Anticontrol of linear systems: New chaotic attractors ⋆

Juan Gonzalo, Barajas-Ramírez, *

* *Instituto Potosino de Investigación Científica y Tecnológica A. C., División de Matemáticas Aplicadas, Camino a la Presa San José 2055, Col. Lomas 4a Secc. C.P. 78216, San Luis Potosí, S.L.P., México. (e–mail: jgbarajas@ipicyt.edu.mx).*

Abstract: We propose a design methodology to construct piecewise linear systems with attractors of different number and distribution of scrolls. We investigate the underlying chaos generating mechanisms of benchmark chaotic systems, then we propose variants based in these benchmark structures that result in new alternative attractors. Unlike previous methods our method is not based on adding equilibrium points on the system, instead we manipulate the eigenvalues and eigenspaces of the existing linear subsystems of the systems to produce or inhibit the formation of scrolls. To illustrate the effectiveness of our proposed method we show new attractors with different symmetries and number of scrolls based on some well-known chaotic systems.

Keywords: Anticontrol, linear systems, chaotic attractors.

1. INTRODUCTION

In recent years, the scientific community has placed increasing attention on the design of simple circuits and systems with chaotic behavior Chen and Ueta (2002); Campos-Cantón *et al* (2012). Different authors have shown that piecewise linear (PWL) systems, *i.e.*, linear systems with a PWL function in their mathematical description, can generate chaotic behaviors Sprott (2000); Lü *et al.* (2000); Campos *et al.* (2010). One of the simplest chaotic system proposed in the literature is Chua's circuit Chua *et al.* (1993). A particularly interesting interpretation of Chua's circuit is as a PWL system, that is, as three linear time-invariant (LTI) systems continuously connected through a PWL function, which is usually called the nonlinear resistor, or Chua's diode Chen and Dong (1998). Based on this interpretation different forms of alternative attractors can be generated. In Suykens *et al.* (1991) a family of systems with multiples of the double scroll attractor of Chua's circuit was proposed. Multispiral chaotic attractors using a PWL function were investigated in Aziz-Alaoui (1999). Alternative methods allow for the generation of n-scroll hyperchaotic attractors Cafagna and Grassi (2003), and even attractors with scrolls ordered on directed grids of one, two, and three dimensions Yalçin et al. (2005). A review of alternatives approaches can be found in Lü and Chen (2006), while recently in Díaz-Gonzĺez *et al* (2017, 2016); Anzo-Hernández *et al* (2018) alternative construction of PWL chaotic systems have been proposed.

In most of the works referred above, the approach basically consists on adding breakpoints into the original PWL function of the system, effectively adding equilibrium points around which oscillations occur. In Elhadj and Sprott (2010) the authors propose a two-variable version to the Chua's diode as a way to obtain three scrolls from a single Chua's circuit without increasing the number of equilibrium points and show numerically that other variants are possible. However, no general method is presented for arbitrary PWL systems.

In this contribution, we propose a methodology to generate attractors with different distributions and number of scrolls without adding equilibrium points to the original system. To this end, we define a modulating parameter approach such that no additional breakpoints are introduced in the original PWL function of the system, instead the geometry of the eigenspaces of its linear subsystems is manipulated such that oscillations around the existing equilibrium points are generated or inhibited.

2. PIECEWISE LINEAR CHAOTIC SYSTEMS

Consider a controlled linear dynamical system

$$\dot{x}(t) = Ax(t) + u_f(x(t)) \qquad (1)$$

where $x(t) = [x_1(t), x_2(t), x_3(t)]^\top \in \mathbf{R}^3$ is the state variable of the system; $A = \{a_{ij}\} \in \mathbf{R}^{3\times 3}$ is a constant matrix; and $u_f(x(t)) : \mathbf{R}^3 \to \mathbf{R}^3$ is a piecewise continuous controller. We propose that the controller be given by

