Chaos synchronization by an observer–based active control

Jesus R. Pulido–Luna * **Jorge A. López–Rentería** ** **Nohe R. Cazarez–Castro** *

* *Departamento de Ingeniería Eléctrica y Electrónica, Tecnológico Nacional de México–Instituto Tecnológico de Tijuana, Tijuana, México. (e-mail: jesus.pulido19@tectijuana.edu.mx, nohe@ieee.org).*
** *Departamento de Ingeniería Eléctrica y Electrónica, Tecnológico Nacional de México–Instituto Tecnológico de Tijuana, CONACyT, Tijuana, México.(e-mail: jorge.lopez@tectijuana.edu.mx).*

Abstract: In this work, a control law for a master–slave synchronization scheme is proposed in order to synchronize different non–homogeneous systems with chaotic behaviour. The proposed control law is designed using the error state feedback with a Lyapunov function to guarantee stability. Also, the control law is used to synchronize two systems with different number of scrolls in their dynamics and defined in one or multiple pieces.

Keywords: Chaos, synchronization, active control, observed based, UDS.

1. INTRODUCTION

E. Lorenz designed a mathematical model to predict the apparently random behaviour of weather, (see Lorenz (1963)), even though his model was rather simplified he discovered what is defined today as a strange attractor, this attractor exhibits what we know as chaotic behaviour, which means that the solutions of the system leads to orbits that converge to a chaotic region. The main characteristics of the chaotic systems defined in Devaney (1989) are the sensitive dependence of initial conditions, transitivity and dense periodic points. This kind of chaotic behaviour has been widely studied and documented in Chua et al. (1986), Mees and Chapman (1987) and Silva (1993).

The synchronization of systems that exhibit chaotic behaviour has had a great development due to the vast amount of possible applications in science and engineering. Agiza and Yassen (2001); Yassen (2005); Mahmoud et al. (2007); Wu et al. (2007) have addressed the issue using *active control* to synchronize two different dynamical systems, this means that the dynamics of the systems is being modified by the controller in order to achieve the synchronization needed. In other hand Zhang and Feng (2007); Mkaouar and Boubaker (2012) use the same approach in piecewise linear systems and Oancea et al. (2009) use a single controller based on the error dynamics, this approach is more similar to an observer.

The main contribution of this paper is to follow the methodology for master–slave synchronization of systems defined in one piece with active control, generalize it and complement the controller with an observer–based dynamics to ensure minimum error between the master and slave systems, this will be also applied to systems defined in multiple pieces and the possible combinations, this includes the synchronization scheme between systems with multiple pieces and the synchronization scheme for systems defined in one piece with systems defined in multiple pieces.

The rest of the paper is organized as follows: In Section 2 the description of the systems is given. In Section 3 four synchronization schemes are presented: i) generalized Lorenz system–generalized Lorenz system, ii) unstable dissipative system–unstable dissipative system, iii) generalized Lorenz system–unstable dissipative system and iv) unstable dissipative system–generalized Lorenz system, in all cases a numerical simulation is presented. Finally conclusions are presented in Section 4.

2. PRELIMINARIES

This section presents in a non exhaustive way the dynamical systems that will be used in the rest of the paper.

2.1 The generalized Lorenz system

Consider the generalized Lorenz system (GLS) defined by Ĉelikovský and Chen (2002) as:

$$\begin{aligned}\dot{x} &= Ax + f(x)\\ &= \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & a_{33}\end{pmatrix} x + x_1 \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix} x,\end{aligned} \quad (1)$$

where $x = (x_1, x_2, x_3)^T$. Four typical chaotic systems can be specified from (1):

(i) Classical Lorenz system:

$$a_{12} = -a_{11} = a, \qquad a_{21} = c,$$
$$a_{22} = -1, \qquad a_{33} = -b$$

(ii) Chen system:

$$a_{12} = -a_{11} = a, \qquad a_{21} = c - a,$$
$$a_{22} = c, \qquad a_{33} = -b$$

(iii) Lü system:

$$a_{12} = -a_{11} = a, \qquad a_{21} = 0,$$
$$a_{22} = c, \qquad a_{33} = -b$$

(iv) Unified chaotic system:

$$a_{12} = -a_{11} = 25 + \alpha, \qquad a_{21} = 28 - 35\alpha,$$
$$a_{22} = 29\alpha - 1, \qquad a_{33} = -\frac{8 + \alpha}{3}$$

