On the pinning synchronization of switching networks ⋆

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Abstract: We investigate the pinning synchronization problem for networks that arbitrarily switch between a set of admissible connection topologies. We consider that both connections and controller gains can arbitrarily switch, then we investigate three scenarios to achieve network synchronization. In the first case, both topology and controller arbitrarily change at different times. In the second case, both switch at the same time. While in the third, the controller remains fixed for all time. In all scenarios, the controller is applied only to the node with the largest node degree of each admissible topology. Then, we can easily compare the control requirements of the different scenarios. We find that a fixed controller requires a smaller control gain than an arbitrary switching gain. However, if both topology and gains commute at the same time, it is possible to use even smaller control gains since one can use the minimum for each admissible topology. We illustrate our results with numerical simulation of switching networks of Lorenz systems.

Keywords: Pinning control, Synchronization, Dynamical networks, Switching dynamics.

1. INTRODUCTION

The emergence of synchronized behavior is one of the most studied phenomena in dynamical networks (Boccaletti *etal.*, 2006; Chen, 2014). Two or more systems are said to be synchronized if their states become coordinated in time. That is the stability of a particular solution of the network where the state of the nodes coincides determines whether or not a network achieves synchronization (Chen *etal.*, 2014). Although synchronization can sometimes emerge naturally, there are situations where is necessary to design a controller to achieve the desired synchronized behavior, this approach is usually called controlled synchronization (Blekhman *etal.*, 1997). A convenient way to control networked systems is the so-called pinning control approach. That is, one designs local controllers for only a very small number of nodes, this control action drives the entire network to the desired behavior, which usually is a common equilibrium point of the network (Li *etal.*, 2004; Chen *etal*, 2007). Pinning control has been utilized to impose a synchronized behavior on a network using different design techniques including neural networks and adaptive controllers (Yu *etal.*, 2009; Zhou *etal.*, 2008; Vega *etal.*, 2020).

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Most of the investigations discussed above only consider networks where the connection topology is fixed. However, in many real-world situations, the coupling structure of a network changes for different reasons. In particular, the interaction between a given pair of nodes may change abruptly at some specific time instant, which results in a commutation between two distinct connection topologies. In a switching dynamical network model, there is only one active connection topology at each time instant, which is chosen from a set of admissible connections. In this model, the switching law is a piecewise constant and continuous from the right function that indicates arbitrarily which of the admissible topologies is active at a given time instant (Xiao & Wang, 2006; Hill & Chen, 2006).

In the literature there are different approaches to establish the stability of a synchronized solution for this type of switched system, including the common Lyapunov function, dwell time, and multiple Lyapunov functions (Liberzon, 2003; Yao *etal.*, 2006; Liu *etal.*, 2010; Wang & Wang, 2011; Zhao *etal*, 2011). It is worth noting that in (Kim & Hill, 2008), the problem is addressed using a linearized analysis under the assumption that all the outer coupling matrices are simultaneously triangularizable. While in (Zhao *etal.*, 2009), the synchronization problem was studied with switched coupling using the average dwell time method. In (Du *etal.*, 2015) pinning synchronization of switching networks was investigated assuming that both topology and control gains switch at the same time, and a synchronization condition was derived by choosing an appropriate switching law.

In this contribution, we consider three scenarios. A first one, both the controller and the topology switch arbitrarily at different times. In our second scenario, both switch at the same time. While in the final scenario, the controller remains fixed. Considering that the node dynamics are bounded by their error and that all admissible topologies are diffusive. We propose a solution to the pinning synchronization problem for the switching network where the controller is only applied to a single node with the largest node degree in each admissible topology. For this simplified design, we compare the control requirements of the different scenarios.

In the following section, the pinning synchronization problem for switching networks is described in detail. While our proposed solution is presented in Section 3. Our results are illustrated with numerical simulations in Section 4. Then, the contribution is closed with some comments and conclusions.

