Memorias del Congreso Nacional de Control Automático
ISSN: 2594-2492

Discrete-Time Sliding Mode Output Tracking Control for a Class of Nonlinear Perturbed Systems ⋆

Cristian C. Zapata-Zuluaga and Alexander G. Loukianov

Department of Electrical Engineering, Advanced Studies and Research Center (CINVESTAV), National Polytechnic Institute (IPN), Guadalajara Campus, Jalisco, México, (e-mail: {cczapata,louk}@gdl.cinvestav.mx).

Abstract: The output tracking problem for a class of discrete-time nonlinear systems exposed in Nonlinear Block Controllable (NBC) form is faced. This paper considers both matched and unmatched perturbations. First, the sliding manifold is designed taking into account the Block Control procedure combined with a perturbation estimation. The impact of unmatched perturbation is attenuated with help the perturbation estimation. Therefore, a discrete-time sliding mode controller is synthesized such that the system state is driven toward a vicinity of the designed sliding manifold and stays there for all sampled time instants, avoiding chattering and reducing the matched perturbation effect. The effectiveness of the proposed technique is confirmed by simulation.

Keywords: robust control, nonlinear systems, nonlinear control, discrete-time systems.

1. INTRODUCTION

In recent years, the rise of digital technology and its extensive use have allowed the implementation of sophisticated control methods. Indeed, enhancements in microcontrollers have improved the discrete-time control schemes, showing significant advantages such as: easy programming, reliability, and simple integration with other digital systems. For this reason, controller synthesis based on the discrete-time model of the system has generated a wild interest, which has raised an active investigative activity in the field of discrete-time control.

In the last three decades diversified tools have been developed, see e.g. Monaco and Normand-Cyrot (1983); Mattioni et al. (2017) and references therein. One of the most effective of these tools is Sliding-Mode Control (SMC) (Utkin et al., 2009), which has been extensively studied first in continuous-time framework and it exhibits notable advantages such as robustness and computational efficiency.

Since modern control systems are implemented by computers, the investigation of discrete time (DT) SMC has been an important topics of the SMC theory (Drakunov and Utkin, 1989; Utkin et al., 2009; Levant and Livne, 2015; Koch et al., 2016; Huber et al., 2016).

In the DT setting, SMC, similar to the continuous time case, enables decomposition of the control design problem into two independent stages:

- **Problem A.** Selection of sliding manifold with the desired sliding motion, and
- **Problem B.** Design a reaching control law to force the sliding mode along this manifold.

The main attention has been payed to **Problem B** and numerous significant results has been obtained in a reaching control law design (Wang et al., 2009). In the proposed DT reaching laws the switching term was preserved from continuous time SMC to suppress effect of matched bounded perturbations. However, this term can produce undesired numerical chattering phenomenon in the vicinity of sliding manifold. This effect can be suppressed by implicit Euler discretization of the discontinuous term (Huber et al., 2016) or by increasing the relative degree and correspondingly the order of sliding mode (Koch et al., 2016). To avoid the chattering problem, continuous reaching controllers has been proposed in (Bartoszewicz and Latosinski, 2016), including Equivalent-Control-Based (Drakunov and Utkin, 1989) and adaptive (Bartoszewicz and Adamiak, 2018) SM controllers. However, as a result of the lack of perturbation for calculating the equivalent control, sliding manifold reaches a boundary layer $\mathcal{O}(\tau)$ with τ as the sample period. In order to mitigate this obstacle, in some researches as Su et al. (2000); Abidi et al. (2007), an estimator of perturbation using its previous step has been designed,

⋆ This work was supported by CONACyT, México, under grant 252405. Cristian Zapata-Zuluaga acknowledges CONACyT, México for the scholarship number 455411.

and an accuracy of $\mathcal{O}(\tau^2)$ in the boundary layer of the sliding manifold has been achieved. On the other hand, **Problem A**, namely, the design of the desired sliding manifold is of great interest. Basely, a classical linear sliding manifold has been synthesized for linear time-invariant (LTI) systems in Utkin et al. (2009); Acary et al. (2012), and those with matched perturbation in Wang et al. (2009); Huber et al. (2016); Koch et al. (2016).

This work deals with the aforementioned approach, we consider a SM output tracking problem for a class of nonlinear systems presented in Nonlinear Block Controllable (NBC) form (Loukianov, 2002) with both matched and unmatched unmodeled perturbations. The considered class is quite wide and includes, for example, electromechanical systems. The principal aim is, first, to use Block Control (BC) feedback linearization (FL) technique combined with the perturbation estimation, for the design of the desired nonlinear sliding manifold on which the system motion satisfies a specified transient response, and the effect of unmatched perturbation on the output tracking error is reduced. Then, a discrete-time SM controller, based on the equivalent control combined with the perturbation estimation, is formulated such that the system state is driven into a smaller bounding layer of the designed sliding manifold and stays there for all sampled time instants, avoiding chattering and reducing the matched perturbation effect. The performance of the designed strategy is validated by simulation.