2.2 Unstable dissipative systems

Now consider an unstable dissipative systems (UDS) defined by Campos-Cantón et al. (2012) as:

$$\dot{\chi} = A\chi + B, \tag{2}$$

where $\chi = (\chi_1, \chi_2, \chi_3)^T$, $A = (\alpha_{ij})_{i,j=1}^n$ and B contains the switching law of the form

$$B = \begin{cases} B_1 & \text{if } x \in D_1 \\ B_2 & \text{if } x \in D_2 \\ \vdots & \quad\vdots \\ B_{k-1} & \text{if } x \in D_{k-1} \\ B_k & \text{if } x \in D_k \end{cases}$$

with $B_k = (b_{k1}, b_{k2}, b_{k3})$.

Definition 1. A system given by (2) with eigenvalues λ_i, $i = 1, 2, 3$, is said to be an UDS Type I (UDS–I), if $\sum_{i=1}^{3} \lambda_i < 0$ and one eigenvalue λ_i is negative real and the other two are complex conjugate with a positive real part, and it said to be Type II (UDS–II) if one of its eigenvalues λ_i is positive real and the other two are complex conjugate with a negative real part.

3. SYNCHRONIZATION SCHEME

In this section four synchronization schemes are presented:

i. GLS–GLS
ii. UDS–UDS
iii. GLS–UDS
iv. UDS–GLS

In each case a numerical simulation is presented in order to test this schemes.

3.1 Master GLS–slave GLS

The master system for this synchronization scheme is

$$\begin{aligned} \dot{x} &= Ax + f(x) \\ &= \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x + x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x, \end{aligned} \tag{3}$$

while the slave system is defined as

$$\begin{aligned} \dot{y} &= By + g(y) \\ &= \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} y + y_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} y + u. \end{aligned} \tag{4}$$

From the error vector $e = y - x$, we obtain

$$\begin{aligned} \dot{e} = &\begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} y - \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (y_1 y - x_1 x) + u, \end{aligned} \tag{5}$$

where $u = (u_1, u_2, u_3)^T$ is the controller. In order to stabilize the error system, the proposed Lyapunov function is

$$L = \frac{e_1^2 + e_2^2 + e_3^2}{2}, \tag{6}$$

whose derivative is

$$\begin{aligned} \dot{L} = &\, e_1(b_{11}y_1 + b_{12}y_2 - a_{11}x_1 - a_{12}x_2 + u_1) + \\ &e_2(b_{21}y_1 + b_{22}y_2 - a_{21}x_1 - a_{22}x_2 - y_1y_3 + \\ &x_1x_3 + u_2) + e_3(b_{33}y_3 - a_{33}x_3 + y_1y_2 - \\ &x_1x_2 + u_3). \end{aligned} \tag{7}$$

Let us to design the control law as

$$\begin{aligned} u = &-Pe - \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} y + \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x \\ &- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (y_1 y - x_1 x), \end{aligned} \tag{8}$$

where $e = (e_1, e_2, e_3)^T$ is the error vector, and (7) can be rewritten as

$$\dot{L} = -Pe^2, \tag{9}$$

where $P = P^T = diag\{p_1^2, p_2^2, p_3^2\} \succ 0$ is a diagonal matrix of parameters selected in order to ensure the negativeness of (9).

Using the controller designed with $A = \left(\begin{smallmatrix} -16 & 16 & 0 \\ 45.6 & -1 & 0 \\ 0 & 0 & -4 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} -16 & 16 & 0 \\ 45.6 & -1 & 0 \\ 0 & 0 & -4 \end{smallmatrix}\right)$, the initial conditions: $x_0 = (2, 3.5, 18.4)^T$,

$y_0 = (-1, 10.2, 8.3)^T$ and $P = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$, the synchronization is achieved in $t = 0.06$, the dynamics of the error between the master and the slave systems are shown in Fig. 1.