2. PRELIMINARIES

Consider a network of N diffusely coupled identical n-dimensional dynamical systems described by

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} \ell_{ij}^{\sigma(t)} \Gamma x_j(t) + u_i(t), \tag{1}$$

for $i = 1, 2, \ldots, N$ where $x_i(t) = [x_{i1}(t), \ldots, x_{in}(t)]^\top \in \mathbf{R}^n$ and $u_i(t) \in \mathbf{R}^n$ are the state variable and controller of the i-th node, respectively; $f(\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear Lipschitz function that describes the dynamics of an isolated node, to have a solution for all nodes in the network. We also assume that it satisfies the following inequality:

$$[x-y]^\top (f(x) - f(y)) \leq [x-y]^\top \epsilon [x-y] \tag{2}$$

with $x, y \in \mathbf{R}^N$ and $\epsilon \in \mathbf{R}$.

The internal coupling is described by the matrix $\Gamma = diag(\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbf{R}^{n\times n}$, with $\gamma_i = 1$ if the nodes are coupled through their i-th state and $\gamma_i = 0$ otherwise. The uniform coupling strength is given by $c \in \mathbf{R}$. The connection between nodes is described by the external coupling matrix $L^{\sigma(t)} = (\ell_{ij}^{\sigma(t)}) \in \mathbf{R}^{N\times N}$ where $\ell_{ij}^{\sigma(t)} = 1$ if the i-th and j-th nodes are coupled at time t, with $\ell_{ij}^{\sigma(t)} = 0$ if they are not connected. At each time instant the outer coupling matrix $L^{\sigma(t)}$ is one of a set of admissible Laplacian matrices

$$\mathcal{M} = \{L^{\sigma(t)} = L^\alpha \in \mathbf{R}^{N\times N}, \alpha \in M\} \tag{3}$$

with $M = \{1, 2, \ldots, m\}$.

The selection of the currently active Laplacian matrix is given by the switching law:

$$\sigma(t) : [0, \infty) \to M \tag{4}$$

which is a piecewise constant and continuous from the right function (Kim & Hill, 2008).

Is important to remark that the external coupling topology of (1) is given by a single Laplacian matrix at each time instant which changes arbitrarily from one admissible Laplacian to any other in $\mathcal{M}$. Additionally, the switching network is diffusive and connected at all times, that is, for any value of $\sigma(t)$ the sum of by row and by columns of the entries is the active Laplacian matrix is zero, and the following equation is satisfied (Du *etal.*, 2015):

$$\ell_{ii}^{\sigma(t)} = -\sum_{i=1, i\neq j}^{N} \ell_{ij}^{\sigma(t)}, \quad i = 1, 2, \cdots, N. \tag{5}$$

Therefore, the eigenspectrum of every admissible Laplacian matrix in $\mathcal{M}$ can be order as:

$$0 = \lambda_1^{\sigma(t)} > \lambda_2^{\sigma(t)} \leq \cdots \leq \lambda_N^{\sigma(t)} \tag{6}$$

with $\lambda_i^{\sigma(t)}$ the i-th eigenvalue of $L^{\sigma(t)}$.

The network in (1) asymptotically becomes synchronized if all nodes evolve towards the same dynamics. That is, as $t \to \infty$ all nodes move towards a synchronized state

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t) \tag{7}$$

where $s(t) \in \mathbf{R}^n$ is the solution of an isolated node; that is,

$$\dot{s}(t) = f(s(t)). \tag{8}$$

Let the controller $u_i(t)$ be a linear feedback controller of the form

$$u_i(t) = -c\kappa_i^{\nu_i(t)} \Gamma (x_i(t) - s(t)) \tag{9}$$

where $\kappa_i^{\nu_i(t)}$ is one of a set of controller gains to be designed

$$\mathcal{N}_i = \{\kappa_i^{\nu_i(t)} = \kappa_i^\beta > 0 \in \mathbf{R}, \beta \in \bar{M}_i\} \tag{10}$$

with $\bar{M}_i = \{1, 2, \ldots, \bar{m}_i\}$.

The corresponding control gain is assigned according to the controller switching law

$$\nu_i(t) : [0, \infty) \to \bar{M}_i \tag{11}$$

which is also a piecewise constant and continuous from the right function.