The rest of the paper is organized as follows: Section 2 presents the class of nonlinear systems considered in this work. Based on this and exploiting the considered system structure, sliding manifold design is presented in Section 3. Section 4 investigates the discrete-time controller synthesis. The closed-loop system motion over the manifold is studied in Section 5. To prove the effectiveness of the proposed discrete-time sliding mode controller, it is applied to the continuous-time magnetic levitation system in Section 6 and simulations investigate the closed-loop system behavior. Final comments conclude the paper in Section 7.

2. PROBLEM STATEMENT

Consider the following continuous time nonlinear system subject to uncertainty:

$$\dot{x} = h(x) + G(x)u + g(x,t), \tag{1}$$

where $x \in X \subset \mathbb{R}^n$ is the state vector, $u \in U \subset \mathbb{R}^m$ is the control vector. It is assumed the vector field $h(x)$ and the columns of $G(x)$ are smooth and bounded mappings of class $C^{\infty}_{[t,\infty)}$, $h(0) = 0$, and $\mathrm{rank}(G(x)) = m$ for all $x \in X$ and $t \geq 0$. The unknown mapping $g(x,t)$ characterizes external disturbances and parameter variations.

In this work, we assume that system (1) can be presented (possible under an appropriate nonsingular smooth transformation) in the following Nonlinear Block Controllable (NBC) form form with perturbation:

$$\begin{aligned}
\dot{x}_1 &= h_1(x_1) + G_1(x_1)x_2 + g_1(x_1,t) \\
&\vdots \\
\dot{x}_i &= h_i(\bar{x}_i) + G_i(\bar{x}_i)\bar{x}_{i+1} + g_i(\bar{x}_i,t) \\
&\vdots \\
\dot{x}_r &= h_r(x) + G_r(x)u + g_r(x,t)
\end{aligned} \tag{2}$$

$$y = x_1 \tag{3}$$

where $x = \left[x_1^\top,\ldots,x_r^\top\right]^\top$, $\bar{x}_i = \left[x_1^\top,\ldots,x_i^\top\right]^\top$ for $i = 1,\ldots,r-1$; $\dim(x_j) = n_j$ for $j = 1,\ldots,r$; $n = \sum_{j=1}^r n_j$, $\mathrm{rank}(G_j(\bar{x}_j)) = n_j\ \forall x_j \in \mathbb{R}^{n_j}$.

Now, applying explicit Euler method to the system (2)-(3), the sampled-date system becomes

$$\begin{aligned}
x_{1,k+1} &= f_1(x_{1,k}) + B_1(x_{1,k})x_{2,k} + d_1(x_{1,k},k) \\
&\vdots \\
x_{i,k+1} &= f_i(\bar{x}_{i,k}) + B_i(\bar{x}_{i,k})\bar{x}_{i+1,k} + d_i(\bar{x}_{i,k},k) \\
&\vdots \\
x_{r,k+1} &= f_r(x_k) + B_r(x_k)u_k + d_r(x_k,k)
\end{aligned} \tag{4}$$

$$y_k = x_{1,k} \tag{5}$$

with

$$\begin{aligned}
f_1(x_k) &= x_{1,k} + \tau h_1(x_k), & B_1(x_k) &= \tau G_1(x_k), \\
f_i(\bar{x}_{i,k}) &= x_{i,k} + \tau h_i(\bar{x}_{i,k}), & B_i(\bar{x}_{i,k}) &= \tau G_i(\bar{x}_{i,k}) \\
d_1(x_k,k) &= \tau g_1(x_k,k), & d_i(\bar{x}_{i,k},k) &= \tau g_i(\bar{x}_{i,k},k)
\end{aligned}$$

for $i = 2,\ldots,r$, where $k \in \mathbb{Z}^+ \cup \{0\}$ denotes the discrete time with $\mathbb{Z}^+$ the set of the positive integers and $x_k, x_{1,k},\ldots,x_{r,k}$, are the discrete approximation of $x(t), x_1(t),\ldots,x_r(t)$, respectively. The control objective is to force the output y_k (5) to track a reference signal y_k^{ref}, reducing the effects of unmatched $d_i(\bar{x}_{i,k},k)$, $i = 1,...,r-1$, and matched $d_r(x_k,k)$ perturbations.

The following assumption is considered hereinafter.

