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Robust Error Feedback Sliding Mode Regulation of Nonlinear Systems ⋆

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Abstract: In this paper, the problem of nonlinear sliding mode (SM) regulation is addressed for nonlinear affine control system subject to unmodelled disturbance. In particular, the error feedback SM regulator problem is defined, taking the concepts related to the zero output tracking submanifold as a starting point. Applying the internal model concept to the time-invariant SM equation, the solvability conditions to the problem are derived. A proportional-integral (PI) nonlinear observer is proposed, and using the observer state, a sliding manifold on which the tracking error is ultimately bounded, is formulated. A SM control algorithm is proposed to ensure the designed manifold to be attractive, achieving robustness with respect to allowed uncertainties. The effectiveness of the proposed method is demonstrated by the application to the Pendubot system.

Keywords: Sliding Mode Control, Error Feedback Regulation, Proportional-Integral Observer, Nonlinear perturbed systems, Output tracking.

1. INTRODUCTION

The regulation problem for nonlinear systems is defined as designing a feedback law in order to asymptotically track a reference signal while rejecting disturbances, both provided by an external system, also called exosystem. In the classical setting, the Francis-Isidori-Byrnes equation is used, along with the internal model principle, which generates the feedback control law that is capable of producing the desired steady state behaviour, Isidori (1955). However, in practice, control plants are affected additionally by unmodelled perturbations. As results, the corresponded nonlinear regulator equation cannot be solved since the unmodelled perturbation is usually unknown. An alternative approach for dealing with this problem is to combine the output regulation theory with the SM control technique, Utkin et al. (2009), which allows decomposing and simplifying the regulator design procedure and imposing robustness properties with respect to matched perturbation, Draženović (1969), El-Ghezawi et al. (2007). The output regulation problem solution via the SM technique has been broadly studied in the last two decades by several authors (see, among others, Jeong and Utkin (1999), Elmali and Olgac (1992), DZ et al. (2001), Zheng and Zhong (2013), Govindaswamy et al. (2014)) mainly for *minimum phase* systems. Few works were addressed to *non-minimum phase* systems, however, just for the case of linear systems (Jeong and Utkin (1999), Utkin and Utkin (2014)), and for the particular case of nonlinear systems with unitary relative degree, Bonivento et al. (2001). By combining SM technique with regulation theory, in Loukianov et al. (2018), a robust regulator is proposed that is able to compensate a matched time-varying perturbation, and using the equivalent control technique Utkin et al. (2009), the autonomous nonlinear regulator equation can be used. It is, however, a state feedback regulator, and an error feedback regulator is left out of this work. In Bonivento et al. (2001), an error feedback SM regulator is addressed but only for systems with unitary relative degree.

In this paper, an Error Feedback SM Regulator problem for nonlinear systems subject to matched unmodelled perturbations is formulated and solution existence conditions are derived using a Regulator Equation that corresponds to the SM equation. A solution of the Regulator Equation is used to define a control error and a local center manifold on which the output error is zeroed. To estimate the unmeasured control error and exosystem state, a nonlinear Proportional-Integral (PI) observer, Beale and Shafai (1988), is designed. Based on the obtained control error estimation, a sliding manifold is formulated, and a reaching control law is proposed. It is shown that a solution of the closed-loop system in the case of nonvanishing perturbation is ultimately bounded and the tracking error is driven to a small bound.

The rest of this work is organized as follows. In Section 2, the error feedback SM regulation problem is formulated

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and conditions for the solution are presented. In Section 3, a nonlinear PI observer, sliding manifold and SM control algorithm are designed and the the closed-loop system solution stability is analyzed. In Section 4, the problem is revisited for a class of nonlinear systems presented in Regular form. To show the effectiveness of the proposed method, the SM regulator is designed for Pendubot dynamical system in Section 5. Finally, Section 6 concludes this work.

2. PROBLEM STATEMENT

Consider a nonlinear system affine in control subject to perturbations

$$\begin{aligned} \dot{x} &= f(x) + B(x)(u + \delta(x,t)) + D(x)w \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in X \subset \Re^n$ is the state of the system, $u \in \Re^m$ is the control input, w represents modelled perturbations, $y \in \Re^p$ represents the output, and $\delta(x,t)$ is a $m-$dimensional vector of bounded unmodelled perturbations.

For an error feedback regulator with the objective of achieving the tracking error

$$e = h(x) - q(w) \tag{2}$$

be equal to zero, an exosystem that generates the desired reference signal $q(w)$ is defined as

$$\dot{w} = s(w) \tag{3}$$

where $w \in W \subset \Re^q$.

