Tracking Moving Objects with Colored Speed Noise Using State Differencing ⋆

J.A Ortega-Contreras, Eli Pale Ramon, J.A. Andrade-Lucio, Yuriy Shmaliy * **Shunyi Zhao** **

* *Electronics Engineering Dept., Universidad de Guanajuato, Salamanca, Gto., Mexico*
eligpale@gmail.com, {ja.ortegacontreras} {andrade} {shmaliy} @ugto.mx
** *Key Lab. Advanc. Process Control, Jiangnan University Wuxi, Jiangsu, China*
shunyi.s.y@gmail.com

Abstract: To reduce errors while tracking a moving object trajectory with a colored speed noise, the Kalman and unbiased finite impulse response filtering algorithms are modified assuming the Gauss-Markov noise nature. The state differencing approach is employed, requires solving a nonsymmetric algebraic Riccati equation to avoid matrix augmentation. In this way, the system matrix is modified for colored process noise (CPN). The higher accuracy of the modified algorithms are validated using a simulated tracking model.

Keywords: Colored process noise, state differencing, Kalman filter, unbiased FIR filter

1. INTRODUCTION

In tracking moving objects Bar-Shalom et al. (2004); Liang et al. (2015) and speech processing Ki Yong Lee et al. (1997); You et al. (2011); Zao and Coelho (2011), the process noise becomes colored when a control mechanism is ignored and not included in the state equation. An example is the human's speed, which was shown to be colored in many investigations Sieranoja and Kinnunen (2016); Chen et al. (1995). A state estimator should thus be used such that its algorithm is modified for CPN. There are three approaches on how to induce the colourless to the estimation algorithm.

The state augmentation can be provided as proposed by Bryson in Bryson and Johansen (1965); Bryson and Henrikson (1968). The measurement differencing can be employed using the *Bryson algorithm* Bryson and Johansen (1965); Bryson and Henrikson (1968), which makes noise white in two phases (smoothing and filtering), or by the *Petovello algorithm* algorithm Petovello et al. (2009), which makes noise white in one phase (filtering). Yet the state differencing can be used to provide estimation under the CPN as shown in Shmaliy et al. (2019).

For CPN, state augmentation does not make the Kalman filter (KF) ill-conditioned. Therefore, this approach is considered as a standard solution Gibbs (2011); Gelb and Corporation (1974); Simon (2006). A common mistake is that the KF can cause extra errors and singularities may appear in the unbiased finite impulse response (UFIR) filter Shmaliy (2011); Shmaliy et al. (2017) as a result of the state augmentation. Moreover, in the UFIR filter, an increased number K of the states may make the optimal horizon N_{opt} smaller than K that leads to errors.

Measurement and state differencing hence still attract the attention of the researches. In Kalman filtering, the problem with colored and time-correlated noise is solved by either re-deriving the Kalman gain Brown and Hwang (1997) or de-correlating noise vectors Bar-Shalom et al. (2004) and it was shown that the Bryson, Pelovello, and alternative Pelovello algorithm derived by Chang are equivalent Chang (2014). Applications of the state estimation algorithms modified for colored noise and other developments can be found in Kim and Suk (2012); Chen and Ma (2011); Chang (2014); Liu (2015); Lee and Johnson (2017); Tong and Ye (2017); Chang et al. (2018).

In this paper, we are focused on the moving object state estimation with colored speed noise. Therefore, we first introduce the KF and UFIR filter modifications for the CPN using state differencing and then apply the algorithms to the Global Positioning System (GPS) measurements of humans walking assuming colored speed noise.

⋆ This investigation was supported by the Mexican CONACyT-SEP Project A1-S-10287, Funding CB2017-201

2. MOVING OBJECT STATE-SPACE MODEL WITH CPN

A moving object can be modeled in discrete-time state-space with a set of polynomial state equations, where the first value in the vector state is the distance, the second is the velocity, the third is the acceleration, and so on. If the measurement is also linear and the object noise is colored, one may assume that the CPN is Gauss-Markov and represent an object with the following state and observation equations

$$\mathbf{x}_n = \mathbf{A}_n\mathbf{x}_{n-1} + \mathbf{B}_n\mathbf{w}_n \,, \tag{1}$$

