their Lipschitz constants are denoted by L_A and L_φ.
A3 The unknown function $\varepsilon(t)$ is essentially bounded, i.e., $\exists \delta_\varepsilon > 0 \quad \text{Ess.sup.}_{t \geq 0} \|\varepsilon(t)\| \leq \delta_\varepsilon$

Since the state is confined to the bounded set X, one can assume the Lipschitz prolongations of the nonlinearities, using smooth saturation functions. In the following, it is assumed that the prolongations have been carried out and that the functions $A(u(t), x(t))$ and $\varphi\left(x(t), u(t)\right)$ are provided from these prolongations. This allows to conclude that for any bounded input $u \in U$, the functions $A(u, x)$ and $\varphi\left(x, u\right)$ are globally Lipschitz w.t.r. x and are bounded for all $x \in \mathbb{R}^n$.

3. CONTINUOUS-TIME OBSERVER

The observer design for system (1) with a continuous-time output is provided. This system is represented as:

$$\begin{cases} \dot{x}(t) = A\left(u(t), x(t)\right) x(t) + \varphi\left(x(t), u(t)\right), \\ y(t) = Cx\left(t\right) = x^1(t). \end{cases} \tag{2}$$

Assumption **A1** gives the existence of an upper bound for the state and $A(x(t), u(t))$, these are defined as:

$$x_M = \sup_{t \geq 0} \|x(t)\|\,; \qquad \tilde{a} = \sup_{t \geq 0} \|A(u(t), x(t))\| \tag{3}$$

The candidate observer is given by

$$\begin{aligned} \dot{\hat{x}}(t) = {} & A(u(t), \hat{x}(t))\hat{x}(t) + \varphi(u(t), \hat{x}(t)) \\ & - \theta \Delta_\theta^{-1} S^{-1}(t) C^T \left(C\hat{x}(t) - y(t)\right) \end{aligned} \tag{4}$$

where $\hat{x} = (\hat{x}^1 \ldots \hat{x}^q)^T \in \mathbb{R}^n$ with $\hat{x}^k \in \mathbb{R}^{n_k}$, u and y are respectively the input and the output of the system (2) and $S(t)$ is a SPD matrix governed by the following linear Lyapunov differential equation:

$$\dot{S}(t) = \theta\left(-S(t) - A(u(t), \hat{x}(t))^T S(t) - S(t) A(u(t), \hat{x}(t)) + C^T C\right) \tag{5}$$

with $S(0) = S^T(0) > 0$ and $\theta > 0$ is a scalar design parameter. We define the diagonal matrix Δ_θ as:

$$\Delta_\theta = \text{diag}\left[\, I_{n_1} \;\; I_{n_2}/\theta \;\; \cdots \;\; I_{n_q}/\theta^{q-1} \,\right] \tag{6}$$

The nonlinear system (2) is not necessarily uniformly observable, its observability depends on the input and the state. We need to guarantee observability at arbitrarily short times for the observer design. A specific excitation is necessary, it is called local regularity, it qualifies the behavior of the input for small times Farza et al. [2015]. In order to adopt an additional hypothesis required for the observer design, we need to define some variable. Let $\Phi(t,s)$ be the state transition matrix of the state-affine system:

$$\dot{\xi}(t) = A(u(t), \hat{x}(t))\xi(t) \tag{7}$$

where $\xi \in \mathbb{R}^n$. Where u and $\hat{x}$, are the input and the state of the dynamical system (4). Recall that the matrix $\Phi_{u,\hat{x}}(t,s)$ is defined as:

$$\frac{d\Phi_{u,\hat{x}}(t,s)}{dt} = A(u(t), \hat{x}(t))\Phi_{u,\hat{x}}(t,s), \quad \forall t \geq s \geq 0, \tag{8}$$

$$\Phi_{u,\hat{x}}(t,t) = I_n, \quad \forall t \geq 0, \tag{9}$$

where I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$.
We can now adopt the following additional hypothesis:

