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Stability of the memory state of a time-varying memristive neural network model ⋆

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Abstract: Memristors are resistive memory devices, where the resistive memory state is a function of the memristor's initial conditions and the history of the voltage across its terminals. Applications of these devices are in neuromorphic circuits. In particular, as representations of the open-close dynamics of the ionic channels in neurophysiological models. We use a memristive version of the Integrate and Fire neuron to construct a time-varying memristive neural network. In this model, a memory state is a stable unique equilibrium point. We show that the existence of a memory state depends uniformly on properties of the network topology and description of the memristive characteristic function. We illustrate our results using numerical simulations.

Keywords: Memristors, Neural network models, Neuron models, Resistive memory.

1. INTRODUCTION

The brain is capable of information integration and processing incoming from several different organs resulting in capabilities like memory and reasoning. The neuron is the basic processing unit of the brain, its behavior, and emergent properties as they are connected into networks are studied in many different ways. The neuron's electrical behavior is captured by the Hodgkin and Huxley model (HH) (Hodgkin & Huxley, 1949). In particular, the action potential phenomenon is the result of the physiological excitability of the ionic currents in the neuron's membrane. The so-called *Integrate and Fire* (IF) neural model is a simplified model that captures this phenomenon as a charge and discharge of a capacitor (Lapique, 1971).

The electrical representation of neural models required the use of time-varying conductance to model the opening and closing of ion channels in the membrane. No basic discrete electronic component had these features until in 1971 L. O. Chua theorized the existence of a fourth electric basic element called Memristor (Chua, 1971). That name is a contraction of words: resistance and memory.

The memristor is an electronic device characterized by a relation between its electric charge and its magnetic flux. Since these variables are related to the current and voltage across the device via its derivative, the resistance value of the memristor depends on the *history* of the voltage that passed through it, furthermore as the derivative goes to zero the resistance value is maintained in the device without new current been needed. As a result, *non-volatility* is a property of this resistive memory (Chua, 2011).

As presented in (Sah *et.al*, 2016), the memristor is a candidate to represent the time-varying conductances of the neural model (Hodgkin & Huxley, 1949). The possibility of implementing memristive circuits as a representation of biological neurons gives the interest in them as neuromorphic circuits (Chua *et.al*, 2012; Yang *et.al*, 2019). In particular, (Di Marco *et.al*, 2018) proposes a memristive version of the IF neural model.

As biological neurons communicate with each other through synapses, several memristive neurons can be coupled together into networks where dynamical phenomena can emerge. Yet, the dynamical behavior of memristive neurons, in particular Memristive IF neurons (MIFN), needs to be studied further.

An important feature of the MIFN model is that it has a continuum of equilibrium points. However, for a given fixed initial condition there exists only an equilibrium point and further this unique equilibrium point is stable, as such, in the sense of a memristor resistance, this unique stable equilibrium point is the memory state of the MIFN model (Di Marco *et.al*, 2018).

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This contribution aims to establish under what conditions a network on identical MIFN models with a time-varying coupling structure has a memory state. In other words, under what conditions, both in terms of the memristor description and the coupling structure of time-varying connections, of a memristor IF neural network (MIFNN) has a unique stable fixed point for a given fixed initial condition.

The remainder of the contribution is organized as follows: In Section 2, the basic aspects of memristor theory, and the memristive version of the IF neuron model are presented. In Section 3, the memory state problem for the time-varying MIFNN model is described in detail. Section 4 presents our main result, in the form of conditions for existence, uniqueness, and stability of an equilibrium point for the time-varying MIFNN. In Section 5 our results are illustrated with numerical simulations, then the contribution is finished with closing remarks.

2. PRELIMINARIES

2.1 Memristor

In (Chua, 1971) a memristor is defined as a basic electronic element that relates electric charge with magnetic flux, and represented as shown in Fig. 1.