$$\mathbf{w}_n = \Theta_n\mathbf{w}_{n-1} + \mu_n \,, \tag{2}$$

$$\mathbf{y}_n = \mathbf{H}_n\mathbf{x}_n + \mathbf{v}_n \,, \tag{3}$$

where $\mathbf{x}_n \in \mathbb{R}^K$, $\mathbf{y}_n \in \mathbb{R}^M$, $\mathbf{H}_n \in \mathbb{R}^{M\times K}$, $\mathbf{w}_n \in \mathbb{R}^K$ is the colored Gauss-Markov noise, matrices $\mathbf{A}_n \in \mathbb{R}^{K\times K}$, $\mathbf{B}_n \in \mathbb{R}^{K\times K}$, and $\Theta_n \in \mathbb{R}^{K\times K}$ are square and nonsingular, and matrix Θ_n is such that the colored noise $\mathbf{w}_n$ is stationary. It is supposed that noise vectors $\mu_n \sim \mathcal{N}(0,\mathbf{Q}_n) \in \mathbb{R}^K$ and $\mathbf{v}_n \sim \mathcal{N}(0,\mathbf{R}_n) \in \mathbb{R}^M$ are zero mean, $E\{\mu_n\} = \mathbf{0}$ and $E\{\mathbf{v}_n\} = \mathbf{0}$, and white Gaussian with the covariances $E\{\mu_n\mu_k^T\} = \mathbf{Q}_n\delta_{n-k}$ and $E\{\mathbf{v}_n\mathbf{v}_k^T\} = \mathbf{R}_n\delta_{n-k}$ and the property $E\{\mu_n\mathbf{v}_k^T\} = \mathbf{0}$ for all n and k.

2.1 Modification for CPN Using State Differencing

Similarly to measurement differencing Bryson and Henrikson (1968) and aimed at avoiding colored noise $\mathbf{w}_n$ in $\mathbf{x}_n$, the state-space model (1)–(3) can be modified as shown in Appendix A to attain the following form of

$$\chi_n = \mathbf{F}_n\chi_{n-1} + \mathbf{B}_n\mu_n \,, \tag{4}$$

$$\mathbf{z}_n = \mathbf{H}_n\chi_n + \mathbf{v}_n \,, \tag{5}$$

where χ_n is a new state defined by the state difference as

$$\chi_n = \mathbf{x}_n - \Pi_{n-1}\mathbf{x}_{n-1} \,, \tag{6}$$

$\mathbf{z}_n = \mathbf{y}_n - \mathbf{H}_n\Pi_n\mathbf{x}_{n-1}$ is a new observation, matrix Π_n is specified by

$$\Pi_n = \mathbf{F}_{n+1}^{-1}\bar{\Theta}_{n+1}\mathbf{A}_n \,, \tag{7}$$

where $\bar{\Theta}_n = \mathbf{B}_n\Theta_n\mathbf{B}_{n-1}^{-1}$ is a weighted Θ_n, and matrix $\mathbf{F}_n$ is given by the recursions (A.9) and (A.10).

Now observe that $\mathbf{F}_n$ becomes $\mathbf{A}_n$ for white noise if $\Pi_n = \mathbf{0}$ and $\bar{\Theta}_n = \mathbf{0}$ that follows from (A.9). Therefore, choosing the initial $\mathbf{F}_0$ is a critical issue. Let us consider this point in more detail. If we suppose that $\mathbf{F}_0 = \mathbf{A}_0$, then (A.9) gives $\mathbf{F}_1 = \bar{\Theta}_1\mathbf{A}_0\bar{\Theta}_0^{-1}$, which may range far from a proper matrix. In the other extreme, $\mathbf{F}_0 = \bar{\Theta}_0$ produces $\mathbf{F}_1 = \bar{\Theta}_1$, which is impractical. As has been shown in Shmaliy et al. (2019), $\mathbf{F}_0$ can be specified if we think that the process before the initial point at zero is time-invariant and transform (A.10) to the nonsymmetric algebraic Riccati equation (NARE)

$$\mathbf{F}^2 - \mathbf{F}(\mathbf{A} + \bar{\Theta}) + \bar{\Theta}\mathbf{A} = \mathbf{0} \,. \tag{8}$$

A solution to (8) for $\mathbf{F} = \mathbf{F}_0$ could be a proper matrix to run (A.9). However, the problem is that the NARE solution is generally not unique Lancaster and Rodman (2002). Thus, efforts must be made to select a proper solution method, which will be further illustrated with an example.