Let us define $A_0 = \left[\dfrac{\partial f(x)}{\partial x}\right]_{(0)}$, $B_0 = B(0)$, and $C_0 = \left[\dfrac{\partial h(x)}{\partial x}\right]_{(0)}$ to introduce the following assumptions:

- **A1**. The pair A_0, B_0 is stabilizable.
- **A2**. The matrix $S = \left[\dfrac{\partial s(w)}{\partial w}\right]_{(0)}$ has all its eigenvalues on the imaginary axis.

The control error or local center manifold (Isidori (1955)) is consequently introduced as

$$\varepsilon(x,w) = 0,\ \varepsilon = x - \pi(w). \tag{4}$$

The Error Feedback SM Regulation problem can be defined as finding a dynamic discontinuous controller

$$\dot{\xi} = \eta(\xi, u, e) \tag{5}$$

$$u = \begin{cases} u^+(\xi) & \text{if } \sigma(\xi) > 0 \\ u^-(\xi) & \text{if } \sigma(\xi) < 0 \end{cases} \tag{6}$$

with $\xi \in \Xi \subset \Re^n$, and the sliding manifold

$$\sigma(\xi) = 0, \qquad \sigma = [\sigma_1, \ldots, \sigma_m)]^T \tag{7}$$

such that the following conditions are satisfied :

- **C1**. Finite-time convergence of the closed-loop system states to the sliding manifold $\sigma(\xi) = 0$.
- **C2**. Asymptotic stability of the SM dynamics in the absence of the perturbation.
- **C3**.
  - **A.** For the case $\|\delta(\varepsilon,t)\| \le \gamma_1, \gamma_1 > 0$, there is $\gamma_2 > 0$ and a neighborhood $\Omega \subset X \times W \times \Xi$ of the origin, such that for each initial condition $(x_0, w_0, \xi_0) \in \Omega$ the tracking error satisfies $$\|e(t)\| \le \gamma_2, \quad \forall\, t \ge T_0.$$
  - **B.** For the case $\|\delta(\varepsilon,t)\| \le \gamma_3\|\varepsilon\|$, $\gamma_3 > 0$, there is a neighborhood $\Omega \subset X \times W \times \Xi$ of the origin, such that for each initial condition $(x_0, w_0, \xi_0) \in \Omega$ the tracking error satisfies $$\lim_{t\to\infty} e(t) = 0.$$

Here, $\delta(x,t) = \delta(\varepsilon,t)$ for convenience. In the *classical* setup, in absence of the perturbation, that is, $\delta(x,t) = 0$, it has been shown that the solvability of the Regulator problem can be stated in terms of the existence of a pair of mappings $x = \pi(w)$ and $u = c(w)$ with $\pi(0) = 0$ and $c(0) = 0$ which solve the following Regulator Equation:

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} s(w) &= f(w) + B(w)c(w) + D(w)w \\ 0 &= h(w) - q(w). \end{aligned}$$

In the presence of $\delta(x,t)$, the corresponding Regulator Equation

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} s(w) &= f(w) + B(w)(c(w) + \delta(w,t)) + D(w)w) \\ 0 &= h(w) - q(w) \end{aligned}$$

is impossible to solve since $\delta(w,t)$ is unknown and unmodelled.

3. SOLUTION OF ERROR FEEDBACK SLIDING MODE REGULATION PROBLEM

In this section, to analyze the behaviour of the real control error ε and derive the solution existence condition, the SM dynamics will be derived, first, for ε instead of its estimate $\hat{\varepsilon}$ which will be used in the control design. Then, the obtained SM dynamics will be modifyed using the SM error feedback controller (5) - (7).

3.1 Control error dynamics

Using (4), (1) and (3), the following control error dynamics are obtained:

$$\begin{aligned} \dot{\varepsilon} &= f(\varepsilon + \pi(w)) + B(\varepsilon + \pi(w))(u + \delta(\varepsilon + \pi(w), t)) \\ &\quad + D(\varepsilon + \pi(w))w - \frac{\partial \pi(w)}{\partial w} s(w) \\ \dot{w} &= s(w) \\ e &= h(\varepsilon + \pi(w)) - q(w). \end{aligned} \tag{8}$$

Introducing the Jacobian matrices: $S = \left[\frac{\partial s(w)}{\partial w}\right]_{(0)}$, $Q = \left[\frac{\partial q(w)}{\partial w}\right]_{(0)}$, $\Pi = \left[\frac{\partial \pi(w)}{\partial w}\right]_{(0)}$ and $D_0 = D(0)$, the linearized system (8) is presented as

$$\begin{bmatrix}\dot{\varepsilon}\\ \dot{w}\end{bmatrix} = \begin{bmatrix}A_0 & A_0\Pi - \Pi S + D_0\\ 0 & S\end{bmatrix}\begin{bmatrix}\varepsilon\\ w\end{bmatrix} + \begin{bmatrix}B_0\\ 0\end{bmatrix}u + \begin{bmatrix}\phi(\varepsilon, w)\\ \phi_w(w)\end{bmatrix} \quad (9)$$

$$e = [C_0 \;\; C_0\Pi - Q]\begin{bmatrix}\varepsilon\\ w\end{bmatrix} + \phi_e(\varepsilon, w)$$

where $\phi(\varepsilon, w)$, $\phi_w(w)$ and $\phi_e(\varepsilon, w)$ contain nonlinear terms and $\phi(0,0) = 0$, $\phi_w(0) = 0$ and $\phi_e(0,0) = 0$.