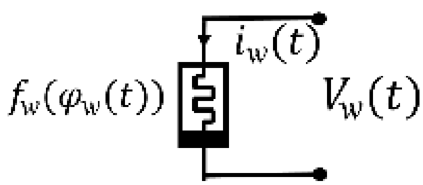

Fig. 1. Memristor's electronic symbol

The memristor fundamental function is defined as:

$$Q_w(t) = f_w(\varphi_w(t)) \tag{1}$$

where $Q_w(t) \in \mathbb{R}$ is the electric charge, and $\varphi_w(t) \in \mathbb{R}$ is the magnetic flux of the memristor. $f_w : \mathbb{R} \to \mathbb{R}$, $f_w(\cdot) \in C^1$ is the memristive characteristic function, it usually considered to be a monotonic increasing function.

We have voltage $V_w(t)$ and current $I_w(t)$ in the memristor are expressed as:

$$V_w(t) = \dot{\varphi}_w(t) \tag{2a}$$

$$I_w(t) = \dot{Q}_w(t) \tag{2b}$$

From the above, the current-voltage relation on the memristor is:

$$I_w(t) = w(\varphi_w)V_w(t) \tag{3}$$

where $w(\varphi_w) = \frac{df_w(\varphi_w)}{d\varphi_w}$ is the memductance of the memristor in Ω^{-1}. By integrating (2a) with respect to time, the magnetic flux $\varphi_w(t)$ is found to be:

$$\varphi_w(t) = \int_{t_0}^{t} V_w(\tau)d\tau + \varphi_w(t_0) \tag{4}$$

where $\varphi_w(t_0)$ is the initial magnetic flux.The magnetic flux described by (4) depends on the *history* of the memristor voltage $V_w(t)$, for this reason if the dynamics of the voltage and magnetic flux converge to a fixed value (Q_w^*, φ_w^*), furthermore if it is stable, one can call (Q_w^*, φ_w^*), the *memory state* of the memristor and $w(\varphi_w^*)$ its memductance.

In (Chua & Kang, 1976) is proposed a general mathematical description called *memristive system* described by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), v(t)) \\ i(t) &= h(\mathbf{x}(t))v(t) \end{aligned} \tag{5}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is vector of state variables of the memristive system, $v(t) \in \mathbb{R}$ is the input associated to the electrical variable voltage, $i(t) \in \mathbb{R}$ is the output associated to the electrical variable current, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in $\mathbb{R}^n \times \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

Memristive systems have been used to characterize the behavior of biological systems, in particular the behavior of time-varying conductances on neuron membrane models as can been seen in (Chua *et.al*, 2012).

2.2 Simplified neural models

There are several reduced models of the HH model (Hodgkin & Huxley, 1949). One of them is the IF (Lapique, 1971), in which electrical circuit representation is depicted in Fig. 2.

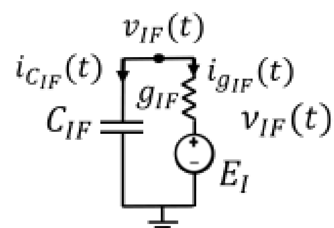

Fig. 2. IF neuron model

From Fig.2, the current $i_{C_{IF}}(t) \in \mathbb{R}$ of the capacitor is defined by:

$$i_{C_{IF}}(t) = C_{IF}\dot{v}_{IF}(t) \tag{6}$$

where $v_{IF}(t) \in \mathbb{R}$ is the voltage across the capacitor and C_{IF} is its capacitance, the current $i_{g_{IF}}(t)$ of the conductance g_{IF} is defined by:

$$i_{g_{IF}}(t) = g_{IF}(v_{IF}(t) - E_I) \tag{7}$$

The dynamical model of the circuit in Fig. 2 is obtained by the Kirchoff's Currents Law.

$$i_{C_{IF}}(t) + i_{g_{IF}}(t) = 0 \tag{8}$$

Substituting equations (6) and (7) in (8) is obtained:

$$C\dot{v}_{IF}(t) = -g_{IF}(v_{IF}(t) - E_I). \tag{9}$$

The IF neuron model is modified by including memristive elements in its description to generate the so-called MIFN model. In the following subsection, its electrical circuit and dynamical model are presented.

2.3 Memristive Integrate and Fire Neuron Model

Consider the Memristive Integrate and Fire neuron model as proposed in (Di Marco *et.al*, 2018) consisting of a memristor M connected in parallel to a capacitor C as represented in Fig. 3.