Example 1: Consider the modified state equation (4) with white Gaussian μ_n and matrix $\mathbf{F}_n$ provided recursively by (A.9) for a proper $\mathbf{F}_0$ selected from possible solutions to (8). To specify the initial χ_0, consider (6) at $n = 0$ as $\chi_0 = \mathbf{x}_0 - \Pi_0\mathbf{x}_{-1}$. Because $\mathbf{x}_{-1}$ is unavailable, let $\mathbf{x}_{-1} = \mathbf{x}_0$ and accept $\chi_0 = (\mathbf{I}-\Pi_0)\mathbf{x}_0 = (\mathbf{I}-\mathbf{F}_1^{-1}\bar{\Theta}_1\mathbf{A}_0)\mathbf{x}_0$ as a reasonable initial state to run (A.10).

Provided a modified state-space model (4) and (5) for uncorrelated white Gaussian noise sources μ_n and $\mathbf{v}_n$, the KF and UFIR state estimators can be modified as in the following.

2.2 KF Algorithm for CPN

There are no special comments on how to modify the KF filtering algorithm for CPN if matrix $\mathbf{F}_n$ is previously determined via (A.10). A pseudo code of the relevant KF algorithm, which is termed as cKF, is listed as Algorithm 1.

Algorithm 1 cKF Algorithm for CPN

Data: $\mathbf{y}_n$, $\hat{\mathbf{x}}_0$, $\mathbf{P}_0$, $\mathbf{Q}_n$, $\mathbf{R}_n$, Θ_n
Result: $\hat{\mathbf{x}}_n$, $\mathbf{P}_n$
begin
  $\hat{\chi}_0 = (\mathbf{I} - \mathbf{F}_1^{-1}\bar{\Theta}_1\mathbf{A}_0)\hat{\mathbf{x}}_0$
  for $n = 1, 2, \cdots$ **do**
    $\mathbf{z}_n = \mathbf{y}_n - \mathbf{H}_n\mathbf{F}_{n+1}^{-1}\bar{\Theta}_{n+1}\mathbf{A}_n\hat{\mathbf{x}}_{n-1}$
    $\mathbf{P}_n^- = \mathbf{F}_n\mathbf{P}_{n-1}\mathbf{F}_n^T + \mathbf{B}_n\mathbf{Q}_n\mathbf{B}_n^T$
    $\mathbf{S}_n = \mathbf{H}_n\mathbf{P}_n^-\mathbf{H}_n^T + \mathbf{R}_n$
    $\mathbf{K}_n = \mathbf{P}_n^-\mathbf{H}_n^T\mathbf{S}_n^{-1}$
    $\hat{\chi}_n^- = \mathbf{F}_n\hat{\chi}_{n-1}$
    $\hat{\chi}_n = \hat{\chi}_n^- + \mathbf{K}_n(\mathbf{z}_n - \mathbf{H}_n\hat{\chi}_n^-)$
    $\hat{\mathbf{x}}_n = \hat{\chi}_n + \mathbf{F}_{n+1}^{-1}\bar{\Theta}_{n+1}\mathbf{A}_n\hat{\mathbf{x}}_{n-1}$
    $\mathbf{P}_n = (\mathbf{I} - \mathbf{K}_n\mathbf{H}_n)\mathbf{P}_n^-$
  end
end

Here, $\mathbf{F}_n$ is obtained recursively via (A.10) for $\mathbf{F}_0$ given by a proper solution of (8). It can easily be observed that, by $\Theta_n = 0$ and $\mathbf{F}_n = \mathbf{A}_n$, algorithm 1 becomes the standard KF and therefore, can be universally applied to processes with and without Gauss-Markov colored noise.

2.3 UFIR Filtering Algorithm

Another option to provided state estimation based on model (4) and (5) is to use the UFIR filter Shmaliy (2011), which does not require any information about zero-mean noise and is more robust than the KF Shmaliy et al. (2017). The UFIR filter provides state estimation over an averaging horizon $[m, n]$ of N points, from $m = n - N + 1$ to n. To produce a near-optimal estimate, this filter requires an optimal averaging horizon N_{opt} to minimize the mean square error (MSE). A pseudo-code of the UFIR algorithm named for CPN as cUFIR is listed as Algorithm 2. To operate on $[m, k]$ and initialize iterations without singularities, algorithm 2 requires a short data vector $\mathbf{Y}_{m,s} = [\,\mathbf{y}_m \;\dots\; \mathbf{y}_s\,]^T$ and matrix