A5 The input u is such that for any trajectory $\hat{x}$ of system (4) starting from $\hat{x}(0) \in X$, $\exists\theta^* > 0$, $\exists\delta_0 > 0$, $\forall\theta \geq \theta^*$ and $\forall t \geq 1/\theta$, the following persistent excitation condition is satisfied

$$\int_{t-1/\theta}^{t} \Phi_{u,\hat{x}}(s,t)^T C^T C \Phi_{u,\hat{x}}(s,t) ds \geq \frac{\delta_0}{\theta\alpha(\theta)}\Delta_\theta^2 \tag{10}$$

where $\alpha(\theta) \geq 1$ is a function satisfying

$$\lim_{\theta\to\infty} \frac{\alpha(\theta)}{\theta^2} = 0 \tag{11}$$

Therefore, we can state the following theorem:

Theorem 1. Consider system (2), satisfying assumptions **A1-A4**. Then, for every bounded input satisfying assumption **A5**, there exists a constant θ^* such that for every $\theta > \theta^*$, system (4) is a state observer for system (2) with an exponential error convergence to the origin for sufficiently high values of θ, i.e. for any initial conditions $(x(0), \hat{x}(0)) \in X$, the observation error $\hat{x}(t) - x(t)$ tends to zero exponentially when $t \to \infty$.

Proof of Theorem 1 We shall first show that the matrix $S(t)$ is SPD and we shall derive a lower bound for its smallest eigenvalue. Indeed, one can show that the transition matrix, $\tilde{\Phi}_{u,\hat{x}}$ of the following state affine system

$$\dot{\xi}(t) = \theta A(u(t), \hat{x}(t))\xi(t) \tag{12}$$

is given by

$$\tilde{\Phi}_{u,\hat{x}} = \Delta_\theta \Phi_{u,\hat{x}}(t,s)\Delta_\theta^{-1} \tag{13}$$

where $\Phi_{u,\hat{x}}$ is defined by (8).
As a result, the matrix $S(t)$, solution of ODE (5), can be expressed as

$$\begin{aligned} S(t) &= e^{-\theta t}\tilde{\Phi}^T_{u,\hat{x}}(0,t)S(0)\tilde{\Phi}_{u,\hat{x}}(0,t) \\ &\quad + \theta \int_0^t e^{-\theta(t-s)}\tilde{\Phi}^T_{u,\hat{x}}(s,t)C^T C\tilde{\Phi}_{u,\hat{x}}(s,t)ds \\ &= e^{-\theta t}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(0,t)\Delta_\theta S(0)\Delta_\theta\Phi_{u,\hat{x}}(0,t)\Delta_\theta^{-1} \\ &\quad + \theta \int_0^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)\Delta_\theta C^T C\Delta_\theta\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \end{aligned} \tag{14}$$

Using the fact that $C\Delta_\theta = C$ and since $S(0)$ is SPD, one gets for $t \geq 1/\theta$

$$\begin{aligned} S(t) &\geq \theta \int_0^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)C^T C\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \\ &\geq \theta \int_{t-\frac{1}{\theta}}^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)C^T C\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \\ &\geq \theta e^{-1} \int_{t-\frac{1}{\theta}}^t e^{-\theta(t-s)}\Delta_\theta^{-1}\Phi^T_{u,\hat{x}}(s,t)C^T C\Phi_{u,\hat{x}}(s,t)\Delta_\theta^{-1}ds \\ &\geq e^{-1}\frac{\delta_0}{\alpha(\theta)}I_n \end{aligned} \tag{15}$$

where δ_0 and $\alpha(\theta)$ are given by assumption **A5**. According to inequality (15), one clearly has

$$\lambda_{\min}(S) \geq \frac{e^{-1}\delta_0}{\alpha(\theta)} \tag{16}$$

We shall now show that $\lambda_{\max}(S)$ is bounded with an upper bound independent of θ. To this end, we shall show that this property is satisfied for each entry of the matrix $S(t)$. Indeed, let us denote by $S_{i,j}$ the block entry of matrix S located at the row i and the column j. Then, according to equation (5), one has

$$\dot{S}_{11} = -\theta(S_{11}(t) - I_p) \tag{17}$$

$$\dot{S}_{1j} = -\theta(S_{1j}(t) + S_{1,j-1}(t)A_{j-1}(u(t), \hat{x}(t))) \quad j = 2, ..., n \tag{18}$$

$$\begin{aligned} \dot{S}_{ij} &= -\theta(S_{ij}(t) + S_{i,j-1}(t)A_{j-1}(u(t), \hat{x}(t)) \\ &\quad + A^T_{i-1}(u(t), \hat{x}(t))S_{i-1,j}(t)) \\ &\quad i = 2, ..., n, j = i, ..., n \end{aligned} \tag{19}$$