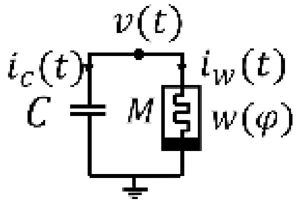

Fig. 3. Electrical circuit of the MIFN model

From Fig. 3, the current $i_c(t) \in \mathbb{R}$ of the capacitor is defined by:

$$i_c(t) = C\dot{v_C}(t) \tag{10}$$

where $v_C(t) \in \mathbb{R}$ is the capacitor voltage and C is its capacitance. The memristor current $i_w(t)$ is defined by:

$$i_w(t) = w(\varphi(t))v_M(t) \tag{11}$$

as before $w(\varphi)$ is the memductance of the memristor. According to Kirchhoff's Voltage Law $v_C(t) - v_M(t) = 0$, therefore $v_C(t) = v_M(t) = v(t)$. The dynamical model of the circuit in Fig. 3 is obtained by the Kirchhoff's Current Law.

$$i_c(t) + i_w(t) = 0 \tag{12}$$

Substituting equations (10) and (11) in (12) and recalling (2a), the following MIFN model equations are obtained:

$$C\dot{v}(t) = -w(\varphi(t))v(t) \tag{13a}$$

$$\dot{\varphi}(t) = v(t) \tag{13b}$$

with initial conditions $\varphi(t_0) = \varphi_0$ and $v(t_0) = v_0$.

The equilibrium points of (13a)-(13b) are a continuum given by

$$\pi_e = \{[0, \varphi]^\top \in \mathbb{R}^2 : \varphi \in \mathbb{R}\}. \tag{14}$$

Is important to note that for a given fixed initial condition there exits a unique equilibrium point $(0, \varphi_e) \in \mathbb{R}^2$ for (13). Furthermore, this equilibrium point if stable, is the *memory state* of the MIFN model.

In the following section, the MIFNN model is presented and investigated under the consideration of time-varying couplings.

3. TIME-VARYING MIFNN MODEL

Consider a set of N identical MIFN (13a-b) called *nodes* $\mathcal{M} = \{m_1, ..., m_N\}$. Where each node has unitary capacitances $C_1 = .. = C_N = 1$ and identical characteristic memristive functions $f_{w_1}(\cdot) = ... = f_{w_N}(\cdot) = f_w(\cdot)$. Therefore, all nodes have identical memductance $w(\cdot) = w_1(\cdot) = ... = w_N(\cdot)$.

If node m_i is connected to node m_j by a *fixed* edge $s_{ij} \in \mathcal{S} \subset \mathcal{M} \times \mathcal{M}$ where $i \neq j$, then $a_{ij} = 1$. Alternatively, if these nodes are not connected $a_{ij} = 0$. Since the edges are undirected, $a_{ij} = a_{ji}\ \forall i, j$ and there are no isolated nodes in the network. The coupling structure is given by the adjacency matrix $A = \{a_{ij}\} \in \mathbb{R}^{N\times N}$.

Let every edge has an associated *time-dependent* connection weight given by a function $c_{ij} : \mathbb{R}_+ \to \mathbb{R}_{>0}$ locally Lipschitz in $\mathbb{R}_+$. The time-varying Laplacian matrix $\mathcal{L}(t) = (\ell_{ij}(t))_{N\times N}$ associated to this connection topology is given by:

$$\ell_{ij}(t) = \begin{cases} \sum_{k=1,k\neq i}^{N} a_{ik}c_{ik}(t), & i = j \\ -a_{ij}c_{ij}(t), & i \neq j \end{cases} \tag{15}$$

The dynamics of the $i-$th node is given by:

$$\dot{v_i}(t) = -w(\varphi_i(t))v_i(t) - \sum_{j=1}^{N} \ell_{ij}(t)v_j(t) \tag{16a}$$

$$\dot{\varphi}_i(t) = v_i(t) \tag{16b}$$

for $i = 1, 2, \cdots, N$.