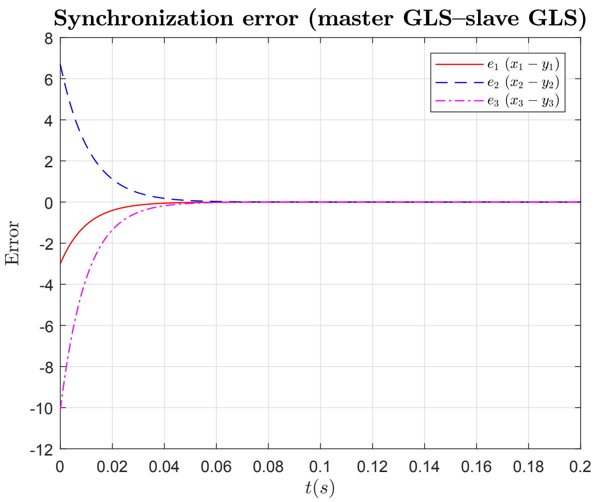

Fig. 1. Synchronization errors between master GLS and slave GLS.

3.2 Master UDS–slave UDS

In this case the master system is

$$\dot{\chi} = A\chi + B = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi + \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix}, \qquad (10)$$

while the slave system can be written as

$$\dot{\psi} = C\psi + D + u = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \psi + \begin{pmatrix} c_{k1} \\ c_{k2} \\ c_{k3} \end{pmatrix} + u. \quad (11)$$

If the error vector is defined as $e = \psi - \chi$, then

$$\begin{aligned} \dot{e} = &\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \psi - \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi \\ &+ \begin{pmatrix} c_{k1} \\ c_{k2} \\ c_{k3} \end{pmatrix} - \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix} + u, \end{aligned} \qquad (12)$$

once again, the proposed Lyapunov function in order to stabilize the error system is

$$L = \frac{e_1^2 + e_2^2 + e_3^2}{2}. \qquad (13)$$

While in this scheme the derivative results in

$$\begin{aligned} \dot{L} &= e_1(\beta_{11}\psi_1 + \beta_{12}\psi_2 + \beta_{13}\psi_3 - \alpha_{11}\chi_1 - \\ &\alpha_{12}\chi_2 - \alpha_{13}\chi_3 + c_{i1} - b_{i1} + u_1) + e_2(\beta_{21}\psi_1 \\ &+\beta_{22}\psi_2 + \beta_{23}\psi_3 - \alpha_{21}\chi_1 - \alpha_{22}\chi_2 - \alpha_{23}\chi_3 \\ &+c_{i2} - b_{i2} + u_2) + e_3(\beta_{31}\psi_1 + \beta_{32}\psi_2 + \beta_{33}\psi_3 \\ &-\alpha_{31}\chi_1 - \alpha_{32}\chi_2 - \alpha_{33}\chi_3 + c_{i3} - b_{i3} + u_3). \end{aligned} \qquad (14)$$

In this occasion the control law is defined as

$$\begin{aligned} u = -Pe - &\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \psi + \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \\ \chi - &\begin{pmatrix} c_{k1} \\ c_{k2} \\ c_{k3} \end{pmatrix} + \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix}, \end{aligned} \qquad (15)$$

where $P = P^T = diag\{p_1^2, p_2^2, p_3^2\} \succ 0$ is a diagonal matrix of parameters designed with the purpose of ensure the negativeness of (14).

Once the control law is described, using $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.15 & 10 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.15 & 10 & 1 \end{pmatrix}$, initial conditions: $\chi_0 = (0, 0, 0)^T$, $\psi_0 = (10, -1, -1)^T$, $P = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ and the switching law for both systems $B = D = \begin{cases} (0, 0, -10)^T & \text{if } x_1 < 1 \\ (0, 0, 0)^T & \text{if } x_1 \geq 1 \end{cases}$, the synchronization is achieved in $t = 0.06$ showing a fast synchronization, in Fig. 2 the error dynamics can be observed.

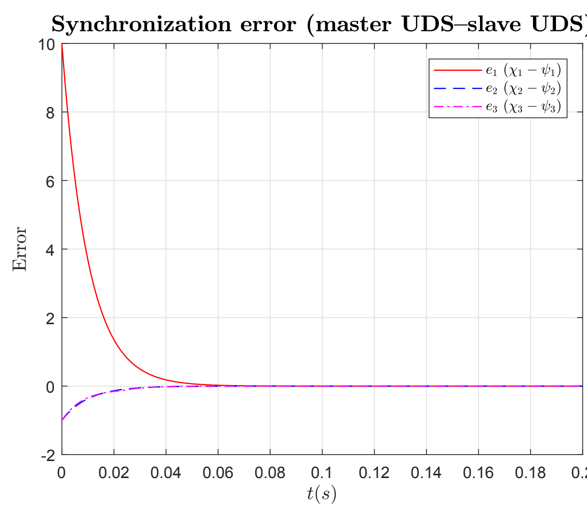

Fig. 2. Synchronization errors between master UDS–II and slave UDS–II.