We assume that the controller (9) is applied only to a small fraction δ ($\delta \ll N$) of the nodes. That is, we use a pinning controller approach. Therefore, synchronization of the switching network becomes a problem in which the number of nodes to control (δ), the controller gains ($\mathcal{N}_i$), and its corresponding switching laws ($\nu_i(t)$) need to be designed such that the closed-loop network (1)-(9) has (7) as its asymptotically stable solution.

Without lost of generality, we assume that the first δ nodes in (1) are controlled with (9), resulting in:

$$\begin{aligned}\dot{x}_i(t) &= f(x_i(t)) + c\sum_{j=1}^{N}\ell_{ij}^{\sigma(t)}\Gamma x_j(t)\\ &\quad -c\kappa_i^{\nu_i(t)}\Gamma(x_i(t)-s(t))\\ &\qquad i=1,2,\ldots,\delta\\ \dot{x}_i(t) &= f(x_i(t)) + c\sum_{j=1}^{N}\ell_{ij}^{\sigma(t)}\Gamma x_j(t)\\ &\qquad i=\delta+1,\delta+2,\ldots,N.\end{aligned} \tag{12}$$

In the following section, we proposed designs for the pinning controller under different switching law scenarios, in which the design is a compromise between the number of nodes to control (δ) and their gains ($\kappa_i^{\nu_i(t)}$), in order to have $\mathbf{S}(t) = [s(t)^\top, s(t)^\top, \ldots, s(t)^\top]^\top \in \mathbf{R}^{Nn}$; as a stable solution of (12).

3. PINNING DESIGNS FOR SYNCHRONIZATION

Lets define the synchronization errors as $e_i(t) = x_i(t) - s(t)$, then from (12) and (8) the error dynamics are given by

$$\dot{e}_i(t) = \begin{cases}\bar{f}(e_i(t)) + c\sum_{j=1}^{N}\ell_{ij}^{\sigma(t)}\Gamma e_j(t) - c\kappa_i^{\nu_i(t)}\Gamma e_i(t)\\ \qquad i=1,2,\ldots,\delta.\\ \bar{f}(e_i(t)) + c\sum_{j=1}^{N}\ell_{ij}^{\sigma(t)}\Gamma e_j(t)\\ \qquad i=\delta+1,\delta+2,\ldots,N.\end{cases} \tag{13}$$

where $\bar{f}(e_i(t)) = f(x_i(t)) - f(s(t))$. Which in vector form can be rewritten as:

$$\dot{\mathbf{E}}(t) = \bar{\mathbf{f}}(\mathbf{E}(t)) + c([L^{\sigma(t)} - K^{\nu(t)}]\otimes\Gamma)\mathbf{E}(t) \tag{14}$$

where $\mathbf{E}(t) = [e_1(t)^\top, \ldots, e_N(t)^\top]^\top \in \mathbf{R}^{Nn}$, $\bar{\mathbf{f}}(\cdot) = [\bar{f}(e_1(t))^\top, \ldots, \bar{f}(e_N(t))^\top]^\top \in \mathbf{R}^{Nn}$, with $K^{\nu(t)} = diag([\kappa_1^{\nu_1(t)}, \ldots, \kappa_\delta^{\nu_\delta(t)}, 0, \cdots, 0]) \in \mathbf{R}^{N\times N}$ and $\otimes$ represents the Kronecker product.

The stability of the null equilibrium point of error dynamics ($\bar{\mathbf{E}} = 0$) can be establish using the common Lyapunov approach (Liberzon, 2003), with the following Lyapunov candidate function

$$V(\mathbf{E}(t)) = \frac{1}{2}\sum_{i=1}^{N} e_i(t)^\top e_i(t). \tag{15}$$

The time derivative of $V(\mathbf{E}(t))$ along the trajectories of (13) is given by