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$$u_f(x(t)) = \begin{cases} M_1 x(t) + B_1, & \text{if } x(t) \in H_1 \\ M_2 x(t) + B_2, & \text{if } x(t) \in H_2 \\ ... & ... \\ M_N x(t) + B_N, & \text{if } x(t) \in H_N \end{cases} \quad (2)$$

with $M_i \in \mathbf{R}^{3\times3}$ and $B_i \in \mathbf{R}^3$ $(i = 1, 2, ..., N)$. The PWL controller is a function defined in terms of state space partitions H_i, such that

$$\bigcup_{i=1}^{N} H_i = \mathbf{R}^3, \text{ and } \bigcap_{i=1}^{N} H_i = \emptyset. \quad (3)$$

Notwithstanding the simplicity of (1)-(2), for a wide range of parameter values the system produces complex trajectories. In fact, letting the switching condition be dependent on a single variable it can express different benchmark chaotic systems. For example, the well-known chaotic system $\dddot{y}(t) + 0.6\ddot{y}(t) + \dot{y}(t) = |y(t)| - 2$ becomes Sprott (2000):

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{31k} & -1 & -0.6 \end{pmatrix} x(t) + u_{f2}(x(t)) \quad (4)$$

$$u_{f2}(t) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, & \text{if } x_1(t) \geq 0 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, & \text{if } x_1(t) < 0 \end{cases} \quad (5)$$

The equilibrium points are $\bar{x}^k = -A_k^{-1} B_k$ ($\bar{x}^1 = [2, 0, 0]^\top$ and $\bar{x}^2 = [-2, 0, 0]^\top$). The eigenvalues associated to each equilibrium point are $\rho(A_1) = \{\gamma_1, \sigma_1 \pm \omega_1\} = \{0.5885, -0.5942 \pm 1.1603i\}$ and $\rho(A_2) = \{\gamma_2, \sigma_2 \pm \omega_2\} = \{-0.8356, 0.1178 \pm 1.0876i\}$. In both cases, $\gamma_i \sigma_i < 0$ and $\omega_i \neq 0$. As such, both equilibriums are focus-saddle, therefore candidates to apply the Shilnikov method to establish the existence of chaos Silva (1993).

An important observation is that around each equilibrium point there are stable and unstable eigenspaces: On the positive side, an unstable eigenline $E_u^1 = \{x \in \mathbf{R}^3 | [2, 0, 0]^\top + t[-0.8259, -0.4860, -0.2860]^\top$, with $t \in \mathbf{R}\}$ associated to $\gamma_1 = 0.5885$, and a stable eigenplane $E_s^1 = \{x \in \mathbf{R}^3 | [2, 0, 0]^\top + t[-0.8259, -0.4860, -0.2860]^\top + s[-0.8259, -0.4860, -0.2860]$, with $t, s \in \mathbf{R}\}$ associated to $\sigma_1 \pm \omega_1 = -0.5942 \pm 1.1603i$. On the negative side, an unstable eigenplane $E_u^2 = \{x \in \mathbf{R}^3 | [-2, 0, 0]^\top + t[0.5128, -0.0618, -0.6282]^\top + s[0.1124, 0.5709, 0]^\top$, with $t, s \in \mathbf{R}\}$ associated to $\sigma_2 \pm \omega_2 = 0.1178 \pm 1.0876i$ and a stable eigenline $E_s^2 = \{x \in \mathbf{R}^3 | [-2, 0, 0]^\top + t[-0.6764, 0.5652, -0.4722]^\top$, with $t \in \mathbf{R}\}$ associated to $\gamma_2 = -0.8356$.