According to (17), one has

$$\begin{aligned} \|S_{11}(t)\| &\leq e^{-\theta t}\|S_{11}(0)\| + \theta\int_0^t e^{-\theta(t-s)}\|I_p\| ds \\ &= \|S_{11}(0)\| + 1\left(1 - e^{-\theta t}\right) \leq \|S_{11}(0)\| + 1 \end{aligned} \tag{20}$$

Now, for $j \geq 2$, let us proceed by induction on j. In order to show that S_{ij} is bounded with a bound that does not depend on θ. We assume that $S_{1,j-1}$ is bounded and let us denote

$$S_M = \sup_{t\geq 0}\|S_{1,j-1}(t)\| \tag{21}$$

Recall that according to assumptions **A1** and **A2**, the matrices $A_k(u,\hat{x}), k = 0,...,q-1$ are bounded. Thus, setting

$$A_M = \sup_{t\geq 0} \|A_k(u(t),\hat{x}(t))\| \qquad (22)$$

Thus, according what we have considered above, we can show that all the entries of the matrix $S(t)$ are bounded with an upper bound independent of θ. As a result the largest eigenvalues of $S(t)$, $\lambda_{\max}(S)$, is also independent of θ.

Now, we prove the exponential convergence to zero of the observation error $\tilde{x}$. Set $\bar{x} = \Delta_\theta \tilde{x}$ where $\tilde{x} = \hat{x} - x$, therefore, the error dynamics is given by

$$\begin{aligned}\dot{\bar{x}}(t) &= \theta\left[A(u(t),\hat{x}(t)) - S^{-1}(t)C^TC\right]\bar{x}(t)\\ &+ \Delta_\theta\left[\tilde{A}(u(t),\hat{x}(t),x(t))x + \tilde{\varphi}(u(t),\hat{x}(t),x(t)) - B\varepsilon(t)\right]\end{aligned} \qquad (23)$$

where $\tilde{A}(u(t),\hat{x}(t),x(t)) = A(u(t),\hat{x}(t)) - A(u(t),x(t))$ and $\tilde{\varphi}(u(t),\hat{x}(t),x(t)) = \varphi(u(t),\hat{x}(t)) - \varphi(u(t),x(t))$.

Let $V(t) = \bar{x}^T(t)S(t)\bar{x}(t)$ be the Lyapunov candidate function, using (5), one gets:

$$\begin{aligned}\dot{V}(\bar{x}(t)) &= -\theta\bar{x}^T(t)S(t)\bar{x}(t) - \theta\bar{x}^T(t)C^TC\bar{x}(t)\\ &+ 2\bar{x}^T(t)S(t)\Delta_\theta\left[\tilde{A}(u(t),\hat{x}(t),x(t))x + \tilde{\varphi}(u(t),\hat{x}(t),x(t))\right]\\ &- 2\bar{x}^T(t)S(t)\Delta_\theta B\varepsilon(t)\end{aligned} \qquad (24)$$

Proceeding as in Farza et al. [2009], one can show that for $\theta > 0$:

$$\begin{aligned}\|2\bar{x}(t)S(t)\Delta_\theta\tilde{A}(u(t),\hat{x}(t),x(t))x(t)\| &\leq 2\sqrt{n}\frac{\sqrt{\lambda_{\max}(S)}}{\sqrt{\lambda_{\min}(S)}}V(\bar{x})L_{\tilde{A}}x_M\\ &\leq 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}V(\bar{x})L_{\tilde{A}}x_M\end{aligned} \qquad (25)$$

$$\begin{aligned}\|2\bar{x}(t)S(t)\Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t))\| &\leq 2\sqrt{n}\frac{\sqrt{\lambda_{\max}(S)}}{\sqrt{\lambda_{\min}(S)}}V(\bar{x})L_{\tilde{\varphi}}\\ &\leq 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}V(\bar{x}(t))L_{\tilde{\varphi}}\end{aligned} \qquad (26)$$

where $L_{\tilde{A}}$ and $L_{\tilde{\varphi}}$ come from Assumption **A2** and considering $\lambda_{\min}(S)$ as in (16).