Algorithm 2 cUFIR Filter Algorithm for CPN

Data: N, $\mathbf{y}_n$, Θ_n
Result: $\hat{\mathbf{x}}_n$
begin
  for $k = N-1, N, \cdots$ **do**
    $m = k - N + 1$, $s = k - N + K$
    $\mathbf{G}_s = (\mathbf{C}_{m,s}^T \mathbf{C}_{m,s})^{-1}$
    $\bar{\mathbf{x}}_s = \mathbf{G}_s \mathbf{C}_{m,s}^T \mathbf{Y}_{m,s}$
    for $l = s+1 : n$ **do**
      $\mathbf{z}_l = \mathbf{y}_l - \mathbf{H}_l \mathbf{F}_{l+1}^{-1} \bar{\Theta}_{l+1} \mathbf{A}_l \hat{\mathbf{x}}_{l-1}$
      $\mathbf{G}_l = [\mathbf{H}_l^T \mathbf{H}_l + (\mathbf{F}_l \mathbf{G}_{l-1} \mathbf{F}_l^T)^{-1}]^{-1}$
      $\mathbf{K}_l = \mathbf{G}_l \mathbf{H}_l^T$
      $\bar{\chi}_l^- = \mathbf{F}_l \bar{\chi}_{l-1}$
      $\bar{\chi}_l = \bar{\chi}_l^- + \mathbf{K}_l (\mathbf{z}_l - \mathbf{H}_l \bar{\chi}_l^-)$
    end
    $\hat{\mathbf{x}}_n = \bar{\chi}_n + \mathbf{F}_{n+1}^{-1} \bar{\Theta}_{n+1} \mathbf{A}_n \hat{\mathbf{x}}_{n-1}$
  end
end

$$
\mathbf{C}_{m,s} = \begin{bmatrix} \mathbf{H}_m (\mathbf{F}_s \dots \mathbf{F}_{m+1})^{-1} \\ \vdots \\ \mathbf{H}_{s-1} \mathbf{F}_s^{-1} \\ \mathbf{H}_s \end{bmatrix} . \tag{9}
$$

By $\bar{\Theta}_n = \mathbf{0}$ and $\mathbf{F}_n = \mathbf{A}_n$, Algorithm 2 becomes the standard UFIR filter Shmaliy et al. (2017) and therefore can also be applied to processes with and without Gauss-Markov color noise.

The error covariance $\mathbf{P}_n = \{(\mathbf{x}_n - \hat{\mathbf{x}}_n)(\mathbf{x}_n - \hat{\mathbf{x}}_n)^T\}$ for the cUFIR filter can be determined knowing that $\varepsilon_n = \mathbf{x}_n - \hat{\mathbf{x}}_n = \epsilon_n \Pi_n \epsilon_{n-1}$, where $\epsilon_n = \chi_n - \hat{\chi}_n$.

Now we can refer to an identity $\mathbf{F}_n \Pi_{n-1} \mathbf{A}_{n-1}^{-1} \bar{\Theta}_n^{-1} = \mathbf{I}$ and transformation error ϵ_n as

$$
\begin{aligned}
\epsilon_n &= \mathbf{F}_n \chi_{n-1} + \mu_n - \mathbf{F}_n \hat{\chi}_{n-1} - \mathbf{K}_n (\mathbf{z}_n - \mathbf{H}_n \mathbf{F}_n \hat{\chi}_{n-1}) \\
&= (\mathbf{I} - \mathbf{K}_n \mathbf{H}_n) \mathbf{F}_n \epsilon_{n-1} + (\mathbf{I} - \mathbf{K}_n \mathbf{H}_n) \mu_n - \mathbf{K}_n \mathbf{v}_n .
\end{aligned}
$$

Accordingly, the covariance $\bar{\mathbf{P}}_n = E\{\epsilon_n \epsilon_n^T\}$ can be represented recursively with

$$
\begin{aligned}
\bar{\mathbf{P}}_n = {} & (\mathbf{I} - \mathbf{K}_n \mathbf{H}_n)(\mathbf{F}_n \bar{\mathbf{P}}_{n-1} \mathbf{F}_n^T + \mathbf{B}_n \mathbf{Q}_n \mathbf{B}_n^T) \\
& \times (\mathbf{I} - \mathbf{K}_n \mathbf{H}_n)^T + \mathbf{K}_n \mathbf{R}_n \mathbf{K}_n^T
\end{aligned} \tag{10}
$$

and we obtain

$$
\mathbf{P}_n = \bar{\mathbf{P}}_n + \Pi_n \mathbf{P}_{n-1} \Pi_n^T , \tag{11}
$$

where $\bar{\mathbf{P}}_n$ is computed recursively by (10).