The strongest restriction to apply Shilnikov's method is the identification of homoclinic orbits or heteroclinic loops Silva (1993). Which is a geometric condition for the existence of a trajectory, with both positive and negative limits at the same equilibrium point (homoclinic) or located at one equilibrium point in one direction and at the other in the opposite direction (heteroclinic). From Figure 1a, one can consider that moving along the unstable plane E_u^2, then switching near the stable line E_u^1 and returning near E_u^2 a homoclinic orbit can be generated for $\bar{x}^2$. The exact description of the homoclinic orbit is a far more complex problem as shown in Tigan and Llibre (2016). Assuming the existence of such homoclinic orbit. Since the eigenvalue inequality of Shilnikov is satisfied $|\gamma_2| > |\sigma_2|$ $(0.8356 > 0.1178)$ for $\bar{x}^2$, then homoclinic chaos can be argued for the system around this equilibrium. However, since the identification of homoclinic/heteroclinic orbits is a prohibitively complex problem.

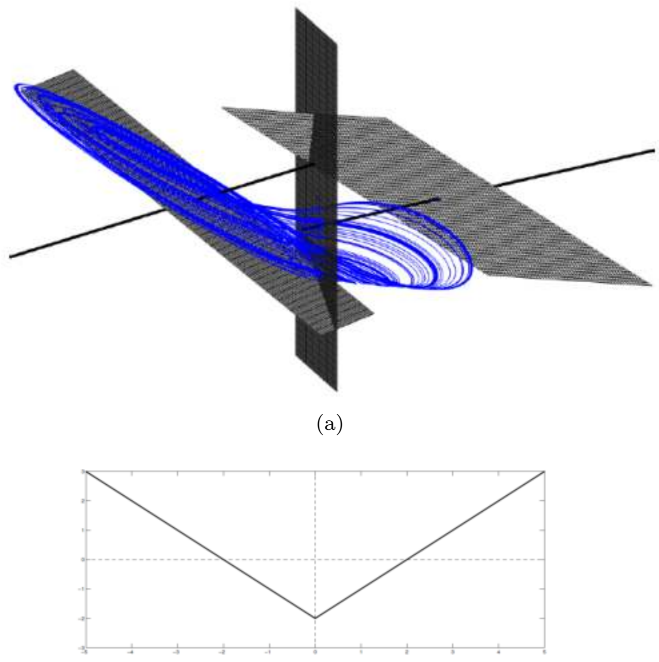
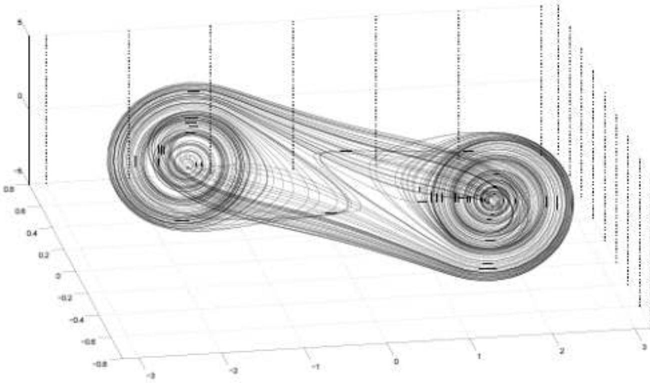


Fig. 1. (a) The Sprott system attractor along with its stable and unstable eigenspaces, switching surface, and (b) its controller function $u_{f2}(x(t))$.

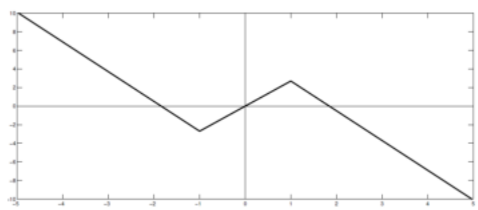
The graphical representation of the Sproot system is shown in Figure 1b. By analogy between these figures the positive slope of $u_{f2}(x(t))$ gives a scroll while the negative slope returns the trajectory to the negative side. This is a very simple and direct way of capturing the eigenspace geometry of the PWL system and the resulting chaotic attractor.