Define the following matrices:

$$\bar{A} = \begin{bmatrix}A_0 & A_0\Pi - \Pi S + D_0\\ 0 & S\end{bmatrix}, \quad \bar{C} = [C_0 \;\; C_0\Pi - Q].$$

With this definitions, the following assumption can be introduced:

- **A3**. The pair $\bar{A}, \bar{C}$ is detectable.

3.2 SM Dynamics

Setting $\sigma(\xi) = \sigma(\varepsilon) = 0$ (7) and following the equivalent control method, the SM dynamics for the variable ε can be derived, first, by calculating u_{eq} from $\dot{\sigma}(\varepsilon) = 0$ as

$$u_{eq} = -[G(\varepsilon)B(\cdot)]^{-1}G(\varepsilon)\big[f(\cdot) + D(\cdot)w - \frac{\partial\pi(w)}{\partial w}s(w)\big] - \delta(\cdot) \quad (10)$$

under condition rank$[G(\varepsilon)B(\cdot)] = m$, where $G(\varepsilon) = \frac{\partial\sigma(\varepsilon)}{\partial\varepsilon}$.

Then, substituting (10) in (8), the SM dynamics become

$$\dot{\varepsilon} = p(\cdot)\left[f(\cdot) + D(\cdot)w - \frac{\partial\pi(w)}{\partial w}s(w)\right] \quad (11)$$

where the nonlinear operator $p(\cdot)$ is defined as $p(\cdot) = I_n - B(\cdot)[G(\varepsilon)B(\cdot)]^{-1}G(\varepsilon)$.

Using (9), the linear approximation of (11) can be rewritten as

$$\dot{\varepsilon} = PA_0\varepsilon + P(A_0\Pi - \Pi S + D_0)w + \phi(\varepsilon, w) \quad (12)$$

where $P = I_n - B_0[G(0)B_0]^{-1}G(0)$.

3.3 Proportional-Integral Observer

To estimate the unmeasured state vector (ε, w) and design the SM error feedback, that is, the dynamical controller (5) - (7), a nonlinear PI observer is proposed as

$$\begin{bmatrix}\dot{\hat{\varepsilon}}\\ \dot{\hat{w}}\end{bmatrix} = \begin{bmatrix}f(\hat{\varepsilon}, \hat{w}) + B(\hat{\varepsilon}, \hat{w})u + D(\hat{\varepsilon}, \hat{w})\hat{w} - \frac{\partial\pi(\hat{w})}{\partial\hat{w}}s(\hat{w})\\ s(\hat{w})\end{bmatrix} + L_1(e - \hat{e}) + L_2\xi_0 \quad (13)$$

$$\dot{\xi}_0 = e - \hat{e} \quad (14)$$

where $\xi_0 = \int(e - \hat{e})dt$; $\hat{\varepsilon}$ and $\hat{w}$ are the estimates of ε and w, respectively, $\hat{e} = h(\hat{\varepsilon}, \hat{w}) - q(\hat{w})$, and L_1, L_2 are the observer gain matrices.

Using the systems (8) and (13) with its linearization (9), the linear observer error dynamics result in

$$\begin{bmatrix}\dot{\tilde{\varepsilon}}\\ \dot{\tilde{w}}\\ \dot{\xi}_0\end{bmatrix} = R\begin{bmatrix}\tilde{\varepsilon}\\ \tilde{w}\\ \xi_0\end{bmatrix} + \delta_1(\varepsilon, \tilde{\varepsilon}, \tilde{w}, t) \quad (15)$$

where $[\tilde{\varepsilon}, \tilde{w}]^T = [\varepsilon - \hat{\varepsilon}, w - \hat{w}]^T$,

$$R = \begin{bmatrix}\bar{A} - L_1\bar{C} & L_2\\ -\bar{C} & 0\end{bmatrix}$$

$$\delta_1(\varepsilon, \tilde{\varepsilon}, \tilde{w}, t) = \begin{bmatrix}\phi(\tilde{\varepsilon}, \tilde{w}) - B_0\delta(\varepsilon, t) - L_{11}\phi_e(\tilde{\varepsilon}, \tilde{w})\\ \phi_w(\tilde{w}) - L_{12}\phi_e(\tilde{\varepsilon}, \tilde{w})\\ -\phi_e(\tilde{\varepsilon}, \tilde{w})\end{bmatrix}.$$

Under Assumption **A3**, the matrices $L_1 = [L_{11}, L_{12}]^T$ and $L_2 = [L_{21}, L_{22}]^T$ can be chosen such that the matrix R in (15) is Hurwitz .