Assumption 1. The matrix $B_i\left(\bar{x}_{i,k+1}\right)$ can be decomposed as

$$\begin{aligned}
B_1\left(x_{1,k+1}\right) &= B_1\left(\phi_{1,k}\right) + \Delta B_1\left(d_1(x_{1,k},k)\right), \\
\bar{B}_i\left(\bar{x}_{i,k+1}\right) &= \bar{B}_i\left(\phi_{i,k}\right) + \Delta\bar{B}_i\left(d_i(\bar{x}_{i,k},k)\right)
\end{aligned}$$

for $i = 2,\ldots,r-1$ where $\bar{B}_i(\bar{x}_{i,k}) = \bar{B}_{i-1}(\phi_{i-1,k})\bar{B}_i(\bar{x}_{i,k})$, $\bar{B}_1(x_{1,k}) = B_1(x_{1,k})$, $\phi_{i,k} = f_i(\bar{x}_{i,k}) + B_i(\bar{x}_{i,k})\bar{x}_{i+1,k}$; $B_i\left(\phi_{i,k}\right)$ is known and $\Delta B_i\left(d_i(\bar{x}_{i,k},k)\right)$ is unknown.

3. SLIDING MANIFOLD DESIGN

In this section, the concept of discrete-time Block Control method is used to design a sliding manifold on which the effect of the perturbation unmatched part is reduced.

The sliding manifold design procedure consists of a step-by-step construction of a new system with states

$$z_{i,k} = B_{i-1}(\bar{x}_{i-1,k})x_i - \alpha_{i,k}, \quad i = 1,\ldots,r, \tag{6}$$

where $\alpha_{i,k}$ is the desired value for $x_{i,k}$, which will be defined by such construction.

Step 1. On the first step, the output tracking error is defined as

$$z_{1,k} = e_k = x_{1,k} - \alpha_{1,k}, \tag{7}$$

with $\alpha_{1,k} = y_k^{ref}$ the reference value for $x_{1,k}$, having from (4) the following dynamics:

$$z_{1,k+1} = f_1(x_{1,k}) + B_1(x_{1,k})x_{2,k} + \bar{d}_{1,k} \tag{8}$$

where $\bar{d}_{1,k} = d_1(x_{1,k}, k) - \alpha_{1,k+1}$.

To impose the following desired dynamics:

$$z_{1,k+1} = K_1 z_{1,k} + z_{2,k} + \delta_{1,k} \tag{9}$$

with K_1 a Schur matrix and

$$\delta_{1,k} = \bar{d}_{1,k} - \bar{d}_{1,k-1}, \tag{10}$$

the new state variable $z_{2,k}$ is defined from (8)-(10) of the form

$$\begin{aligned} z_{2,k} &= B_1(x_{1,k})x_{2,k} - \alpha_{2,k}, \\ \alpha_{2,k} &= K_1 z_{1,k} - f_1(x_{1,k}) - \bar{d}_{1,k-1}. \end{aligned} \tag{11}$$

The perturbation $\bar{d}_{1,k-1}$ in (11) is calculated from (8) as

$$\bar{d}_{1,k-1} = z_{1,k} - f_1(x_{1,k-1}) - B_1(x_{1,k-1})x_{2,k-1}. \tag{12}$$

Step i. Iterating these steps, the i-th variable $z_{i,k}$ in (6) is $z_{i,k} = \bar{B}_{i-1}(\bar{x}_{i-1,k})x_i - \alpha_{i,k}$, and using the Assumption 1, its respective dynamics become

$$z_{i,k+1} = \bar{f}_i(\bar{x}_{i,k}) + \bar{B}_i(\bar{x}_{i,k})x_{i+1,k} + \bar{d}_{i,k} \tag{13}$$

where

$$\begin{aligned} \bar{f}_i(\bar{x}_{i,k}) =& B_{i-1}(\phi_{i-1,k})f_i(\bar{x}_{i,k}), \\ \bar{B}_i(\bar{x}_{i,k}) =& B_{i-1}(\phi_{i-1,k})B_i(\bar{x}_{i,k}), \\ \bar{d}_{i,k} =& B_{i-1}(\phi_{i-1,k})d_i(\bar{x}_{i,k}, k) - \alpha_{i,k+1} + \\ & \Delta B_{i-1}\left(d_{i-1}(\bar{x}_{i-1,k}, k)\right)x_{i,k+1}. \end{aligned}$$

With the desired dynamics defined as

$$z_{i,k+1} = K_i z_{i,k} + z_{i+1,k} + \delta_{i,k} \tag{14}$$

where K_i is a Schur matrix and

$$\delta_{i,k} = \bar{d}_{i,k} - \bar{d}_{i,k-1}, \tag{15}$$

the new state variable $z_{i+1,k}$ in (6) is determined from (13)-(15) by the following expression:

$$\begin{aligned} z_{i+1,k} &= \bar{B}_i(\bar{x}_{i,k})x_{i+1,k} - \alpha_{i+1,k}, \\ \alpha_{i+1,k} &= K_i z_{i,k} - \bar{f}_i(\bar{x}_{i,k}) - \bar{d}_{i,k-1}. \end{aligned} \tag{16}$$