Another example is Chua's circuit Chua *et al.* (1993) $(\dot{x}_1(t) = -10x_1(t) + 10x_2(t) - 10G(x_1(t)); \dot{x}_2(t) = x_1(t) - x_2(t) + x_3(t); \dot{x}_3(t) = -14x_2(t)$, with $G(x_1(t)) = -0.6x_1(t) - 0.3(|x_1(t) + 1| - |x_1(t) - 1|))$, which can be written as a PWL system of the form:

$$\dot{x}(t) = \begin{cases} \bar{A}_1 x(t) + B_1, & \text{if } x_1(t) > 1 \\ \bar{A}_2 x(t) + B_2, & \text{if } |x_1(t)| \leq 1 \\ \bar{A}_3 x(t) + B_3, & \text{if } x_1(t) < -1 \end{cases} \quad (6)$$



(a)



(b)

Fig. 2. (a) The Chua's circuit attractor, and (b) its PWL controller function.

where $\bar{A}_k = \begin{pmatrix} \bar{a}_{11k} & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14 & 0 \end{pmatrix}$, $B_k = \begin{pmatrix} b_{1k} \\ 0 \\ 0 \end{pmatrix}$ with $\bar{a}_{111} = \bar{a}_{113} = -4.0$, $\bar{a}_{112} = 2$, $b_{11} = 6.0$, $b_{12} = 0$, and $b_{13} = -6.0$. In this case there are three equilibrium points $\bar{x}^1 = [1.5, 0, -1.5]^\top$, $\bar{x}^2 = [0, 0, 0]^\top$ and $\bar{x}^3 = [-1.5, 0, 1.5]^\top$. With the eigenvalues $\rho(\bar{A}_1) = \rho(\bar{A}_3) = \{\gamma_1, \sigma_1 \pm \omega_1\} = \{\gamma_3, \sigma_3 \pm \omega_3\} = \{-5.4272, 0.2136 \pm 3.2051i\}$ and $\rho(A_2) = \{\gamma_2, \sigma_2 \pm \omega_2\} = \{-3.1643, -1.0822 \pm 2.7708i\}$, that is, all are focus-saddle equilibriums.

In a similar way as above, the properties of the eigenspaces can be reinterpret in terms of the corresponding PWL controller function. For illustration purposes, in Figure 2a we show the chaotic attractor of Chua's circuit, while the corresponding PWL controller is shown in Figure 2b.

In the PWL version of the Chua's circuit in the form of (1) we have $H_1 = \{x(t) \in \mathbf{R}^3 | x_1(t) < 1\}$, $H_2 = \{x(t) \in \mathbf{R}^3 || x_1(t)| \leq 1\}$ and $H_2 = \{x(t) \in \mathbf{R}^3 | x_1(t) < -1\}$ with two switching surfaces at $\Sigma_1 = \{x(t) \in \mathbf{R}^3 | x_1(t) = 1\}$ and $\Sigma_{-1} = \{x(t) \in \mathbf{R}^3 | x_1(t) = -1\}$. Then

$$\dot{x}(t) = Ax(t) + u_f(x(t))$$

with $A = \begin{pmatrix} 0 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14 & 0 \end{pmatrix}$ and

$$u_f(x(t)) = \begin{cases} [-4x_1(t) + 6, 0, 0]^\top, & \text{if } x_1(t) > 1 \\ [2x_1(t), 0, 0]^\top, & \text{if } |x_1(t)| \leq 1 \\ [-4x_1(t) - 6.0]^\top, & \text{if } x_1(t) < -1 \end{cases}$$

Considering $u_f(x(t))$ as the controller of the attractor structure a positive slope means no scroll, while a negative slope generated scrolls. From this observation in what follows we propose a design methodology to generate new chaotic attractors where scrolls are added or remove from the attractors of PWL systems.