3.4 Solution existence conditions

Lemma 1. *The relation*

$$p(\pi(w))\left[f(\pi(w)) + D(\pi(w))w - \frac{\partial\pi(w)}{\partial w}s(w)\right] = 0$$

is true if and only if there are $\pi(w)$ and $\lambda(w)$, such that

$$f(\pi(w)) + d(\pi(w))w - \frac{\partial\pi(w)}{\partial w}s(w) = B(\pi(w))\lambda(w).$$

The proof of this Lemma comes from Draženović (1969) and El-Ghezawi et al. (2007).

From **Lemma 1**, the solvability conditions are established in the following proposition:

Proposition 1 *Under assumptions **A1-A3**, if there exist a $C^k(k \geq 2)$ mapping $x = \pi(w)$, with $\pi(0) = 0$, defined in a neighborhood W of the origin, satisfying the following conditions*

$$f(\pi(w)) + d(\pi(w))w - B(\pi(w))\lambda(w) = \frac{\partial\pi(w)}{\partial w}s(w)$$

$$h(\pi(w)) - q(w) = 0$$

then, the Error Feedback SM Regulation problem, as defined above, is solvable.

Proof 1. Choosing the manifold $\sigma(\xi) = \sigma(\hat{\varepsilon}) = 0$ (7) as

$$\sigma(\hat{\varepsilon}) = C_s\hat{\varepsilon}, \quad (16)$$

where $C_s = G(0)$, and using (13), the following system can be obtained:

$$\dot{\sigma}(\hat{\varepsilon}) = \bar{f}(\hat{\varepsilon}, \hat{w}) + \bar{B}(\hat{\varepsilon}, \hat{w})u \quad (17)$$

where $\bar{f}(\hat{\varepsilon}, \hat{w}) = C_s[f(\hat{\varepsilon}, \hat{w}) + D(\hat{\varepsilon}, \hat{w})\hat{w} - \frac{\partial\pi(\hat{w})}{\partial\hat{w}}s(\hat{w}) + L_{11}(e - \hat{e}) + L_{21}\xi]$ and $\bar{B}(\hat{\varepsilon}, \hat{w}) = C_sB(\hat{\varepsilon}, \hat{w})$.

Setting $\dot{\sigma}(\hat{\varepsilon}) = 0$, the equivalent control is calculated as

$$\hat{u}_{eq}(\hat{\varepsilon}, \hat{w}) = \bar{B}(\hat{\varepsilon}, \hat{w})^{-1}\bar{f}(\hat{\varepsilon}, \hat{w}). \quad (18)$$

Using (18), the SM error feedback controller is designed accordingly

$$u = \hat{u}_{eq}(\hat{\varepsilon}, \hat{w}) - \bar{B}(\hat{\varepsilon}, \hat{w})^{-1}(k_1\,\mathrm{sign}(\sigma(\hat{\varepsilon}))). \quad (19)$$

Then, the closed-loop system (17) - (19) becomes

$$\dot{\sigma}(\hat{\varepsilon}) = -k_1\,\mathrm{sign}(\sigma(\hat{\varepsilon})).$$

It can be easily seen that for $k_1 > 0$, a SM motion occurs on $\sigma(\hat{\varepsilon}) = 0$ (16) in finite time, satisfying **C1**.

Without loss of generality, we assume that using the relation $[\tilde{\varepsilon}, \tilde{w}]^T = [\varepsilon - \hat{\varepsilon}, w - \hat{w}]^T$ (15) and $u_{eq}(\varepsilon, w)$ (10), the control $\hat{u}_{eq}(\hat{\varepsilon}, \hat{w})$ (18) can be represented as

$$\hat{u}_{eq}(\hat{\varepsilon}, \hat{w}) = u_{eq}(\varepsilon, w) + \delta_2(\tilde{\varepsilon}, \tilde{w}, t). \tag{20}$$

Then, by substituting (20) in (8) and using the linear approximation (11) of (12), the SM dynamics have the form

$$\begin{aligned}\dot{\varepsilon} &= PA_0\varepsilon + P(A_0\Pi - \Pi S + D_0)w + \phi(\varepsilon, w) \\ &\quad + \delta_2(\tilde{\varepsilon}, \tilde{w}, t).\end{aligned} \tag{21}$$