A vectorial form of (16) is:

$$\dot{V}(t) = -W(\phi(t))V(t) - \mathcal{L}(t)V(t) \tag{17a}$$

$$\dot{\phi}(t) = V(t) \tag{17b}$$

where $V(t) = [v_1(t), ..., v_N(t)]^\top \in \mathbb{R}^N$ and $\phi(t) = [\varphi_1(t), ..., \varphi_N(t)]^\top \in \mathbb{R}^N$ are the voltage and magnetic flux vectors, respectively. With the memductance matrix $W(\phi(t)) = \text{diag}(w(\varphi_1(t)), .., w(\varphi_N(t))) \in \mathbb{R}^{N\times N}$.

From (15) we have that the time-varying Laplacian matrix of the MIFNN is uniformly diffusive, that is, the sum by row and by columns is zero at all times. As a consequence the eigenvalues of $\mathcal{L}(t)$, denoted as $\lambda_i(t)$ $(i = 1, \cdots, N)$ can be arrange as (Lü & Chen, 2005):

$$\lambda_N(t) \geq \cdots \lambda_2(t) \geq \lambda_1 = 0 \tag{18}$$

In other words, the coupling of the MIFNN is captured by a Laplacian matrix that is semipossitive at each time instant.

As before, for (16a)-(16b) there is a continuum of the equilibrium points defined by the set

$$\beta_e = \{[0, \phi]^\top \in \mathbb{R}^{2N} : \phi \in \mathbb{R}^N\}. \tag{19}$$

The equilibrium point $[0, \phi^*]^\top \in \beta_e$ is a memory state of the MIFNN model, if for a given fixed initial condition $[V_0, \phi_0]^\top \in \mathbb{R}^{2N}$, $[0, \phi^*]^\top$ is unique and stable equilibrium point to which the network model converges, as shown in the following section, where we determine the conditions for the existence and stability of a unique equilibrium point of (16a-b) for a given fixed initial condition.

$$\mathcal{L}(t) = \begin{pmatrix} \ell_{11}(t) & -1.5 & -6 & -0.5+\exp(2-0.1t) & -0.5-t\exp(-0.1t) & -0.1t^2 \\ -1.5 & \ell_{22}(t) & -\exp(0.1t-6) & -6\cos^2(t)\sin(0.1\pi t)-7 & -\exp(-0.5t) & -\exp(0.1t) \\ -6 & -\exp(0.1t-6) & \ell_{33}(t) & -0.1t & -\cos(t)-1.1 & -\arctan(t) \\ -0.5-\exp(2-0.1t) & -6\cos^2(t)\sin(0.1\pi t)-7 & -0.1t & \ell_{44}(t) & -10 & -3 \\ -0.5-t\exp(-0.1t) & -\exp(-0.5t) & -\cos(t)-1.1 & -10 & \ell_{55}(t) & -\sin(\pi t)-1.1 \\ -0.1t^2 & -\exp(0.1t) & -\arctan(t) & -3 & -\sin(\pi t)-1.1 & \ell_{66}(t) \end{pmatrix} \tag{20}$$

$$\begin{aligned} \ell_{11}(t) &= 1.5+6+0.5+\exp(2-0.1t)+0.5+t\exp(-0.1t)+0.1t^2 \\ \ell_{22}(t) &= 1.5+6\cos^2(t)\sin(0.1\pi t)+\exp(0.1t-6)+7+\exp(-0.5t)+\exp(0.1t) \\ \ell_{33}(t) &= 6+\exp(0.1t-6)+0.1t+\cos(t)+1.1+\arctan(t) \\ \ell_{44}(t) &= 0.5+\exp(2-0.1t)+6\cos^2(t)\sin(0.1\pi t)+7+0.1t+10+3 \\ \ell_{55}(t) &= 0.5+t\exp(-0.1t)+\exp(-0.5t)+\cos(t)+1.1+10+\sin(\pi t)+1.1 \\ \ell_{66}(t) &= 0.1t^2+\exp(0.1t)+\arctan(t)+3+\sin(\pi t)-1.1 \end{aligned} \tag{21}$$

4. MAIN RESULTS

To establish the existence and stability of a unique equilibrium point for the MIFNN model given a fixed initial condition $[V_0, \phi_0]^\top$, we start considering the voltage equation (16a).