3.3 Master GLS–slave UDS

For this synchronization scheme the master system proposed is

$$\begin{aligned}\dot{x} &= Ax + f(x) \\ &= \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x + x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x,\end{aligned} \tag{16}$$

with the slave system defined by

$$\dot{\chi} = A\chi + B + u = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi + \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix} + u. \tag{17}$$

Defining the error as $e = \chi - x$, we can obtain

$$\begin{aligned}\dot{e} = &\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi + \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix} - \\ &\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x - x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x + u,\end{aligned} \tag{18}$$

the proposed Lyapunov function this time is

$$L = \frac{e_1^2 + e_2^2 + e_3^3}{2}, \tag{19}$$

with derivative

$$\begin{aligned}\dot{L} = \; & e_1(\alpha_{11}\chi_1 + \alpha_{12}\chi_2 + \alpha_{13}\alpha_3 + b_{i1} - a_{11}x_1 \\ & -a_{12}x_2 + u_1) + e_2(\alpha_{21}\chi_1 + \alpha_{22}\chi_2 + \alpha_{23}\chi_3 \\ & -a_{21}x_1 - a_{22}x_2 + x_1x_3 + u_2) + e_3(\alpha_{31}\chi_1 \\ & +\alpha_{32}\chi_2 + \alpha_{33}\chi_3 - a_{33}x_3 - x_1x_2 + u_3).\end{aligned} \tag{20}$$

The design of the proposed control law is

$$\begin{aligned}u = -Pe - &\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi - \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix} \\ + &\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x + x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x.\end{aligned} \tag{21}$$

Similar to previous schemes, using the controller previously designed this control law with $A = \begin{pmatrix} -16 & 16 & 0 \\ 45.6 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.15 & 10 & 1 \end{pmatrix}$, initial conditions: $x_0 = (2, 3.5, 18.4)^T$, $\chi_0 = (1.1, -1, -1)^T$, $P = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix}$ and the switching law from Section 3.2. In this case, in order to achieve synchronization, is necesary to design $p_i > 100$. The error dynamics of this scheme can bee seen in Fig. 3. It is important to highlight that the UDS–II system has only one scroll and it is described by multiple pieces, while the Lorenz system has two scrolls in one piece, in other words, a slave system with only one scroll adopt the dynamics of the master system with two scrolls, this can be appreciated in Fig. 4, also, the initial conditions on the slave system does not affect the synchronization,

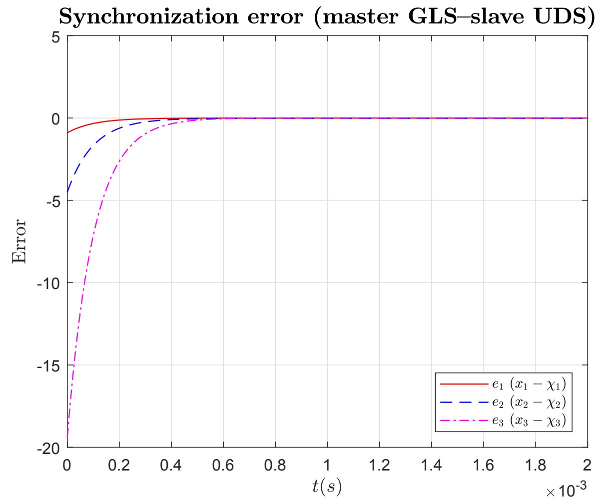

Fig. 3. Synchronization errors between master GLS and slave UDS–II.

in Fig. 5 is posible to appreciate the fact that two slave systems with two different initial conditions synchronize to the same master system, Fig. 5(b) shows a UDS system with initial conditions $\chi_0 = (1.1, -1, -1)^T$, Fig. 5(c) show the same UDS system with initial conditions $\chi_0 = (10, -1, -1)^T$ and Fig. 5(d) and Fig. 5(e) show both systems synchronized with the master GLS.