$$\dot{V}(\mathbf{E}(t)) = \sum_{i=1}^{N} e_i(t)^\top \begin{cases}\bar{f}(e_i(t)) + c\sum_{j=1}^{N}\ell_{ij}^{\sigma(t)}\Gamma e_j(t)\\ \quad -c\kappa_i^{\nu_i(t)}\Gamma e_i(t)\\ \quad i=1,\ldots,\delta.\\ \bar{f}(e_i(t)) + c\sum_{j=1}^{N}\ell_{ij}^{\sigma(t)}\Gamma e_j(t)\\ \quad i=\delta+1,\ldots,N.\end{cases}$$

which in vector form becomes

$$\begin{aligned}\dot{V}(\mathbf{E}(t)) &= \mathbf{E}(t)^\top\bar{\mathbf{f}}(\mathbf{E}(t)) + \\ &\quad c\mathbf{E}(t)^\top([L^{\sigma(t)} - K^{\nu(t)}]\otimes\Gamma)\mathbf{E}(t)\end{aligned}$$

Using the conditions (2) we have the following inequality

$$\dot{V}(\mathbf{E}(t)) \le \mathbf{E}(t)^\top Q\mathbf{E}(t) \tag{16}$$

where $Q = \epsilon I_{Nn} + c[L^{\sigma(t)} - K^{\nu(t)}]\otimes\Gamma$.

The network asymptotically achieves synchronization if the matrix Q is negative definitive for all possible combinations of the switching laws $\sigma(t)$ and $\nu_i(t)$.

In this contribution we consider the following scenarios:

I. The controller gains $\kappa_i^{\nu_i(t)}$ and the Laplacian matrix $L^{\sigma(t)}$, have different switching laws, that is, $\exists t$, such that $\sigma(t) \neq \nu_i(t)$ for at least one i, with $i \in \{1,2,...,\delta\}$.
II. The controller gains $\kappa_i^{\nu_i(t)}$ and the Laplacian matrix $L^{\sigma(t)}$, all switch at the same time, that is, $\nu_i(t) = \sigma(t), \forall t$ and $\forall i \in \{1,2,...,\delta\}$.
III. All controller gains are identical for all time, that is $\kappa_i^{\nu_i(t)} = \kappa$, $\forall t$ and $\forall i \in \{1,2,...,\delta\}$.

Lets start by considering the first scenario, in it the resulting Q matrix has a large number of possible components, since it can be different for every value of $\sigma(t)$ and $\kappa_i^{\nu_i(t)}$ with $i \in \{1,2,...,\delta\}$.

From the definition above, we have that $L^{\sigma(t)}$ is always negative semidefinite with only a single zero eigenvalue, while the diagonal controller matrix $K^{\nu_i(t)}$ is positive semidefinite with $\delta \geq 1$ nonzero diagonal elements. It follows that the matrices $L^{\sigma(t)} - K^{\nu(t)}$ are negative definitive. Then Q becomes negative definite if c is large enough to overtake ϵI_{Nn}. In that case, the switching network (12) achieves synchronization.

Notice that in most of the previous results in the literature it is assumed that both the topology and all controllers switch at the same time (Du *etal.*, 2015). Usually, a further requirement is that all admissible Laplacian matrices be simultaneously triangularizable so that a local linear approach can be utilized for the switching system (Yao *etal.*, 2006; Zhao *etal.*, 2009).

For Q to become negative definite there is a compromise between the number of nodes to control and their corresponding gains. In order to establish an appropriate value for δ, we consider that as shown in (Chen *etal*, 2007), it is sufficient to have a single controller ($\delta = 1$) to make $L^{\sigma(t)} - K^{\nu(t)}$ negative definitive, yet one must have a sufficiently large value of $\kappa_1^{\nu_1(t)}$. Then, even for a fixed $c > 0$ we get Q to be definitive negative.

In the following section we use a single controller with both fixed and switching gains as described above.

4. NUMERICAL SIMULATIONS

Let us consider a switching network (1) where each node is a Lorenz system of the form

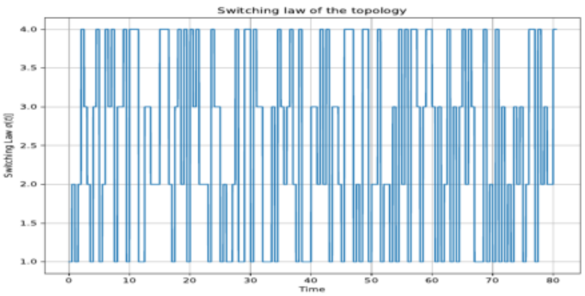

Fig. 1. A time evolution of the arbitrary switching law $\sigma(t)$ (4).