Proceeding as in Bouraoui et al. [2015], one can show that for $\theta > 0$:

$$\|2\bar{x}^T(t)S(t)\Delta_\theta B\varepsilon(t)\| \leq \frac{2}{\theta^{q-1}}\sqrt{\lambda_{\max}(S)}\sqrt{V(\bar{x}(t))}\delta_\varepsilon \qquad (27)$$

where δ_ε comes from Assumption **A3** and considering $\lambda_{\min}(S)$ as in (16).

By substituting (25)-(27) in (24), one gets:

$$\begin{aligned}\dot{V}(t) &\leq -\theta V(t) + 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}\left(L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right)V(t)\\ &+ 2\left(\frac{\delta_\varepsilon}{\theta^{q-1}}\sqrt{\lambda_{\max}(S)}\right)\sqrt{V(t)}\end{aligned} \qquad (28)$$

We can rewrite (28) as:

$$\begin{aligned}\frac{d}{dt}\sqrt{V} &\leq -\theta\left(1 - 2\sqrt{\frac{\alpha(\theta)}{\theta^2}}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}\left(L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right)\right)\sqrt{V}\\ &+ 2\left(\frac{\delta_\varepsilon}{\theta^{q-1}}\sqrt{\lambda_{\max}(S)}\right)\end{aligned} \qquad (29)$$

According to (11), $V(t)$ converges exponentially to values of θ sufficiently large. This ends the proof.

Remark 2. It is worth to note that in the noise free case, the proposed observer has two properties, when there are no uncertainties, i.e. $\delta_\varepsilon = 0$, the observation error converges exponentially to zero. This is not the case with $\delta_\varepsilon \neq 0$, the observation error is finite, since, the observation error converges into a ball centered at the origin with a radius δ_ε, but it can be made smaller by choosing values of tuning parameter θ sufficiently high. However, this is impractical, high values of θ should be avoided in practice, since, the presence of noise measurements is unavoidable.

4. CONTINUOUS-DISCRETE TIME OBSERVER

The continuous-discrete time observer candidate for system (1) is defined by the following set of equations:

$$\dot{\hat{x}}(t) = A(u(t),\hat{x}(t))\hat{x} + \varphi(u(t),\hat{x}(t)) - \theta\Delta_\theta^{-1}S^{-1}(t)C^T\eta(t) \qquad (30)$$

$$\dot{S}(t) = \theta\left(-S(t) - A(u(t),\hat{x}(t))^TS - SA(u(t),\hat{x}(t)) + C^TC\right) \qquad (31)$$

$$\dot{\eta}(t) = -\theta CS^{-1}(t)C^T\eta(t) \qquad t \in [t_k, t_{k+1}[, k \in N \qquad (32)$$

$$\eta(t_k) = C\hat{x}(t_k) - y(t_k) \qquad t = t_k \qquad (33)$$

where $\hat{x} = [\hat{x}^1, ..., \hat{x}^q]^T$ is the state estimate and Δ_θ is a block-diagonal matrix defined in (6) with $\theta > 0$.
We can now present our main result:

Theorem 3. Consider the system (1), satisfying Assumptions **A1-A5**, with u bounded and making $A(u(t),x(t))$ bounded. There exists $\sigma_0 > 0$ such that for all $\sigma \geq \sigma_0$, if Δ_M is such that:

$$\Delta_M < \frac{a_\theta}{b_\theta} \qquad (34)$$

then the state of continuous-discrete time observer with discrete-time measurements (30)-(33) exponentially converges to the state of the state affine nonlinear system (1).

Proof of Theorem 3

Let us now prove the exponential convergence to zero of the observation error. Set $\bar{x} = \Delta_\theta\tilde{x}$ where $\tilde{x} = \hat{x} - x$, the error equation is given by:

$$\begin{aligned}\dot{\bar{x}}(t) &= \theta A(u(t),\hat{x}(t))\bar{x}(t) - \theta S^{-1}(t)C^T\eta(t) - \Delta_\theta B\varepsilon(t)\\ &+ \Delta_\theta\tilde{A}(u(t),\hat{x}(t),x(t))x(t) + \Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t))\\ &= \theta\left[A(u(t),\hat{x}(t)) - S^{-1}(t)C^TC\right]\bar{x}(t) + \theta S^{-1}(t)C^Tz(t)\\ &+ \Delta_\theta\left[\tilde{A}(u(t),\hat{x}(t),x(t))x + \tilde{\varphi}(u(t),\hat{x}(t),x(t)) - B\varepsilon(t)\right]\end{aligned} \qquad (35)$$