3. DESIGNING PWL CONTROLLERS TO GENERATE NEW CHAOTIC ATTRACTORS

Inspired by the observations on Sprott system. We start by proposing to generate a new attractor for this system with two scrolls. We propose modulating the PWL controller function in the following manner:

$$u_{f2}(t) = \begin{cases} \epsilon_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, & \text{if } x_1(t) \geq 0 \\ \epsilon_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, & \text{if } x_1(t) < 0 \end{cases} \quad (7)$$

Choosing different combinations of ϵ_1 and ϵ_2 new attractors are generated. The PWL function of the Sprott system generates scrolls for negative slopes, then by simply choosing $\epsilon_1 = -1$ and $\epsilon_2 = 1$, the two-scroll attractor shown in Figure 3a is obtained. This is achieved by adding a scroll oscillation around the equilibrium point $\bar{x}^1$ where the eigenvalues associated to $\bar{x}^1$ become $\rho(A_1) = \{-0.8356, 0.1178 \pm 1.0876i\}$. Other variants are easily realized by the following combinations: $\epsilon_1 = -1$, $\epsilon_2 = -1$, this flips the conditions of the original PWL controller function resulting on a single scroll attractor for the Sprott system, but now around the opposite equilibrium point (Figure 3c). Playing with the parameters ϵ_1 and ϵ_2 the width of the scroll can be altered, for $\epsilon_1 = -0.75$ and $\epsilon_2 = -1$ the single scroll attractor shown in Figure 3e is generated.

A similar method can be use to generate new tree-scroll attractors. By taking inspiration from Chua's circuit, we propose using the PWL controller function

$$u_f(x(t)) = \begin{cases} [-4\epsilon_1 x_1(t) + 6, 0, 0]^\top, & \text{if } x_1(t) > 1 \\ [2\epsilon_2 x_1(t), 0, 0]^\top, & \text{if } |x_1(t)| \leq 1 \\ [-4\epsilon_3 x_1(t) - 6.0]^\top, & \text{if } x_1(t) < -1 \end{cases} \quad (8)$$

and tuning the modulation parameters ϵ_i one can generate new attractors with one, two or three scrolls. Choosing $\epsilon_1 = 1$, $\epsilon_2 = -0.75$, $\epsilon_3 = 1$ a three scroll attractor is generated (see Figure 4a). In this case, the PWL function is changed to having all three negative slopes, the additional scroll is generated around the equilibrium point $\bar{x}^2 = [0, 0, 0]^\top$ where the associated eigenvalues become $\rho(A_2) = \{-3.0, 0.25 \pm 2.6339i\}$ which satisfy the Shilnikov condition. With $\epsilon_1 = 0.01$, $\epsilon_2 = -0.75$, and $\epsilon_3 = 0.75$ an alternative two scroll attractor is generated (see Figure 4c). Notice that by changing the slope of the PWL function to almost zero the scroll is eliminated. Finally, a single scroll attractor for a tree domains system is generated around the equilibrium point $\bar{x}^2$ if the