Combining (21), (15), (3) and (2), the closed-loop system motion on the manifold $\sigma(\hat{\varepsilon}) = 0$ is described by

$$\begin{aligned}\dot{\varepsilon} &= PA_0\varepsilon + P(A_0\Pi - \Pi S + D_0)w + \phi(\varepsilon, w) \\ &\quad + \delta_2(\tilde{\varepsilon}, \tilde{w}, t)\end{aligned} \tag{22}$$

$$\begin{bmatrix}\dot{\tilde{\varepsilon}} \\ \dot{\tilde{w}} \\ \dot{\hat{\xi}}\end{bmatrix} = R\begin{bmatrix}\tilde{\varepsilon} \\ \tilde{w} \\ \hat{\xi}\end{bmatrix} + \delta_1(\varepsilon, \tilde{\varepsilon}, \tilde{w}, t) \tag{23}$$

$$\begin{aligned}\dot{w} &= Sw + \phi_w(w) \\ e &= h(\varepsilon + \pi(w)) - q(w).\end{aligned}$$

By choosing adequately the matrix C_s, $n-m$ eigenvalues of matrix PA_0 can be assigned to be in C^-, while m eigenvalues are equal to zero. Moreover, the matrices L_1 and L_2 can be chosen such the the matrix R is Hurwitz. Therefore, the equilibrium point $\varepsilon = 0, \tilde{\varepsilon} = 0, \tilde{w} = 0$ of the system (22) - (23) in the absence of the perturbation is asymptotically stable satisfying **C2**.

If the condition given by **Lemma 1** is satisfied, then

$$P(A\Pi - \Pi S + D_0)w + \phi(0, w) = p(\pi(w))\Big[f(\pi(w)) + d(\pi(w))w - \frac{\partial \pi(w)}{\partial w}s(w)\Big] = 0.$$

- For the case $\|\delta(\varepsilon, t)\| \le \gamma_1$, there is $\gamma_3 > 0$ and a neighborhood Ω of the origin where

$$\left\|\begin{matrix}\delta_1(\varepsilon, \tilde{\varepsilon}, \tilde{w}, t) \\ \delta_2(\tilde{\varepsilon}, \tilde{w}, t)\end{matrix}\right\| \le \gamma_3.$$

Therefore, a solution of the perturbed system (22) - (23) is ultimately bounded resulting in

$$\|e(t)\| \le \gamma_2, \quad \forall\, t \ge T_0.$$

Thus, the condition **C3.A** is satisfied.

- For the case $\|\delta(\varepsilon, t)\| \le \gamma_3\|\varepsilon\|$, there is $\gamma_4 > 0$ and a neighborhood Ω of the origin where

$$\left\|\begin{matrix}\delta_1(\varepsilon, \tilde{\varepsilon}, \tilde{w}, t) \\ \delta_2(\tilde{\varepsilon}, \tilde{w}, t)\end{matrix}\right\| \le \gamma_4 \left\|\varepsilon, \tilde{\varepsilon}, \tilde{w}\right\|.$$

Therefore, in this case, a solution of the perturbed system (22) - (23) tends asymptotically to zero. By continuity of $h(\varepsilon, w)$, $\lim_{t\to\infty} e(t) = 0$, satisfying **C3.B**. □

4. REGULAR FORM

In this section, we consider a class of nonlinear systems which can be represented (possibly after a nonlinear transformation) in Regular form, Loukianov and Utkin (1981)

$$\begin{aligned}\dot{z}_1 &= f_1(z_1, z_2) + D_1(z)w \\ \dot{z}_2 &= f_2(z) + D_2(z)w + B_2(z)(u + \delta(z, t)) \\ e &= h(z) - q(w)\end{aligned}$$

where $z = [z_1, z_2]^T$, $z_1 \in \Re^{n-m}$, $z_2 \in \Re^m$, rank $B_2 = m$.

4.1 Conditions of existence for Regular form

Introducing the steady state $\pi_1(w)$ and $\pi_2(w)$, one can define the steady-state error as

$$\begin{aligned}\varepsilon_1 &= z_1 - \pi_1(w) \\ \varepsilon_2 &= z_2 - \pi_2(w).\end{aligned}$$