3.4 Master UDS–slave GLS

Finally, the master UDS is:

$$\dot{\chi} = A\chi + B = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi + \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix}, \tag{22}$$

and similar, to other schemes the slave is

$$\begin{aligned}\dot{x} &= Ax + f(x) + u \\ &= \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x + x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x + u.\end{aligned} \tag{23}$$

Considering the error vector as $e = x - \chi$, then:

$$\begin{aligned}\dot{e} = &\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x + x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x \\ - &\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi - \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix} + u,\end{aligned} \tag{24}$$

similarly to previous schemes, a Lyapunov function

$$L = \frac{e_1^2 + e_2^2 + e_3^2}{2}, \tag{25}$$

is also proposed and its derivative is

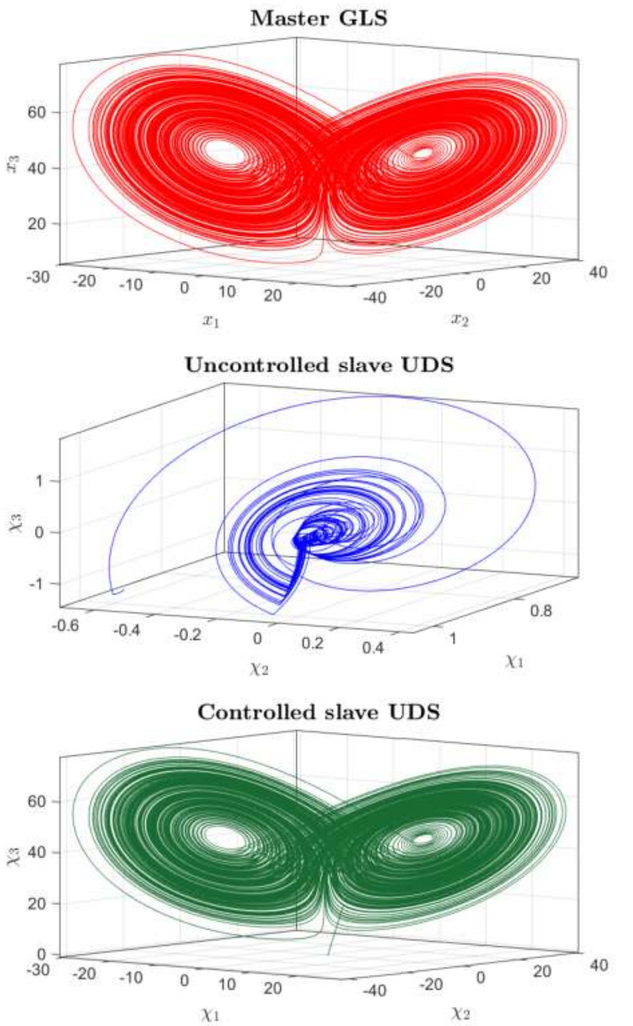

Fig. 4. (From top to bottom) Master GLS, Uncontrolled slave UDS and Controlled slave UDS.