The perturbation $\bar{d}_{i,k-1}$ in (16) is obtained from (13) as

$$\bar{d}_{i,k-1} = z_{i,k} - \bar{f}_i(\bar{x}_{i,k-1}) - \bar{B}_i(\bar{x}_{i,k-1})x_{i+1,k-1}. \tag{17}$$

Step r. Finally, on the r-th step, the variable $z_{r,k} = \bar{B}_{r-1}(\bar{x}_{r-1,k})x_k - \alpha_{r,k}$ in (6) is introduced with dynamics

$$z_{r,k+1} = \bar{f}_r(x_k) + \bar{B}_r(x_k)u_k + \bar{d}_{r,k} \tag{18}$$

where

$$\begin{aligned} \bar{f}_r(x_k) =& B_{r-1}(\phi_{r-1,k})f_r(x_k), \\ \bar{B}_r(x_k) =& B_{r-1}(\phi_{r-1,k})B_r(x_k), \\ \bar{d}_{r,k} =& B_{r-1}(\phi_{r-1,k})d_r\left(x_k, k\right) - \alpha_{r,k+1} + \\ & \Delta B_{r-1}\left(d_{r-1}(\bar{x}_{r-1,k}, k)\right)x_{r,k+1}. \end{aligned}$$

It is worth mentioning that the new variables $z_{i,k}$, $i = 1, \ldots, r$, determine a nonlinear transformation

$$\begin{aligned} z_{1,k} &= x_{1,k} - \alpha_{1,k} := \psi_1(x_{1,k}) \\ &\vdots \\ z_{r,k} &= \bar{B}_{r-1}(\bar{x}_{r-1,k})x_{r,k} - \alpha_{r,k} := \psi_r(x_k). \end{aligned} \tag{19}$$

By means of the transformation $z_k = \psi(x_k)$, $\psi(x_k) = \left[\psi_1^\top \ \psi_2^\top \ \ldots \ \psi_r^\top\right]^\top$, the system (4) is diffeomorphic to

$$\begin{aligned} z_{1,k+1} &= K_1 z_{1,k} + z_{2,k} + \delta_{1,k} \\ &\vdots \\ z_{r,k+1} &= \bar{f}_r(z_k) + \bar{B}_r(z_k)u_k + \bar{d}_{r,k} \end{aligned} \tag{20}$$

where $z_k = \left[z_{1,k}^\top, \ \ldots, \ z_{r,k}^\top\right]^\top$ and $\delta_{j,k} = \bar{d}_{j,k} - \bar{d}_{j,k-1}$ for $j = 1, \ldots, r-1$.

Now, based on Loukianov (2002), a natural choice is the sliding variable

$$\sigma_k = z_{r,k}. \tag{21}$$

This variable will be used to design a sliding mode control in the following section.

4. DISCRETE-TIME SLIDING MODE CONTROL

In this section, a sliding mode controller will be designed for the transformed system (20), such that the reference signal y_k^{ref} is tracked and the perturbation matched part effect is reduced. This will be achieved in presence of constraint on the input

$$\|u_k\| \leq u_{\max}, \qquad u_{\max} > 0. \tag{22}$$

As is usual in SM technique, the control forces the system evolution on a certain manifold which guarantees the achievement of control requirements. Considering the selected sliding variable (21), its dynamics can be obtained from (18) by the following expression:

$$\sigma_{k+1} = \bar{f}_{r,k} + \bar{B}_{r,k}u_k + \bar{d}_{r,k}, \tag{23}$$

with $\bar{f}_{r,k} = \bar{f}_r(z_k)$, $\bar{B}_{r,k} = \bar{B}_r(z_k)$. In order to induce chattering-free sliding mode on the manifold $\sigma_k = 0$, the control u_k can be selected as (Utkin et al., 2009)

$$u_k = u_{eq,k} \tag{24}$$

where the equivalent control value $u_{eq,k}$ is calculated as a solution to $\sigma_{k+1} = 0$ (23) for u_k of the form

$$u_{eq,k} = -\bar{B}_{r,k}^{\dagger}\left(\bar{f}_{r,k} + \bar{d}_{r,k}\right). \tag{25}$$

The control (24) with (25) brings the system trajectories on the sliding manifold $\sigma_k = 0$ (21) in one sampling time period. However, (25) cannot be implemented since the perturbation $\bar{d}_{r,k}$ is unknown. To overcome this problem, the expression (25) is modified as

$$\tilde{u}_{eq,k} = -\bar{B}_{r,k}^{\dagger}\left(\bar{f}_{r,k} + \bar{d}_{r,k-1}\right) \tag{26}$$

where the retarded perturbation $\bar{d}_{r,k-1}$ can be obtained from (23) of the form

$$\bar{d}_{r,k-1} = \sigma_k - \bar{f}_{r,k-1} - \bar{B}_{r,k-1}u_{k-1}. \tag{27}$$