2.4 Simulation of the tracking of a moving object

Concerted of tracking for moving objects, we now simulate a process with a tracking state equation. To investigate state estimation errors produced by the KF, cKF, UFIR filter, and cUFIR filter, we consider a two-state, $K = 2$, a polynomial process described by model (1)–(3) with

$$
\mathbf{A} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \frac{\tau}{2} & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{H} = [\,1 \;\, 0\,] ,
$$

$$
\Theta = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, \mathbf{C}_{m,s} = \begin{bmatrix} \mathbf{H}\mathbf{F}^{-1} \\ \mathbf{H} \end{bmatrix},
$$

and $\tau = 1$. We suppose that white Gaussian $\mu_n \sim \mathcal{N}[\mathbf{0}, \mathrm{diag}(\sigma_\mu^2 \; \sigma_\mu^2)]$ and $v_n \sim \mathcal{N}(0, \sigma_v^2)$ have the standard deviations of $\sigma_\mu = 0.8$ and $\sigma_v = 1$. For each discrete value of the color factor $0 < \theta < 1$, we find optimal ζ_{opt}. We also determine N_{opt} by minimizing the MSE. What was inferred is that ζ_{opt} varies from 1.79 for $\theta = 0.05$ to 0.65 for $\theta = 0.99$, while $N_{\mathrm{opt}} = 4$ holds for $\theta < 1$ in both FIR filters.

Given $K = 2$, we represent (8) with four algebraic equations corresponding to components of $\mathbf{F} \in \mathbb{R}^{2\times 2}$. Next, we find solutions and choose the one $\mathbf{F}_n = \mathbf{F} = \begin{bmatrix} \theta & \zeta \\ 0 & 1 \end{bmatrix}$, in which θ is given and ζ is arbitrary, and notice that other solutions improperly project χ_{n-1} to χ_n. Of importance is that further update of $\mathbf{F}$ by (A.10) is not required, because the model is time-invariant.

The most critical question now is how the modified algorithms improve performance. We, therefore, compute the root MSEs (RMSEs) produced by all four filters and sketch the results in Fig. 1 as functions of θ. What follows from this simulation can be summarized as follows:

- Under the ideal conditions when the model and noise are completely known, the KF and cKF perform more accurately than the UFIR and cUFIR filter, respectively, for all $0 < \theta < 1$.
- The modified cKF and cUFIR filter both improve the performance. However, the improvement appears to be essential only when $\theta > 0.5$. Otherwise, the cKF and cUFIR filters may not demonstrate efficiency.
- When θ approaches unity associated with the Gauss-Markov noise stationarity, the cKF and cUFIR filter perform much better than the KF and UFIR filters. That means that the filters modified for Gauss-Markov CPN are efficient when the process noise is highly colored.

3. TRACKING OF A WALKING HUMAN

Referring to the above, we now consider several human walking trajectories measured each second, $\tau = 1\,\mathrm{s}$, using the GPS loggers and phones as described in Zheng et al. (2009) and notice that data are available from Zheng et al. (2012). Our goal is to estimate the human speed using the standard KF and UFIR filter and the cKF and cUFIR

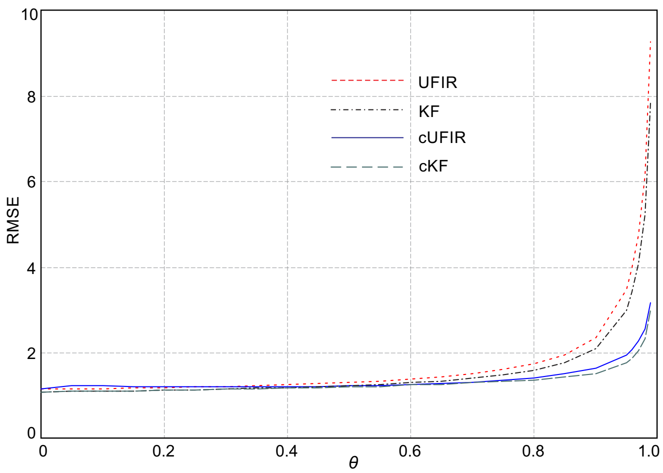

Fig. 1. RMSEs produced by the KF, cKF, UFIR, and cUFIR for a two-state polynomial model with CPN as functions of θ.

filter modified for CPN. Therefore, we refer to Sieranoja and Kinnunen (2016) and suppose that the human speed noise is colored. Under such a supposition, we consider a tracking problem along with the coordinate x employing the same model as in simulations.