Now, the control error dynamics become

$$\dot{\varepsilon}_1 = f_1(\varepsilon_1, \varepsilon_2, w) + D_1(\varepsilon_1, \varepsilon_2, w)w - \frac{\partial \pi_1(w)}{\partial w}s(w) \tag{24}$$

$$\begin{aligned}\dot{\varepsilon}_2 &= f_2(\varepsilon, w) + D_2(\varepsilon, w)w - \frac{\partial \pi_2(w)}{\partial w}s(w) \\ &\quad + B_2(\varepsilon, w)(u + \delta(\varepsilon, w, t))\end{aligned} \tag{25}$$

$$e = h(\varepsilon_1, \varepsilon_2, w) - q(w). \tag{26}$$

The sliding variable is selected as

$$\sigma = \varepsilon_2 + \sigma_0(\varepsilon_1).$$

On the sliding manifold $\sigma = 0$, or $\varepsilon_2 = -\sigma_0(\varepsilon_1)$, the SM equation becomes

$$\begin{aligned}\dot{\varepsilon}_1 &= f_1(\varepsilon_1 + \pi_1(w), \sigma_0(\varepsilon_1) + \pi_2(w)) \\ &\quad + D_1(\varepsilon_1 + \pi_1(w), \sigma_0(\varepsilon_1) + \pi_2(w))w - \frac{\partial \pi_1(w)}{\partial w}s(w).\end{aligned}$$

To estimate ε_1, ε_2 (24)-(25), and w (3), the following nonlinear PI observer is proposed

$$\begin{aligned}\begin{bmatrix}\dot{\hat{\varepsilon}}_1 \\ \dot{\hat{\varepsilon}}_2 \\ \dot{\hat{w}}\end{bmatrix} &= \begin{bmatrix}f_1(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{w}) + D_1(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{w})\hat{w} - \frac{\partial \pi_1(\hat{w})}{\partial \hat{w}}s(\hat{w}) \\ f_2(\hat{\varepsilon}, \hat{w}) + D_2(\hat{\varepsilon}, \hat{w})\hat{w} - \frac{\partial \pi_2(\hat{w})}{\partial \hat{w}}s(\hat{w}) \\ s(\hat{w})\end{bmatrix} \\ &\quad + L_1(e - \hat{e}) + L_2\xi_1 + \begin{bmatrix}0 \\ B_2(\hat{\varepsilon}, \hat{w}) \\ 0\end{bmatrix}u \\ \dot{\xi}_1 &= e - \hat{e}\end{aligned}$$

with $\hat{e} = h(\hat{\varepsilon}, \hat{w}) - q(\hat{w})$.

Defining $\tilde{\varepsilon}_1 = \hat{\varepsilon}_1 - \varepsilon_1$, $\tilde{\varepsilon}_2 = \hat{\varepsilon}_2 - \varepsilon_2$, $\tilde{w} = \hat{w} - w$, the observer error can be represented as in (15), with $L_1 = [L_{11}, L_{12}, L_{13}]$, and $L_2 = [L_{21}, L_{22}, L_{23}]$.

Proposition 2 *Under **A1** - **A3**, if there exist $C^k (k \ge 2)$ mappings $x_1 = \pi_1(w)$ and $x_2 = \pi_2(w)$, with $\pi_1(0) = 0$ and $\pi_2(0) = 0$, defined in a neighborhood W of the origin, satisfying the following conditions:*

$$\begin{aligned}f_1(\pi_1(w), \pi_2(w)) + D_1(\pi_1(w), \pi_2(w))w &= \frac{\partial \pi_1(w)}{\partial w}s(w) \\ h(\pi_1(w), \pi_2(w)) - q(w) &= 0\end{aligned}$$

then, the Error Feedback SM Regulation problem for nonlinear systems in Regular form, as defined above, is solvable.

The proof of this Proposition is similar to Section 3.4.

5. EXAMPLE: APPLICATION TO PENDUBOT

The proposed method is applied here to an underactuated electromechanical system known as the Pendubot, whose mathematical model can be described by

$$\dot{x} = f(x) + B(x)(u(t) + \delta(x,t)) \quad (27)$$
$$y = x_2,$$

where $x = [\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]^T$ represent each joint positions: x_1, x_2, and velocities: x_3, x_4. The input $u(t)$ is a scalar value, the term $\delta(x,t)$ describes an unmodelled perturbation term including parameter variations bounded by $\|\delta(x,t)\| \leq \gamma_0\|x\| + \gamma_1$, and the functions $f(x) = [x_3, x_4, b_3(x_2)p_1(x), b_4(x_2)p_2(x)]^T$, and $B(x) = [0, 0, b_3(x_2), b_4(x_2)]^T$, where $b_3(x_2) = \frac{D_{22}}{D_{11}(x_2)D_{22} - D_{12}^2(x_2)}$, and $b_4(x_2) = -\frac{D_{12}(x_2)}{D_{11}(x_2)D_{22} - D_{12}^2(x_2)}$.

Here, $p_1(x)$, $p_2(x)$, $D_{11}(x_2)$, $D_{12}(x_2)$, and D_{22}, depend on operations between plant parameters and elements of the state.