Replacing (27) in (26) yields

$$\tilde{u}_{eq,k} = \bar{B}_{r,k}^{\dagger}\left[-\bar{f}_{r,k} - \sigma_k + \bar{f}_{r,k-1} + \bar{B}_{r,k-1}\tilde{u}_{eq,k-1}\right] \quad (28)$$

In this case, the closed-loop system (23) and (26) becomes

$$\sigma_{k+1} = \delta_{r,k}, \qquad \delta_{r,k} = \bar{d}_{r,k} - \bar{d}_{r,k-1}. \quad (29)$$

Considering the control constraint (22), the control u_k is selected of the form

$$u_k = \begin{cases} \tilde{u}_{eq,k} & \text{for } \|\tilde{u}_{eq,k}\| \le u_{\max} \\ u_{\max}\frac{\tilde{u}_{eq,k}}{\|\tilde{u}_{eq,k}\|} & \text{for } \|\tilde{u}_{eq,k}\| > u_{\max}. \end{cases} \quad (30)$$

Now, a stability analysis is required for closed-loop system (23) and (30) motion over the manifold $\sigma_k = 0$. First, to reveal the structure of the modified equivalent control (26) and the system (23), let us represent them, by imposing the term $\sigma_k + \psi_{r,k} = 0$, $\psi_{r,k} = \psi_r(x_k)|_{x_k=\psi_r^{-1}(z_k)}$, into

$$\tilde{u}_{eq,k} = -\bar{B}_{r,k}^{\dagger}\sigma_k - \bar{B}_{r,k}^{\dagger}\left(\bar{\psi}_{r,k} + \bar{d}_{r,k-1}\right), \quad (31)$$
$$\sigma_{k+1} = \sigma_k + \bar{\psi}_{r,k} + \bar{B}_{r,k}u_k + \bar{d}_{r,k}. \quad (32)$$

where $\bar{\psi}_{r,k} = \bar{f}_{r,k} + \psi_{r,k}$.

For the case $\|\tilde{u}_{eq,k}\| \le u_{\max}$, the control (26) is applied, yielding motion in the neighborhood of the sliding manifold $\|\sigma_k\| \le \chi_s$ at finite time, where $\chi_s = \|\delta_{r,k}\|$. For the case $\|\tilde{u}_{eq,k}\| > u_{\max}$, it is assumed that the available control resources are sufficient to stabilize the system, i.e.

$$u_{\max} > \sup_k \left(\|\bar{B}_{r,k}^{\dagger}\|\|\bar{\psi}_{r,k} + \bar{d}_{r,k}\|\right). \quad (33)$$

This last assumption implies that the control magnitude is enough to guarantee the control objective requirements.

Substituting the proposed control (30) with (31) in (32) yields

$$\begin{aligned}\sigma_{k+1} =& \sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k} - \frac{u_{\max}(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})}{\|\bar{B}_{r,k}^{\dagger}(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|} \\ =& (\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k})\left(1 - \frac{u_{\max}}{\|\bar{B}_{r,k}^{\dagger}(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|}\right) \\ &+ \frac{u_{\max}\delta_{r,k}}{\|\bar{B}_{r,k}^{\dagger}\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1}\|}.\end{aligned}$$

Thus,

$$\begin{aligned}\|\sigma_{k+1}\| =& \|\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1} + \delta_{r,k}\| \times \\ & \left(1 - \frac{u_{\max}}{\|\bar{B}_{r,k}^{\dagger}(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|}\right) \\ & + \frac{u_{\max}\delta_{r,k}}{\|\bar{B}_{r,k}^{\dagger}(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|} \\ =& \|\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k}\| + \frac{u_{\max}\|\delta_{r,k}\|}{\|\bar{B}_{r,k}^{\dagger}\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1}\|} \\ & - \frac{u_{\max}\|\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1}\| + \|\delta_{r,k}\|}{\|\bar{B}_{r,k}^{\dagger}(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|} \\ \le& \|\sigma_k\| + \|\bar{\psi}_{r,k} + \bar{d}_{r,k}\| - \frac{u_{\max}}{\|\bar{B}_{r,k}^{\dagger}\|} \\ <& \|\sigma_k\|\end{aligned}$$

due to (33).

Thus, as $\|\sigma_k\|$ decreases monotonically, $\tilde{u}_{eq,k}$ (31) does too, and there will be a time $\bar{k}$ such that $\|\tilde{u}_{eq,k}\| \le u_{\max}$, for $k > \bar{k}$. At this time, the equivalent control $\tilde{u}_{eq,k}$ (28) is applied, bringing the closed-loop system trajectory in an $\mathcal{O}\left(\tau^2\right)$-neighborhood of the sliding manifold (Su et al., 2000), i.e.

$$\|\sigma_k\|| = \mathcal{O}(\tau^2), \ k > \bar{k}$$

achieving quasi-sliding mode.

5. SLIDING MODE DYNAMICS

Now, SM dynamics will be investigated, i.e. when the closed-loop system motion appears in the $\mathcal{O}(\tau^2)$ vicinity of the manifold $\sigma_k = z_{r,k} = 0$.