Assuming:

(1) There exists a solution to the magnetic flux equation (16b) which is unique and continuous on $\mathbb{R}$ for each node i.
(2) The memristive characteristic $f_w(\cdot)$ of the neurons is a monotonic and strictly increasing function.

The voltage equation of the MIFNN model (17a) can be written as

$$\dot{V}(t) = -B(t)V(t) \tag{22}$$

where $B(t) = W(\phi(t)) + \mathcal{L}(t)$,

We have the following results:

Theorem 1. The voltage equation (22) has a unique equilibrium point given by

$$V^* = 0 \in \mathbb{R}^N \tag{23}$$

Proof 1. Given that the $\mathcal{L}(t)$ is a positive semidefinite matrix for all time instants, and $W(\phi(t))$ is positive definite due to assumption (2). Their sum is positive definite $\forall t$, and as a consequence $B(t)$ is a non-singular matrix $\forall t$, therefore get that $V^* = 0$ is the only solution to its equilibrium point algebraic equation. □

From the above result we can derive the following:

Theorem 2. Under assumptions (1) and (2) the unique equilibrium point $V^* = 0 \in \mathbb{R}^N$ is uniformly asymptotically stable.

Proof 2. To establish the stability of $V^* = 0$ consider the Lyapunov candidate function, $E(V(t)) = \frac{1}{2}V^\top(t)PV(t)$ with P a constant symmetric and positive definite matrix of appropriate dimensions. Its derivative on the trajectories of (22) is given by

$$\dot{E}(V(t)) = -V^\top(t)\left[PB(t)\right]V(t) \tag{24}$$

If $\exists Q \in \mathbb{R}^{N\times N}$ definite positive for all time, such that $PB(t) > -Q$, would imply that $\dot{E}(V(t)) < 0$, $\forall t$, that is, $V^* = 0$ is uniformly asymptotically stable equilibrium point of (17a). □

As a consequence of Theorem 2 we know that the right side of (17b) will converge to zero. Then, we have the following result:

Corollary 2.1. Under assumptions of theorem 2. If $B(t)$ is positive definite $\forall t$, then the solution of the magnetic flux equation of the MIFNN model (17b) will converge asymptotically to a fixed value $\phi^* \in \mathbb{R}^N$. Furthermore,

$$\phi^* = \lim_{t\to\infty} \int_{t_0}^{t} V(\tau)d\tau + \phi_0 \tag{25}$$

Proof 3. Integrating both sides of (17b) we have

$$\phi(t) = \int_{t_0}^{t} V(\tau)d\tau + \phi_0$$

From the result in Theorem 2 we have that for a sufficiently large T, $V(T) = 0$, regardless of the initial condition $[V_0, \phi_0]^\top \in \mathbb{R}^{2N}$, and given that $V(t)$ is a unique and continuous function, the limit in (25) exists and is a unique fixed value that depends on the history of the voltage across the memristive neurons and the initial conditions.

Finally, combining the above results we have

Theorem 3. Under assumptions of theorem 2, for a given fixed initial condition the time-varying MIFNN model (17a-b) has a unique equilibrium point $[0, \phi^*]^\top \in \mathbb{R}^{2N}$ and it is uniformly asymptotically stable.

Proof 4. It follows from the previous results.

5. SIMULATION EXAMPLE

In this numerical analysis, we illustrate Theorems 2 and 3, by constructing six node network as described in (17a)-(17b), where every node is connected to its five neighbor nodes, the network topology consists of a six-node fully connected network, where its time-varying Laplacian is described in equation (20). Let the initial conditions be:

$$V_0 = [-5.5, 4.5, -3.1, 6.3, -2.2, 5.2]^\top \tag{26}$$

$$\phi_0 = [1, -1, -2, -1.4, -1.6, 3]^\top \tag{27}$$

Let the memductance matrix be:
$W(\phi) = \mathrm{diag}(w(\varphi_1), \cdots, w(\varphi_6)) \in \mathbb{R}^{6\times 6}$, where $w(\varphi_i) = \frac{df_w(\varphi_i)}{d\varphi_i}$ is the memductance function and $f_w(\varphi_i)$ the memristive characteristic function described by:

$$f_w(\varphi_i) = \begin{cases} 0.1\varphi_i - 4, & \varphi_i \leq -2 \\ 2.1\varphi_i, & -2 < \varphi_i < 2 \\ 0.1\varphi_i + 4, & \varphi_i \leq 2 \end{cases} \quad (28)$$