where $z(t) = C\bar{x}(t) - \eta(t)$. Using the fact that $\eta(t)$ is governed by the ODE (32), one can show that:

$$\dot{z}(t) = C[\theta A(u(t),\hat{x}(t))\bar{x}(t) + \Delta_\theta \tilde{A}(u(t),\hat{x}(t),x(t))x(t) + \Delta_\theta \tilde{\varphi}(u(t),\hat{x}(t),x(t))] - C\Delta_\theta B\varepsilon(t) \quad (36)$$

where $C\Delta_\theta B = 0$, we can rewrite (36) as :

$$\dot{z}(t) = C[\theta A(u(t),\hat{x}(t))\bar{x}(t) + \Delta_\theta \tilde{A}(u(t),\hat{x}(t),x(t))x(t) + \Delta_\theta \tilde{\varphi}(u(t),\hat{x}(t),x(t))] \quad (37)$$

A candidate Lyapunov function is given by $V(\bar{x}(t)) = \bar{x}^T(t)S(t)\bar{x}(t)$. Using (31) one gets:

$$\begin{aligned}\dot{V}(\bar{x}(t)) = & -\theta\bar{x}^T(t)S(t)\bar{x}(t) - \theta\bar{x}^T(t)C^TC\bar{x}(t) \\ & + 2\theta\Delta_\theta^{-1}\bar{x}(t)C^Tz(t) - 2\bar{x}^T(t)S(t)\Delta_\theta B\varepsilon(t) \\ & + 2\bar{x}(t)S(t)\Delta_\theta[\tilde{A}(u(t),\hat{x}(t),x(t))x \\ & + \tilde{\varphi}(u(t),\hat{x}(t),x(t))]\end{aligned} \quad (38)$$

We shall now obtain an over-valuation of $|z(t)|$, according to equation (36), we have:

$$|z(t)| \leq \frac{\theta\tilde{a}}{\sqrt{\lambda_{\max}(S)}}\int_{t_k}^{t}\sqrt{V(\bar{x}(s))}ds + \frac{\sqrt{n}L_{\tilde{A}}x_M}{\sqrt{\lambda_{\max}(S)}}\int_{t_k}^{t}\sqrt{V(\bar{x}(s))}ds + \frac{\sqrt{n}L_{\tilde{\varphi}}}{\sqrt{\lambda_{\max}(S)}}\int_{t_k}^{t}\sqrt{V(\bar{x}(s))}ds \quad (39)$$

where x_M and $\tilde{a}$ were defined by (3). It follows that:

$$\|2\bar{x}(t)C^Tz(t)\| \leq 2\theta\frac{\sqrt{V(\bar{x}(t))}}{\lambda_{\max}(S)}\left[\left(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}}\right)\int_{t_k}^{t}\sqrt{V(\bar{x}(s))}ds\right] \quad (40)$$

Combing the equations (38), (25), (26), (27) and (40), one gets:

$$\begin{aligned}\dot{V}(\bar{x}(t)) \leq & -\theta V(\bar{x}(t)) \\ & + 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta}}\left[L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right]V(\bar{x}(t)) \\ & + 2\theta\frac{\sqrt{V(\bar{x}(t))}}{\lambda_{\max}(S)}\times \\ & \left[\left(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}}\right)\int_{t_k}^{t}\sqrt{V(\bar{x}(s))}ds\right] \\ & - \frac{2}{\theta^{q-1}}\sqrt{\lambda_{\max}(S)}\sqrt{V(t)}\delta_\varepsilon\end{aligned} \quad (41)$$

We can rewrite (41) as:

$$\begin{aligned}\frac{d\sqrt{V(\bar{x}(t))}}{dt} \leq & -\theta\sqrt{V(\bar{x}(t))} \\ & + \left[\frac{2\theta\left(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}}\right)}{\lambda_{\max}(S)}\int_{t_k}^{t}\sqrt{V(\bar{x}(s))}ds\right] \\ & + 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta}}\left[L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right]\sqrt{V(\bar{x}(t))} \\ & - \frac{2}{\theta^{q-1}}\sqrt{\lambda_{\max}(S)}\delta_\varepsilon \quad (42) \\ \leq & -a_\theta\sqrt{V(\bar{x}(t))} + b_\theta\int_{t_k}^{t}\sqrt{V(\bar{x}(s))}ds \\ & + \frac{2}{\theta^{q-1}}\sqrt{\lambda_{\max}(S)}\delta_\varepsilon \quad (43)\end{aligned}$$

where:

$$a_\theta = \theta - 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\lambda_{\max}(S)e}{\delta_0}}\left[L_{\tilde{A}}x_M + L_{\tilde{\varphi}}\right] \quad (44)$$

$$b_\theta = \frac{2\theta\left(\theta\tilde{a} + \sqrt{n}L_{\tilde{A}}x_M + \sqrt{n}L_{\tilde{\varphi}}\right)}{\lambda_{\max}(S)} \quad (45)$$