This motion is governed by the reduced order SM equation derived from (20) and (29) of the form

$$\begin{aligned} z_{1,k+1} &= K_1 z_{1,k} + z_{2,k} + \delta_{1,k} \\ &\vdots \\ z_{i,k+1} &= K_i z_{i,k} + z_{i+1,k} + \delta_{i,k} \\ &\vdots \\ z_{r-1,k+1} &= K_{r-1} z_{r-1,k} + \delta_{r-1,k} + \sigma_k \end{aligned}$$

or in compact form

$$\bar{z}_{r-1,k+1} = A_s \bar{z}_{r-1,k} + \delta_k \quad (34)$$

where

$$\begin{aligned}\bar{z}_{r-1,k} &= \left[z_{1,k}^{\top}\ z_{2,k}^{\top}\ \dots\ z_{r-1,k}^{\top}\right]^{\top}, \\ A_s &= \mathrm{diag}(K_1, \dots, K_{r-1}) + \mathbb{I}_a, \\ \mathbb{I}_a &= \mathrm{subdiag}(\mathbb{I}_{n_1}, \dots, \mathbb{I}_{n_{r-1}}), \\ \delta_k &= \left[\delta_{1,k}^{\top}, \dots, \delta_{i,k}^{\top}, \dots, (\delta_{r-1,k}^{\top} + \sigma_k)\right]^{\top}.\end{aligned}$$

Then, a solution of the system (34) is defined by

$$\bar{z}_{r-1,k} = A_s^k \bar{z}_{r-1,0} + \sum_{i=1}^{k-1} A_s^i \delta_{k-i-1}.$$

Since A_s is a Shur matrix, the steady state solution can be estimated by

$$\|\bar{z}_{r-1,k}\| \leq \sum_{i=1}^{k-1} \|A_s^i\| \|\delta_{k-i-1}\|. \quad (35)$$

From $\delta_{i,k} = \mathcal{O}(\tau^2),\ i = 1, ..., r-1$ and $\|\sigma_k\| = \mathcal{O}(\tau^2)$ it follows $\delta_k = \mathcal{O}(\tau^2)$. Selecting the poles of order $\mathcal{O}(1)$ of the matrix A_s, yields

$$\|\bar{z}_{r-1,k}\| = \mathcal{O}(1)\mathcal{O}(\tau^2) = \mathcal{O}(\tau^2), \quad (36)$$

resulting in the tracking error e_k (7) ultimate bound

$$\|e_k\| = \mathcal{O}(\tau^2). \quad (37)$$

The obtained results are formulated in the following theorem.

Theorem 1. Consider the robust tracking problem for the system (4) with the output (5) and the constraint control input (22). Let Assumption 1 with condition (33) are satisfied. Then, a solution of the system (4) closed by the control (30) with (28) is ultimately exponentially bounded by (36), and the tracking error e_k defined in (7) is ultimately bounded by (37).

6. SIMULATION RESULTS

To show the effectiveness of the proposed design, a simulation is carried out. Consider the magnetic levitation system drawn in Fig. 1, which consists of a steel ball suspended in a voltage-controlled magnetic field. The system input is the supplied voltage u, the ball's position is denoted by y; i is the current in the coil of the electromagnet, R is the coil's resistance, L is the coil's inductance, g_c is the gravitational constant, c is the magnetic force constant, and m is the mass of the levitated ball. Let the states be chosen such that $x_1 = y$, $x_2 = \dot{y}$, $x_3 = i$ and $x = [x_1\ x_2\ x_3]^\top$ is the state vector. The continuous-time dynamical model of the system is governed by (Koch et al., 2016)

$$\begin{aligned} \dot{x}_1 &= x_2 + d_1(t) \\ \dot{x}_2 &= g_c - \frac{c}{m}\frac{x_3^2}{x_1^2} + d_2(t) \\ \dot{x}_3 &= -\frac{R}{L}x_3 + \frac{2c}{L}\frac{x_2 x_3}{x_1^2} + \frac{1}{L}u + d_3(t) \\ y(t) &= x_1(t), \end{aligned} \quad (38)$$

with unmatched d_1, d_2 and matched d_3 perturbations.