The desired tracking trajectory, $q(w) = w_2$, is produced by

$$\dot{w} = \begin{bmatrix} 0 & \alpha w_2 \\ -\alpha w_1 & 0 \end{bmatrix}.$$

Hence, the tracking error becomes $e = x_2 - w_2$.

After the transformation, (27) can be represented in Regular form as

$$\begin{aligned} \dot{z}_1 &= z_3 - f_{11}(z) \\ \dot{z}_2 &= z_4 \\ \dot{z}_3 &= f_{31}(z) \\ \dot{z}_4 &= f_{41}(z) + b_4(z_2)(u(t) + \delta(z,t)) \\ y &= z_2 \end{aligned}$$

where the expressions $f_{11}(z) = \frac{D_{22}}{D_{12}(z_2)}z_4$, $f_{31}(z) = b_3(z_2)p_1(z) + \frac{D_{22}}{D_{12}(z_2)}b_4(z_2)p_2(z) + \frac{D_{22}}{D_{12}(z_2)^2}C_3(z_2, z_4)$, and $f_{41}(z) = b_4(z_2)p_2(z)$.

Defining the steady-state error as $\varepsilon_i = z_i - \pi_i(w), i = 1, ..., 4$, the error dynamics are described by

$$\begin{aligned} \dot{\varepsilon}_1 &= \varepsilon_3 + \pi_3(w) - f_{11}(\varepsilon + \pi(w)) - \frac{\partial \pi_1}{\partial w}s(w) \\ \dot{\varepsilon}_2 &= \varepsilon_4 + \pi_4(w) - \frac{\partial \pi_2}{\partial w}s(w) \\ \dot{\varepsilon}_3 &= f_{31}(\varepsilon + \pi(w)) - \frac{\partial \pi_3}{\partial w}s(w) \\ \dot{\varepsilon}_4 &= f_{41}(\varepsilon + \pi(w)) + b_4(\varepsilon_2, w)(u(t) + \delta(\varepsilon, w, t)) \\ &\quad - \frac{\partial \pi_4}{\partial w}s(w) \end{aligned} \quad (28)$$

where $\pi(w) = [\pi_1(w), \ldots, \pi_4(w)]^T$.

From (28), the Regulator Equation is given by

$$\frac{\partial \pi_1}{\partial w}s(w) = \pi_3(w) - f_{11}(\pi(w)) \quad (29)$$
$$\frac{\partial \pi_2}{\partial w}s(w) = \pi_4(w) \quad (30)$$
$$\frac{\partial \pi_3}{\partial w}s(w) = f_{31}(\pi(w)) \quad (31)$$
$$0 = \pi_2(w) - w_2. \quad (32)$$

From (32) and (30), it follows $\pi_2(w) = w_2$, and $\pi_4(w) = -\alpha w_1$. Since the solution of $\pi_1(w)$, and $\pi_3(w)$, involves solving partial differential equations; we take a simpler approach such as proposing a polynomial approximation for $\pi_1(w)$ as

$$\pi_1(w) = a_0 + a_1 w_1 + a_2 w_2 + a_3 w_1 w_2 + a_4 w_1^2 + a_5 w_2^2 + a_6 w_1 w_2^2 + a_7 w_1^2 w_2 + a_8 w_1^3 + a_9 w_2^3 + O^4(\|w\|),$$

with $\alpha = 1.5$, $a_0 = 1.57$, $a_1 = -0.008$, $a_2 - 0.98$, $a_3 = a_4 = a_5 = 0$, $a_6 = -0.57e-04$, $a_7 = 0.01$, $a_8 = -2.57e-05$, $a_9 = -0.007$; and from (29):

$$\pi_3(w) = \frac{\partial \pi_1}{\partial w}s(w) - \frac{D_{22}}{D_{12}(w_2)}\alpha w_1. \quad (33)$$

Based on (13)-(14), the proposed observer takes the form:

$$\begin{bmatrix} \dot{\hat{\varepsilon}}_1 \\ \dot{\hat{\varepsilon}}_2 \\ \dot{\hat{\varepsilon}}_3 \\ \dot{\hat{\varepsilon}}_4 \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} \hat{\varepsilon}_3 + \pi_3(\hat{w}) - f_{11}(\hat{\varepsilon} + \pi(\hat{w})) - \frac{\partial \pi_1}{\partial \hat{w}}s(\hat{w}) \\ \hat{\varepsilon}_4 + \pi_4(\hat{w}) + \alpha \hat{w}_1 \\ f_{31}(\hat{\varepsilon} + \pi(\hat{w})) \\ f_{41}(\hat{\varepsilon} + \pi(\hat{w})) + b_4 u(t) - \frac{\partial \pi_4}{\partial \hat{w}}s(\hat{w}) \\ \alpha \hat{w}_2 \\ -\alpha \hat{w}_1 \end{bmatrix} + L_1(e - \hat{e}) + L_2\xi$$
$$\dot{\xi} = e - \hat{e},$$

where $e = z_2 - w_2 = \varepsilon_2$, and $\hat{e} = \hat{\varepsilon}_2$.