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} p_1(x_1(t) - x_2(t)) \\ (p_2 - x_3(t))x_1(t) - x_2(t) \\ x_1(t)x_2(t) - p_3x_3(t) \end{bmatrix} \tag{17}$$

Which is chaotic for the parameter values $p_1 = 10$, $p_2 = 28$, $p_3 = \frac{8}{3}$.

We consider a switching network with fifteen nodes connected in four different admissible topologies $\mathcal{M} = \{L^1, L^2, L^3, L^4\}$ with a switching law $\sigma(t) : (0, \infty] \to \{1, 2, 3, 4\}$ which commutes arbitrarily as shown in Figure 1.

The objective is to design the controller gains in (9) such that synchronization is achieved. The resulting pinned controlled switching network is (12) with a fixed uniform coupling strength $c = 1.0$, the inner connection matrix $\Gamma = diag([1, 0, 0])$, and the single controller at the node with the largest node degree ($\delta = 1$).

For scenario I. As presented above ($\nu_1(t) \neq \sigma(t)$) the switching law of the controller is $\nu(t) : (0, \infty] \to \{1, 2, 3\}$, which also commutes arbitrarily between the gains. In our numerical simulation, for the initial forty units of time ($t < 40$) $\kappa_1^1 = \kappa_1^2 = \kappa_1^3 = 0\}$. After that time, the controller gain switches between $\{\kappa_1^1 = 275, \kappa_1^2 = 250, \kappa_1^3 = 175\}$, which are chosen such that $L^{\sigma(t)} - K^{\nu(t)}$ for becomes negative definite in all possible combinations. That is, even for the admissible Laplacian matrix with the smallest $\lambda_2^{\sigma(t)}$ eigenvalue the matrix $K^{\nu(t)} = diag([175, 0, \cdots, 0])$ is sufficient to make $\epsilon I_{Nn} + c[L^{\sigma(t)} - K^{\nu(t)}] \otimes \Gamma$ a negative definite matrix. As shown in Figure 2, once the controller gains are activated, the states of the network converge to the synchronized solution describe by (7) and (8).

For scenario II. The controller switches at the same time as the topology ($\nu_1(t) = \sigma(t)$). In this case, the controller gains can be chosen as the minimum required for each admissible Laplacian matrix. In this case, the smallest $\lambda_2^{\sigma(t)}$ requires $\kappa_1^{\sigma(t)} = 175$, while for the other values of $\sigma(t)$ it can be smaller. In this case, for the largest $\lambda_2^{\sigma(t)}$ is sufficient to have $\kappa_1^{\sigma(t)} = 100$. In particular, in our numerical simulation the controller gain switched between the values $\{\kappa_1^1 = 100, \kappa_1^2 = 125, \kappa_1^3 = 150, \kappa_1^4 = 175\}$. The resulting synchronization errors (13) for this scenario are shown in Figure 3. In this case, the controller gains switch

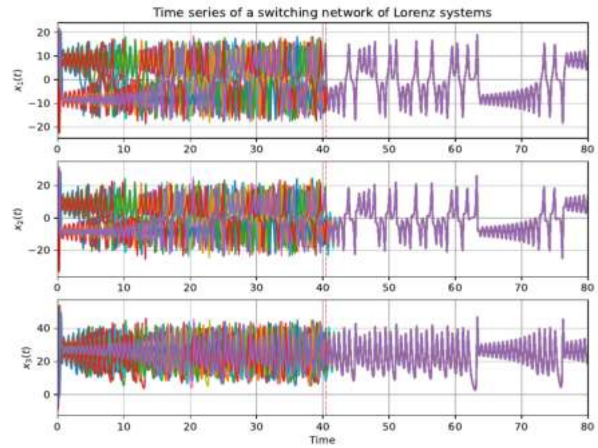

Fig. 2. A time evolution of a switching ($\sigma(t)$) network of Lorenz systems with arbitrary switching controller law ($\nu(t)$).