Now, using the following local diffeomorphism:

$$\chi = [\chi_1,\ \chi_2,\ \chi_3]^\top = \left[x_1,\ x_2,\ g_c - \frac{c}{m}\left(\frac{x_3}{x_1}\right)^2\right]^\top, \quad (39)$$

the dynamical model (38) can be expressed in the new coordinates (39) as [1]

$$\begin{aligned} \dot{\chi}_1 &= \chi_2 + d_1(t) \\ \dot{\chi}_2 &= \chi_3 + d_2(t) \\ \dot{\chi}_3 &= f(\chi) + g(\chi)u + d_3(t) \\ y(t) &= \chi_1(t) \end{aligned} \quad (40)$$

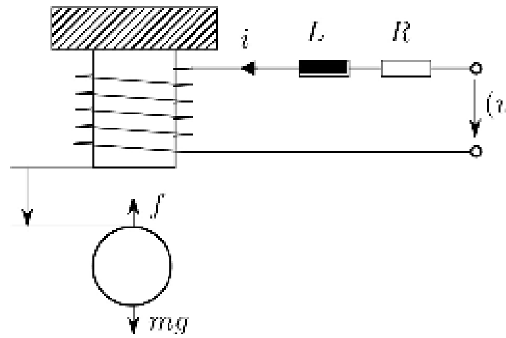

Fig. 1. Schematic diagram of the magnetic levitation system Koch et al. (2016).

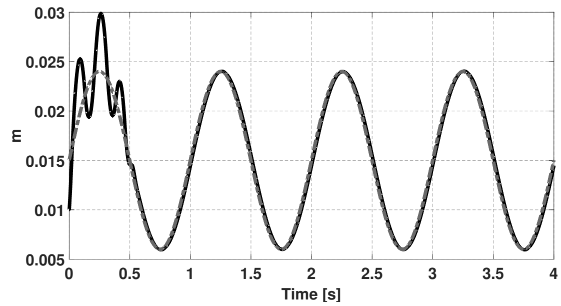

Fig. 2. Position response $x_{1,k}$ with proposed controller.

The system parameters are defined as $g_c = 9.8\text{m/s}^2$ $m = 66.87 \times 10^{-3}\text{kg}$, $c = 13.632 \times 10^{-5}\text{kg m s}^{-2}\text{ A}^{-2}$, $L = 1.08\text{H}$ and $R = 18\Omega$; while the perturbations are defined as $d_1(t) = 0.005\cos(12t) + 0.06\sin(4t) + 0.2$, $d_2(t) = 0.3\sin(30t) + 0.2\cos(10t) + 1$ and $d_3(t) = 2\cos(14t) + 0.5\sin(2t)$. The aim is to force the position ball y to track a reference signal despite to matched and unmatched perturbations. Applying the explicit Euler's method to discretize the system (40), yields

$$\begin{aligned} \chi_{1,k+1} &= \chi_{1,k} + \tau\chi_{2,k} + \tau d_{1,k} \\ \chi_{2,k+1} &= \chi_{2,k} + \tau\chi_{3,k} + \tau d_{2,k} \\ \chi_{3,k+1} &= \chi_{3,k} + \tau f(\chi_k) + g(\chi_k)u_k + \tau d_{3,k} \\ y_k &= \chi_{1,k} \end{aligned} \quad (41)$$

where τ is the sampling time, $\chi_{1,k}$, $\chi_{2,k}$ and $\chi_{3,k}$ are the discrete approximation of $\chi_1(t)$, $\chi_2(t)$ and $\chi_3(t)$, respectively. The sample perturbations are $d_{1,k} = d_1(k\tau)$, $d_{2,k} = d_2(k\tau)$ and $d_{3,k} = d_3(k\tau)$. The control gains are settled as $K_1 = 0.9$ and $K_2 = 0.7$, and the control input is saturated at $u_{\max} = 70\text{V}$. Initial conditions of the system are adjusted in $\chi_0 = [0.01\ 0\ 0.7]^\top$. The reference is stated as a time-variant signal to be tracked which is defined as $y^{ref} = \chi_1^{ref} = 0.015 + 0.009\sin(2\pi k\tau)$. Sample time is established as $\tau = 1\text{ms}$.

Fig. 2 shows the position response and the reference signal when the proposed controller is implemented. It can be seen that the controller drives the system output to track the desired value after the time 0.5s. Respectively,

[1] Due to space limitations, the functions $f(\chi)$ and $g(\chi)$ are omitted. For more details of those functions see Koch et al. (2016).

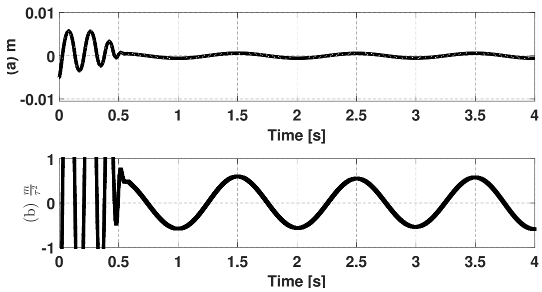

Fig. 3. (a) Position error $z_{1,k}$, (b) zoom of the same graphic on (a).