Because the ground truth is not available in our experiment, we will analyze the results by comparing the filter outputs. Measurement data do not contain sufficient information about the process and noise. We therefore refer to the average human speed of 5 km/hour or 1.4 m/s and assign $\sigma_\mu = 0.8\,\mathrm{m/s}$. We also notice that the GPS standard positioning service is available with errors less than 15 m with the probability of 95% in the 2σ sense and accept $\sigma_v = 3.75\,\mathrm{m}$. Although these values are not exact for the model and the measurement, they affect all filtering algorithms equivalently that allows comparing errors.

Provided the filters tuning, we next run Algorithm 2.2 and Algorithm 2.3 for several measured trajectories. At a test stage, we have found $\theta = 0.99$ and $\zeta = 2.9$ for the KF. For the UFIR filter, the optimal averaging horizon was measured as $N_{\mathrm{opt}} = 6$. A bit smaller value of $N_{\mathrm{opt}} = 4$ was found for the cUFIR filter (Algorithm 2.3) to produce visually minimum possible errors. Extensive investigations of the human trajectories were then conducted. Accordingly, in Fig. 2 we show different human speeds (data), which were computed by applying the derivatives to the measured trajectories.

Estimates provided by all filters are also shown here that allows making the following conclusions:

- It is neatly seen (Fig. 2a) that noise in both experiments is definitely not white. This confirms that the human speed noise is colored, although its coloredness is a topic for special investigations.
- The KF and UFIR filter produce consistent estimates, although the UFIR filter generates a bit larger excursions, as expected Shmaliy et al. (2017).
- The cKF and cUFIR estimates also appear to be consistent, although with a bit smaller CPN weights at the outputs.

Several particular observations can also be made. As can be seen in Fig. 1a, the human speed varies quasi harmonically and it follows that the cKF and cUFIR filter both have smaller sensitivities to the colored speed noise. The impulsive speed changes are shown in Fig. 2b, which are reminiscent of the unit pulses and impulse responses. What we see here is that the cKF and cUFIR filter generate shorter excursions and are thus less sensitive to the colored speed noise.

4. CONCLUSIONS

Estimates of the walking human speed provided by the cKF and cUFIR filter modified for Gauss-Markov CPN have shown a better performance of the estimation algorithms, especially when the speed noise color factor is large. The results were achieved by applying the state differencing for the square and nonsingular system matrix, process noise matrix, and noise color matrix. That allowed providing an accurate state estimation by solving the NARE to specify the modified system matrix. Extensive investigations of GPS-based data have confirmed the efficiency of the modified algorithms.

Appendix A. MODIFICATION OF THE STATE-SPACE MODEL USING STATE DIFFERENCING

Consider a new state χ_n represented with state differences,

$$\chi_n = \mathbf{x}_n - \Pi_n \mathbf{x}_{n-1} \quad \text{(A.1a)}$$

$$= \mathbf{A}_n \mathbf{x}_{n-1} + \mathbf{B}_n \mathbf{w}_n - \Pi_n \mathbf{x}_{n-1}$$

$$= \mathbf{F}_n \chi_{n-1} + \bar{\mathbf{w}}_n - \mathbf{F}_n \chi_{n-1} - \bar{\mathbf{w}}_n$$

$$+ (\mathbf{A}_n - \Pi_n)\mathbf{x}_{n-1} + \mathbf{B}_n \mathbf{w}_n \,, \quad \text{(A.1b)}$$

where Π_n, $\mathbf{F}_n$, and $\bar{\mathbf{w}}_n$ are still to be determined.

Rewrite the state equation as

$$\chi_n = \mathbf{F}_n \chi_{n-1} + \bar{\mathbf{w}}_n \,, \quad \text{(A.2)}$$

where claim that noise $\bar{\mathbf{w}}_n \sim \mathcal{N}(0, \bar{\mathbf{Q}}_n) \in \mathbb{R}^K$ must be zero mean and white Gaussian. To make it possible, define matrix $\mathbf{F}_n$ such that the remaining part of (A.1b) becomes zero,

$$-\mathbf{F}_n \chi_{n-1} - \bar{\mathbf{w}}_n + (\mathbf{A}_n - \Pi_n)\mathbf{x}_{n-1} + \mathbf{B}_n \mathbf{w}_n = 0 \,. \quad \text{(A.3)}$$

To find $\mathbf{F}_n$, substitute (A.1a) into (A.2), combine with $\mathbf{x}_{n-2} = \mathbf{A}_{n-1}^{-1}(\mathbf{x}_{n-1} - \mathbf{B}_{n-1}\mathbf{w}_{n-1})$ taken from (1) and write