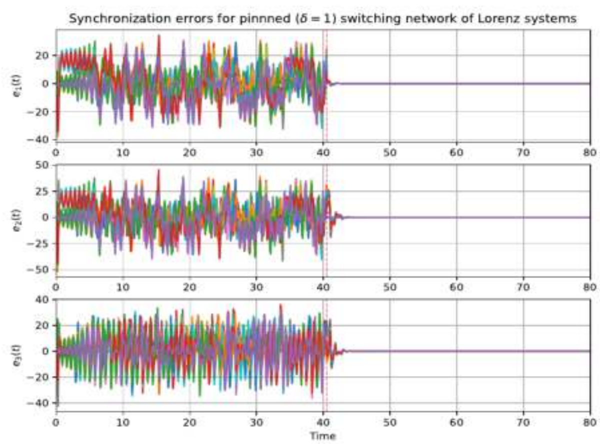

Fig. 3. A time evolution of synchronization errors for a switching network of Lorenz systems with a switching controller with the same switching law as the topology ($\nu(t) = \sigma(t)$).

at the same time as the topology starting after forty time units ($t > 40$).

For scenario III, the controller does not switch. That is, the gain $\kappa_1^{\nu(t)} = \kappa$ is the same for all time. In this case, we use the smallest $\lambda_2^{\sigma(t)}$ of the admissible Laplacian matrices to place $\kappa = 175$ regardless of the value of $\sigma(t)$. The resulting states of the pinned synchronized switching network of Lorenz systems are shown in Figure 4.

5. CONCLUSION

We investigate the synchronization of networks that commutes instantaneously between a set of admissible connections, which are connected and diffusive, as such, they are irreducible, symmetric, and negative semidefinite at each time instant. In this contribution, we investigate

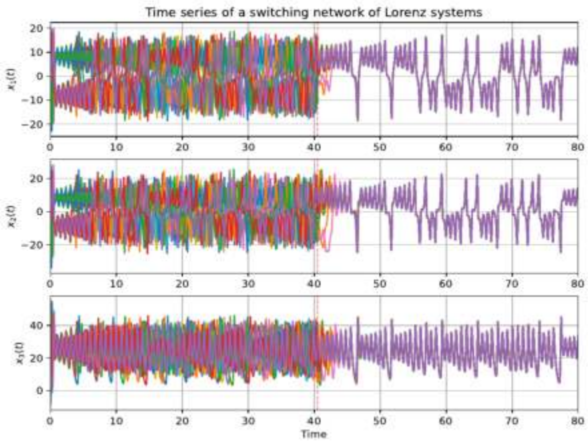

Fig. 4. A time evolution of a switching network of Lorenz systems with a fixed controller ($\kappa_1^{\nu(t)} = \kappa$).

the pinning synchronization for switching networks under three different scenarios. We consider that the controller can switch arbitrarily, at the same time as the topology, and finally as a fixed controller. Furthermore, to simplify the comparisons between scenarios we consider that only a single node is controlled, our choice is the node in the network with the largest node degree. Based on our results, as one compares the required controller gains for each of these scenarios we have that in the scenario I the smallest $\lambda_2^{\sigma(t)}$ results in a minimum acceptable controller gain and the controller can switch arbitrarily amount larger gain values. In scenario II, the minimum gain for each admissible Laplacian pair can be used. While for scenario III, the required controller gain for the smallest $\lambda_2^{\sigma(t)}$ can be set as fixed for the entire switching network.

From the above, one can conclude that a single fixed controller is more effective than a single arbitrary switching controller, yet if both the gains and the topology commute at the same time, is possible to use smaller gains since one can use the minimum for each admissible Laplacian.

In future works, we will address comparisons where a larger number of nodes are controlled ($\delta > 1$). However, in that case, the choice of which nodes to control will make determining the required gains $\kappa_i^{\nu(t)}$ to make Q negative definitive a very elaborate optimization problem.

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