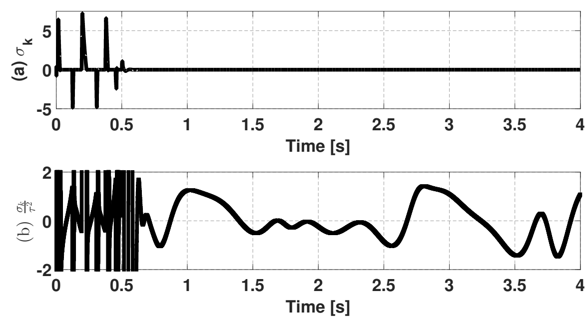

Fig. 4. (a) Sliding variable σ_k, (b) zoom of the same graphic on (a).

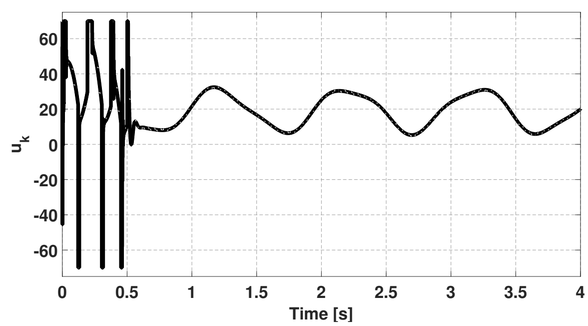

Fig. 5. Control input applied to the system.

position error $z_{1,k}$ is depicted in Fig. 3. Sliding variable $\sigma_k = z_{3,k}$ response is presented in Fig. 4. It can be seen that it remains in a boundary layer with thickness $\mathcal{O}(\tau^2)$; this result agrees with the theoretical design. Finally, Fig. 5 illustrates the control applied to the system, which stays in the control constraint (22) $|u_k| \leq 70$V.

7. CONCLUSIONS

A novel discrete-time sliding mode controller was proposed for a system in NBC form with both matched and unmatched perturbations. First, the sliding manifold design was formulated by using the BC iterative technique for this purpose. On the designed manifold, the influence of unmatched perturbation is reduced due to an approximation based on its previous step. Then, a DT SM controller based on the improved equivalent control is designed to achieve the desired manifold be attractive, and the effect of the matched perturbation is also reduced achieving $\mathcal{O}(\tau^2)$ order precision of the tracking error.

REFERENCES

Abidi, K., Xu, J.X., and Xinghuo, Y. (2007). On the discrete-time integral sliding-mode control. *IEEE Transactions on Automatic Control*, 52(4), 709–715.

Acary, V., Brogliato, B., and Orlov, Y.V. (2012). Chattering-free digital sliding-mode control with state observer and disturbance rejection. *IEEE Transactions on Automatic Control*, 57(5), 1087–1101.

Bartoszewicz, A. and Adamiak, K. (2018). Model reference discrete-time variable structure control. *International Journal of Adaptive Control and Signal Processing*, 32(10), 1440–1452.

Bartoszewicz, A. and Latosinski, P. (2016). Discrete time sliding mode control with reduced switching - a new reaching law approach. *International Journal of Robust and Nonlinear Control*, 26(1), 47–68.

Drakunov, S. and Utkin, V. (1989). On discrete-time sliding modes. *IFAC Proceedings Volumes*, 22(3), 273–278.

Huber, O., Acary, V., and Brogliato, B. (2016). Lyapunov stability and performance analysis of the implicit discrete sliding mode control. *IEEE Transactions on Automatic Control*, 61(10), 3016–3030.

Koch, S., Reichhartinger, M., Horn, M., and Fridman, L.M. (2016). Discrete implementation of sliding mode controllers satisfying accuracy level specifications. 2016 14th International Workshop on Variable Structure Systems (VSS), 154–159. Nanjing, China.

Levant, A. and Livne, M. (2015). Uncertain disturbances attenuation by homogeneous multi-input multi-output sliding mode control and its discretisation. *IET Control Theory Applications*, 9(4), 515–525.

Loukianov, A.G. (2002). Robust block decomposition sliding mode control design. *Mathematical Problems in Engineering*, 8(4-5), 349–365.

Mattioni, M., Monaco, S., and Normand-Cyrot, S.D. (2017). Lyapunov stabilization of discrete-time feedforward dynamics. IEEE 56th Annual Conference on Decision and Control (CDC), 4272–4277. Melbourne, Australia.

Monaco, S. and Normand-Cyrot, D. (1983). The immersion under feedback of a multidimensional discrete-time non-linear system into a linear system. *International Journal of Control*, 38(1), 245–261.

Su, W.C., Drakunov, S.V., and Özgüner, Ü. (2000). An $\mathcal{O}(t^2)$ boundary layer in sliding mode for sampled-data systems. *IEEE Transactions on Automatic Control*, 45(3), 482–485.

Utkin, V., Guldner, J., and Shi, J. (2009). *Sliding Mode Control in Electro-Mechanical Systems, Second Edition.* CRC Press, New York, United States. ISBN 1420065610.

Wang, B., Yu, X., and Chen, G. (2009). Zoh discretization effect on single-input sliding mode control systems with matched uncertainties. *Automatica*, 45(1), 118 – 125.