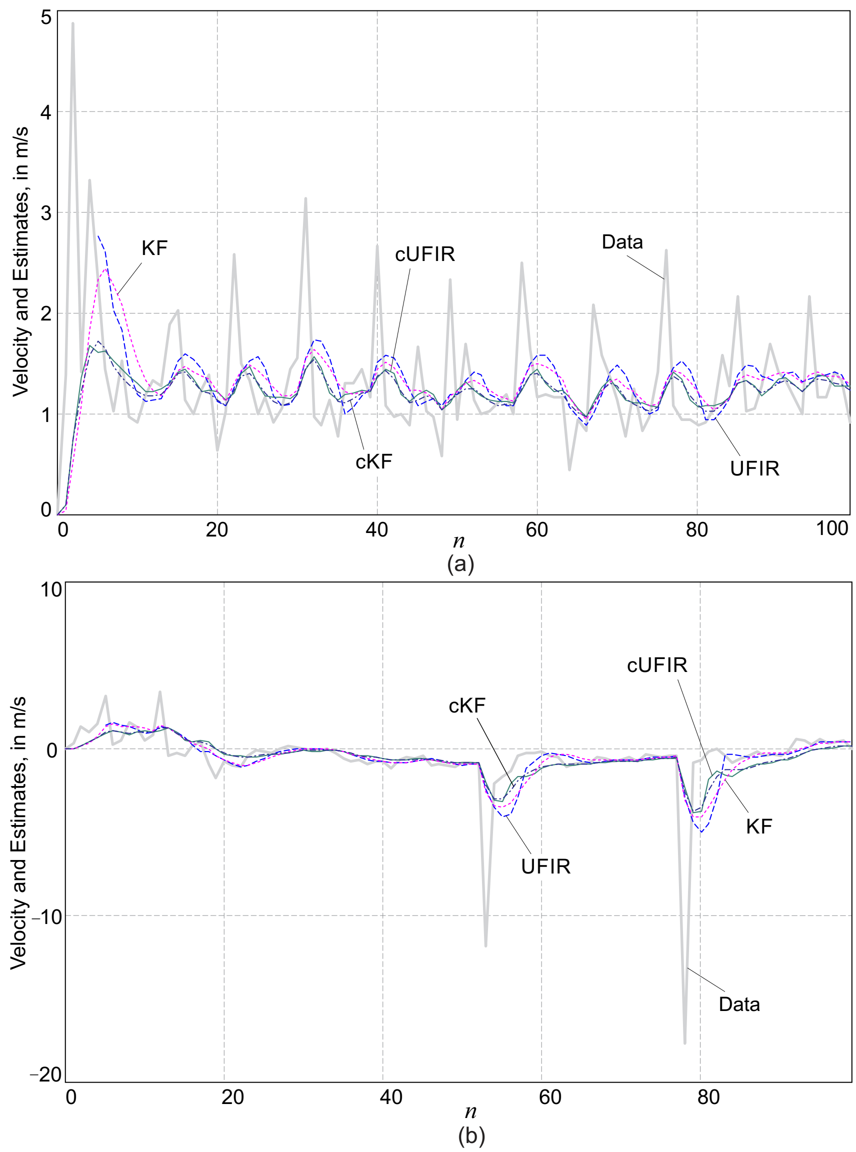

Fig. 2. Different walking human speeds measured by GPS (data) and estimated by the KF, UFIR filter, cKF, and cUFIR filter assuming CPN: (a) quasi harmonic variations and (b) impulsive changes.

$$(\mathbf{A}_n - \Pi_n - \mathbf{F}_n + \mathbf{F}_n\Pi_{n-1}\mathbf{A}_{n-1}^{-1})\mathbf{x}_{n-1} = \bar{\mathbf{w}}_n - \mathbf{B}_n\mu_n - (\mathbf{B}_n\Theta_n - \mathbf{F}_n\Pi_{n-1}\mathbf{A}_{n-1}^{-1}\mathbf{B}_{n-1})\mathbf{w}_{n-1}\,. \tag{A.4}$$