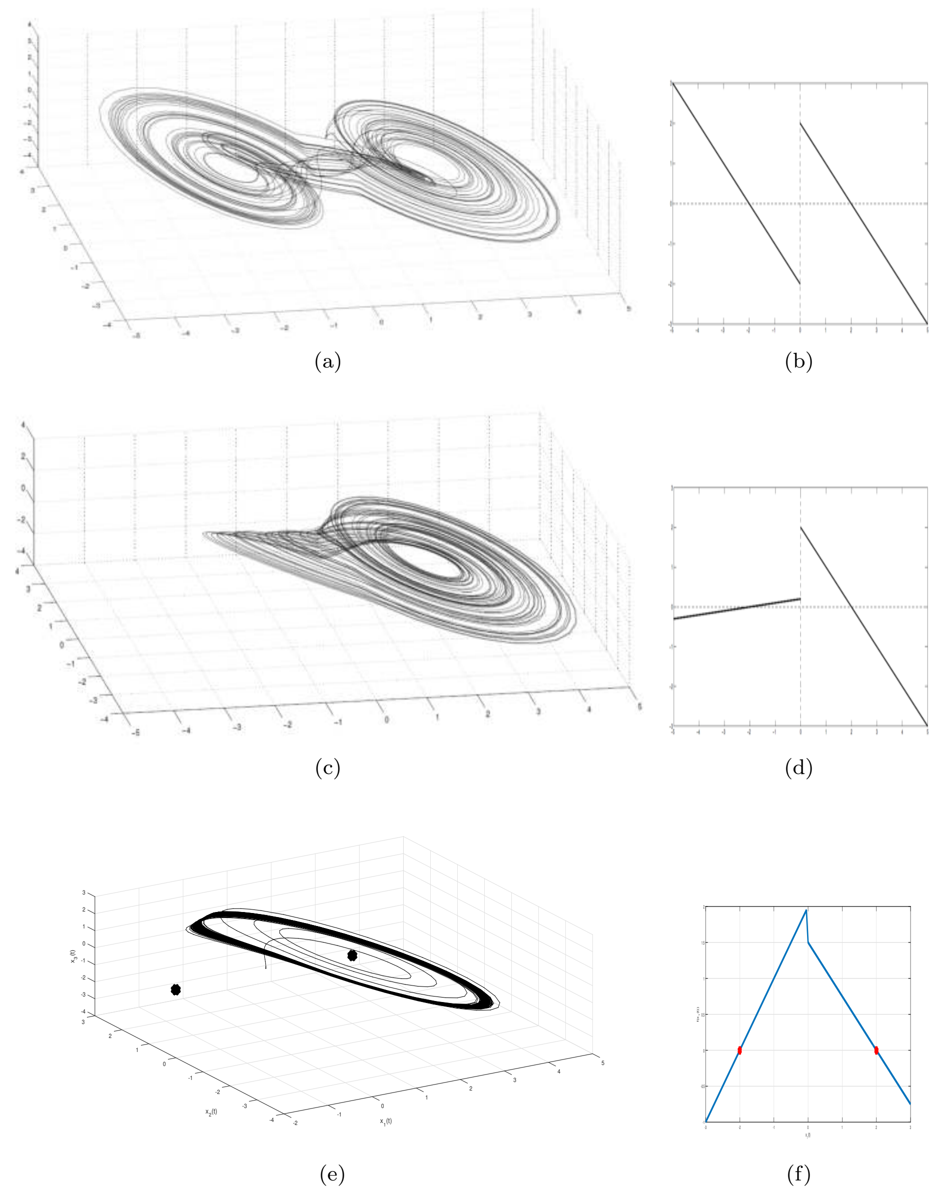


Fig. 3. New attractors of two domains: (a) Double scroll attractor, and (b) its PWL controller functions ($\epsilon_1 = -1$ $\epsilon_2 = 1$) (c) Single scroll attractor, and (d) its PWL controller functions ($\epsilon_1 = -1$, $\epsilon_2 = -1$) (e) Single scroll attractor, and (f) its PWL controller functions ($\epsilon_1 = -0.75$, $\epsilon_2 = -1$).

modulation parameters are set to $\epsilon_1 = 0.01$, $\epsilon_2 = -0.75$ and $\epsilon_3 = 0.01$ as shown in Figure 4e.

As illustrated in Figures 3 and 4, using the proposed method new attractors can be found in terms of the parameters ϵ_i, by modifying the stability properties of the trajectories around the system's equilibrium points. However, these new PWL controller functions do not introduce new equilibrium points. In fact, to a large extend, the original chaotic systems retain their structure. Therefore, implementations of these new chaotic systems can easily be realized electronically.

4. CONCLUSIONS

This contribution we proposed a method to generate alternative chaotic attractors for PWL systems with two and three domains. Our method takes inspiration from two well-known chaotic systems, Sproot and Chua's circuit. Our proposal consists in modulating a parameter make the slopes of their corresponding PWL functions to have negative slope, which modifies the stability properties of the linear subsystems and allows for the formation of scrolls. As a result scrolls can be generated or inhibited with in the original chaotic structure.