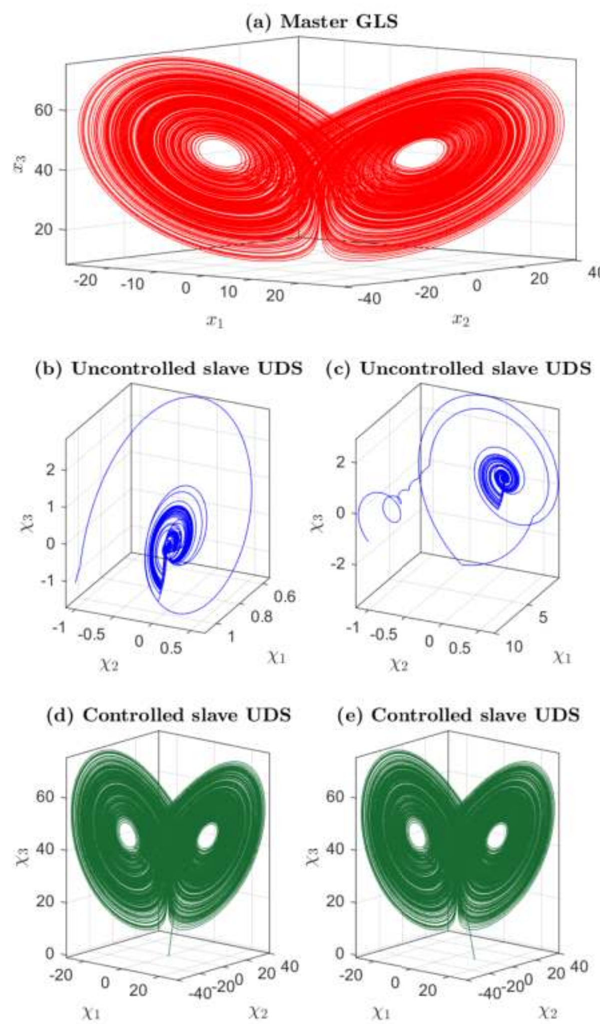

Fig. 5. Two slave systems with different initial conditions synchronized with the same master system.

$$\begin{aligned}\dot{L} = {} & e_1(a_{11}x_1 + a_{12}x_2 - \alpha_{11}\chi_1 - \alpha_{12}\chi_2 - \alpha_{13}\chi_3 \\ & -b_{i1} + u_1) + e_2(a_{21}x_1 + a_{22}x_2 - x_1x_3 \\ & -\alpha_{21}\chi_1 - \alpha_{22}\chi_2 - \alpha_{23}\chi_3 - b_{i2} + u_2) \\ & +e_3(a_{33}x_3 + x_1x_2 - \alpha_{31}\chi_1 - \alpha_{32}\chi_2\alpha_{33}\chi_3 \\ & -b_{i3} + u_3),\end{aligned} \tag{26}$$

however, in this scheme the control law is modified to be

$$\begin{aligned}u = {} & -Pe - \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x - x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x \\ & + \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \chi + \begin{pmatrix} b_{k1} \\ b_{k2} \\ b_{k3} \end{pmatrix},\end{aligned} \tag{27}$$

recalling that $P = P^T = diag\{p_1^2, p_2^2, p_3^2\} \succ 0$ is the diagonal matrix of values, designed to ensure the stability of (24).

Finally using controller designed with $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.15 & 10 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -16 & 16 & 0 \\ 45.6 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$, initial conditions: $\chi_0 = (10, -1, -1)^T$, $x_0 = (2, 3.5, 18.4)^T$, $P = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix}$ and the same switching law from section 3.2. Similarly to the previously case, the synchronization is achieved for $p_i > 100$ as can be observed in Fig. 6. Is important to highlight the fact that the GLS has two scrolls and its described in one piece, while the UDS–II has one scroll described in multiple pieces, in other words, the slave system with two

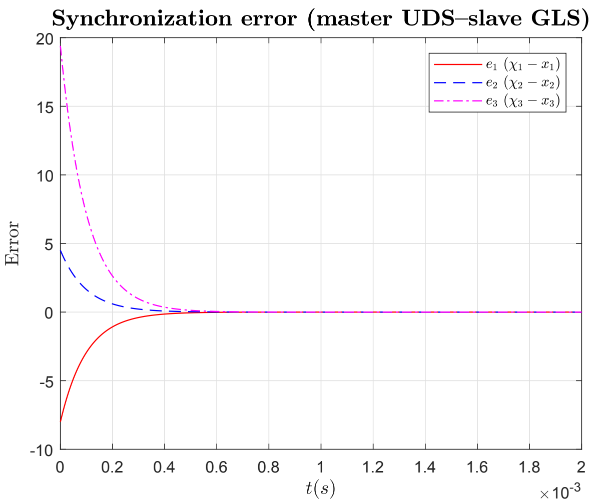

Fig. 6. Synchronization errors between master UDS–II and slave GLS.

scrolls adopt the dynamic of the master system with one scroll.

4. CONCLUSION

In this paper a control law to synchronize two different systems with chaotic behaviour was designed, ensuring Lyapunov stability. These control laws provide a fast synchronization with a minimum error as long as the parameters p_i are adequate. In other words, the control law provides the dynamic of the master system to the slave system with the objective of synchronization.

The controls designed can be used in systems described in one or multiple pieces, and all the combinations (i.e. multiple pieces master–one piece slave), another characteristic of the provided control laws are the robustness to changes in the initial conditions of both master and slave systems, making the synchronization scheme suitable in a large number of applications in sciences and engineering,

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