The sliding variable is selected as $\sigma = \hat{\varepsilon}_4 + C_s[\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3]^T$, and the controller is chosen as in (19).

6. NUMERICAL EVALUATION AND SIMULATION RESULTS

Using Table 1, the linear matrices A_{11}, A_{12}, are numerically evaluated, and selecting $C_s = [49.64, 48.64, 8.0203]^T$, the matrix $(A_{11} - A_{12}C_s)$ has the eigenvalues $(-0.99, -6.21 + 0.44i, -6.21 - 0.44i)$.

On the other hand the matrix gains L_1, and L_2 are computed as in Beale and Shafai (1988), obtaining $L_1 = (1 \times 10^4)[-1.4097, 0.0260, -8.8378, 1.9860, 0.5930, -0.4517]$, $L_2 = (1 \times 10^4)[-0.0386, 0.0100, -1.6190, -0.0186, 0.6034, -0.0168]$, which ensures the matrix R has the eigenvalues $(-150, -70, -18, -13, -7, -2.5, -2)$.

The Pendubot model was simulated using MATLAB, using the parameters from Table 1, with $x_1(0) = 1.57$rad, $x_2(0) = x_3(0) = x_4(0) = 0$, and the external perturbation

	Link 1	Link 2
Mass (kg)	0.0551	0.0237
Length (m)	0.0825	0.2197
Center of mass distance (m)	0.0523	0.0799
Moment of inertia (kg m^2)	6.272e-05	1.759e-04
Friction coefficient (kg m^{2}s^{-1})	5.5286e-04	9.8895e-05

Table 1. Nominal values for parameters

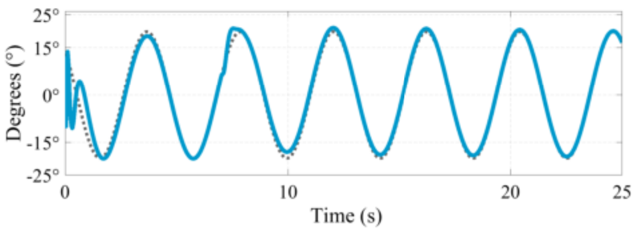

Fig. 1. Output tracking signal z_2 (solid blue), and reference w_2 (dotted gray).

as $\delta(t) = 0.0001 \sin(0.1t)$ is present from $t > 7$. Fig. 1 shows the output tracking signal and its reference, where the performance of the controller shows good results. The control signal $u(t)$ and $u_{eq}(t)$ are shown in Fig. 2 (a), and (b) respectively.

Fig. 3 shows a comparison with a Luenberger observer, it can be seen how it is not robust, and even has a finite-time escape.

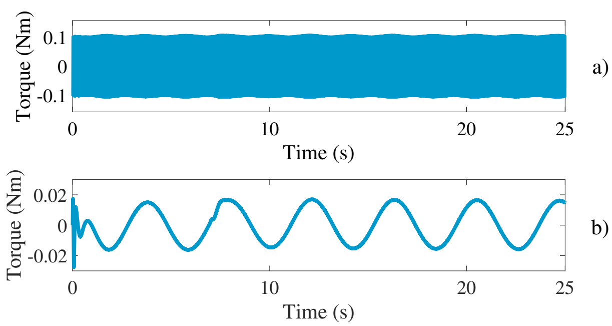

Fig. 2. a) Discontinuous control action $u(t)$. b) $u_{eq}(t)$ control action.

7. CONCLUSIONS

The SM error feedback regulation problem for nonlinear systems with unmodelled external matched perturbations was analyzed, and conditions for existence of solution were derived. Using a nonlinear PI observer, an error feedback regulator was designed in order to track a signal produced by a given exosystem. The effectiveness of the proposed method is mostrated by application to Pendubot system .

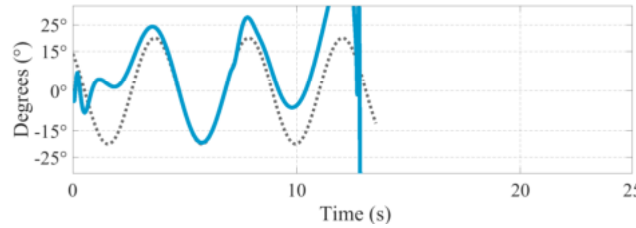

Fig. 3. Comparison with Luenberger observer. z_2 (solid blue), and reference w_2 (dotted gray).

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