Therefore, assumption (2) is satisfied, and we proceed to solve the system of the six node network, described above via numerical integration using Matlab® software, through *Runge-Kutta* method. First is verified Theorem 2, that is, asymptotic convergence of nodes voltages towards zero solution as shown in Fig. 4.

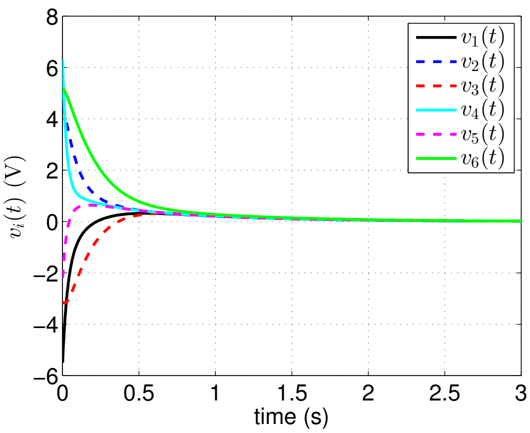

Fig. 4. Numerical integration of the MIFNN network described by (17a-b). Plot of node voltages $V(t) = [v_1(t), v_2(t), v_3(t), v_4(t), v_5(t), v_6(t)]$

Subsequently, to verify the results of Corollary 2.1, that is, the asymptotic convergence of nodes magnetic fluxes towards different constant values, dependent on the initial conditions $[V_0, \phi_0]^\top \in \mathbb{R}^{12}$, as shown in Fig. 5.

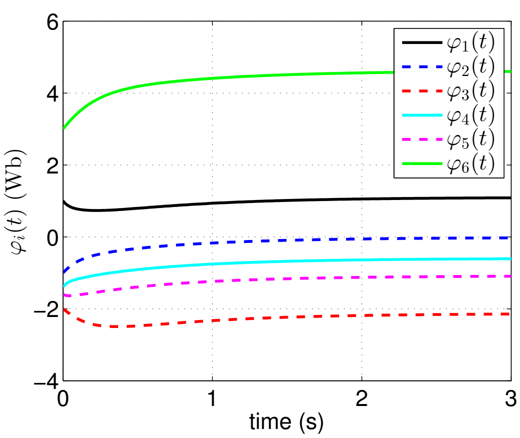

Fig. 5. Numerical integration of the six nodes network described by (17a-b). Plot of nodes magnetic fluxes $\phi(t) = [\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t), \varphi_5(t), \varphi_6(t)]$

As shown in Fig. 5 the corresponding equilibrium point ϕ^* as described in equation (25) in Corollary 2.1 is:

$$\phi^* = [1.1002, -0.0152, -2.1316, -0.5967, -1.0827, 4.6169]^\top \quad (29)$$

As a result of the numerical analysis performed in Fig. 4 and Fig.5 is verified Theorem 3, in which is shown that for the initial condition (26), the equilibrium point $[0, \phi^*]^\top$, where ϕ^* is given in (29), is an asymptotic stable equilibrium point of (17a-b).

6. DISCUSSION OF RESULTS

In this contribution, we proposed a time-varying MIFNN model and derived simple conditions to establish the existence of a memory state, e.g. a unique stable equilibrium point for each initial condition. The conditions for the existence of a memory state are the increasing monotonicity of the memristive characteristic function and uniform dissipation of the time-varying Laplacian matrix to describe the neuron connections. Further, we show using the Lyapunov approach that the voltage equation of the MIFNN model converges to the zero solution as its only equilibrium point and that is uniformly asymptotically stable. As a consequence of this, for a given initial condition the entire time-varying network has a unique stable equilibrium point, which represents the network's memory of its initial conditions.

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