Applying **Lemma 1** in Farza et al. [2014b] with $a = a_\theta$ and $b = b_\theta$ gives the result. This ends the proof.

5. EXAMPLE

Consider the 1-DOF helicopter system:

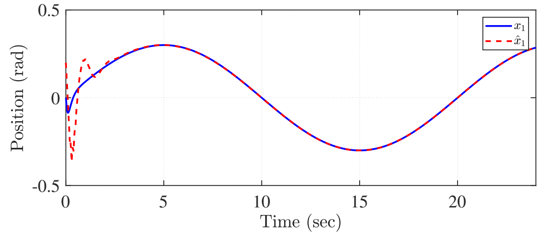
$$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{lg}{I}Fe - \frac{lF_g}{I}sin(x_1) - \frac{l\beta}{I}x_2 \\ y(t_k) = x_1(t_k)\end{cases} \quad (46)$$

where x_1 is the angular position , x_2 is the angular velocity, length of the bar $l = 0.25m$, gravity force $g = 9.81\frac{m}{s^2}$, gravity force applied on the motor $F_g = mg = 0.6210N$, inertial mass-moment $I = 0.0109N$, $Fe(t)$ is the pushing force by the helix and β is the friction coefficient, which shall be treated as time-varying parameter with unknown dynamics. Therefore, the objective consists in estimating this key parameter. Thus, the system (46) is rewritten as a state-affine nonlinear system of the form (1), as follows:

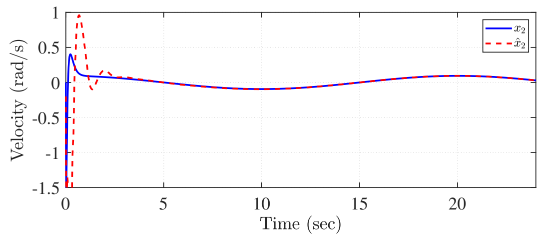
$$\dot{x} = \begin{bmatrix}0 & 1 & 0 \\ 0 & 0 & \frac{-lx_2}{I} \\ 0 & 0 & 0\end{bmatrix}x + \begin{bmatrix}0 \\ -\frac{lF_g}{I}sin(x_1) + \frac{glFe}{I} \\ 0\end{bmatrix}$$
$$y(t_k) = x_1(t_k) \quad (47)$$

there, the state is $x=[x_1\ x_2\ x_3]^T$, since, we consider x_3 as the unknown parameter β.

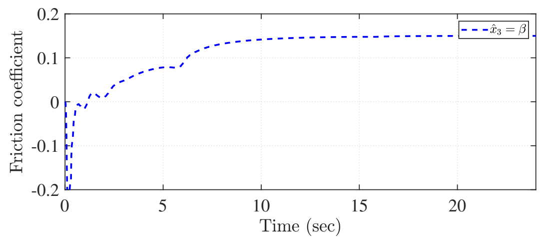
System (47) is simulated by considering a PID controller, which computes the input $Fe(t)$ and we consider the unknown uncertain term $\beta = 0.15$. Obviously, the observer ignores this value. In order to verify the validity of the observer design performance, we consider that the output is sampled at sampling instants $\Delta_M = 0.1sec$. Figure 1 illustrates the estimations of the state.



(a) Estimation state x_1



(b) Estimation state x_2



(c) Estimation state x_3

Fig. 1. Estimation of the state x: $\theta = 1.8$ and $\Delta_k = 0.1$s, where $\beta = 0.15$.

6. CONCLUSION

We consider the problem of observer design for a general class of non uniformly observable state affine nonlinear systems. A simple observer design for the continuous case based on a high-gain structure is provided. This first observer is redesigned in order to tackle the sampled output problem. The proposed algorithm is further validated on simulation for a model of 1-DOF helicopter.

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