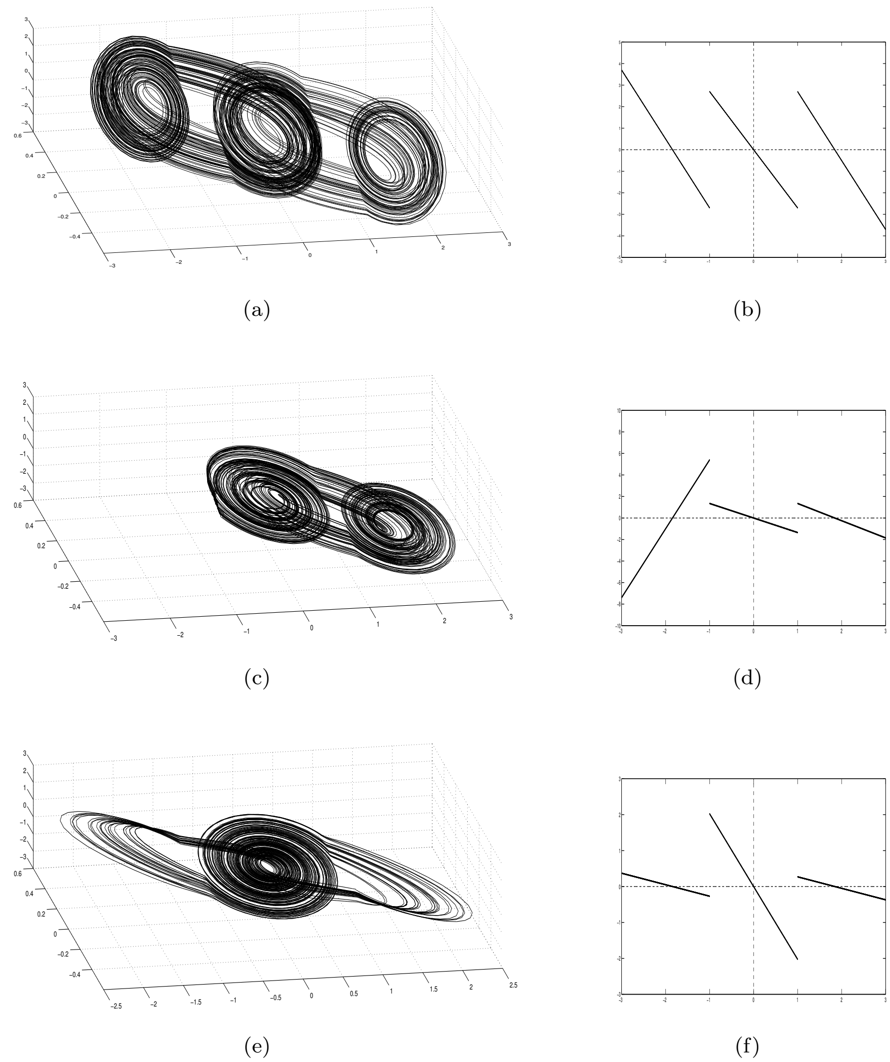


Fig. 4. New attractors of three domains: (a) Three scroll attractor, (b) its PWL controller function with $\epsilon_1 = 1$, $\epsilon_2 = -0.75$, $\epsilon_3 = 1$; (c) Two scroll attractor, (d) its PWL controller function with $\epsilon_1 = 0.01$, $\epsilon_2 = -0.75$, $\epsilon_3 = 0.75$, (e) Single scroll attractor, (f) its PWL controller function with $\epsilon_1 = 0.01$, $\epsilon_2 = -0.75$, $\epsilon_3 = 0.01$.

The approach presented in this paper is simple and viable for different families of PWL systems. Although here we are focused on generating new attractors in the same original system, the proposed approach can easily be generalized to multi-scroll chaotic attractor of composed PWL systems.

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