Because the case of $\mathbf{x}_{n-1} = 0$ is isolated for arbitrary n, find a solution to (A.4) by considering three equations

$$\mathbf{F}_n = \mathbf{A}_n - \Pi_n + \mathbf{F}_n\Pi_{n-1}\mathbf{A}_{n-1}^{-1}\,, \tag{A.5}$$

$$\mathbf{B}_n\Theta_n = \mathbf{F}_n\Pi_{n-1}\mathbf{A}_{n-1}^{-1}\mathbf{B}_{n-1}\,, \tag{A.6}$$

$$\bar{\mathbf{w}}_n = \mathbf{B}_n\mu_n\,, \tag{A.7}$$

where (A.7) suggests that $\bar{\mathbf{w}}_n$ is white Gaussian, as required. Now, for nonsingular $\mathbf{F}_n$, $\mathbf{A}_n$, and Θ_n, (A.6) yields

$$\Pi_n = \mathbf{F}_{n+1}^{-1}\bar{\Theta}_{n+1}\mathbf{A}_n\,, \tag{A.8}$$

where $\bar{\Theta}_n = \mathbf{B}_n\Theta_n\mathbf{B}_{n-1}^{-1}$ is a weighted Θ_n.

Substitute (A.8) into (A.5), providing the transformations, and arrive at recursions for $\mathbf{F}_n$,

$$\mathbf{F}_n = \mathbf{A}_n - \mathbf{F}_{n+1}^{-1}\bar{\Theta}_{n+1}\mathbf{A}_n + \bar{\Theta}_n\,, \tag{A.9}$$

$$\mathbf{F}_{n-1} = \mathbf{A}_{n-1} - \mathbf{F}_n^{-1}\bar{\Theta}_n\mathbf{A}_{n-1} + \bar{\Theta}_{n-1}\,,$$

$$\mathbf{F}_n^{-1}\bar{\Theta}_n\mathbf{A}_{n-1} = \mathbf{A}_{n-1} - \mathbf{F}_{n-1} + \bar{\Theta}_{n-1}\,,$$

$$\mathbf{F}_n^{-1}\bar{\Theta}_n = \mathbf{I} - (\mathbf{F}_{n-1} - \bar{\Theta}_{n-1})\mathbf{A}_{n-1}^{-1}\,,$$

$$\mathbf{F}_n = \bar{\Theta}_n[\mathbf{I} - (\mathbf{F}_{n-1} - \bar{\Theta}_{n-1})\mathbf{A}_{n-1}^{-1}]^{-1}\,. \tag{A.10}$$

REFERENCES

Bar-Shalom, Y., Li, X., and Kirubarajan, T. (2004). *Estimation with Applications to Tracking and Navigation: Theory, Algorithms and Software*.

Brown, R.G. and Hwang, P.Y.C. (1997). *Introduction to random signals and applied kalman filtering: with MATLAB exercises and solutions; 3rd ed.* Wiley, New York, NY.

Bryson, A. and Johansen, D. (1965). Linear filtering for time-varying systems using measurements containing colored noise. *IEEE Transactions on Automatic Control*, 10(1), 4–10.

Bryson, A.E. and Henrikson, L.J. (1968). Estimation using sampled data containing sequentially correlated noise. *Journal of Spacecraft and Rockets*, 5(6), 662–665.

Chang, G. (2014). On kalman filter for linear system with colored measurement noise. *Journal of Geodesy*, 88, 1163–1170.

Chang, G., Xu, T., Yao, Y., and Wang, Q. (2018). Adaptive kalman filter based on variance component estimation for the prediction of ionospheric delay in aiding the cycle slip repair of gnss triple-frequency signals. *Journal of Geodesy*.

Chen, J. and Ma, L. (2011). Particle filtering with correlated measurement and process noise at the same time. *Radar, Sonar Navigation, IET*, 5, 726–730.

Chen, W., Zhou, G., and Giannakis, G.B. (1995). Velocity and acceleration estimation of doppler weather radar/lidar signals in colored noise. In *1995 International Conference on Acoustics, Speech, and Signal Processing*, volume 3, 2052–2055.

Gelb, A. and Corporation, T.A.S. (1974). *Applied Optimal Estimation*. The MIT Press.

Gibbs, B.P. (2011). *Advanced Kalman filtering, least-squares and modeling: a practical handbook* . Wiley, Hoboken NJ.

Ki Yong Lee, Byung-Gook Lee, and Souguil Ann (1997). Adaptive filtering for speech enhancement in colored noise. *IEEE Signal Processing Letters*, 4(10), 277–279.

Kim, D.M. and Suk, J. (2012). Gps output signal processing considering both correlated/white measurement noise for optimal navigation filtering. *International Journal of Aeronautical and Space Sciences*, 13, 499–506.

Lancaster, P. and Rodman, L. (2002). *Algebraic Riccati Equations*.

Lee, K. and Johnson, E.N. (2017). State estimation using gaussian process regression for colored noise systems. In *2017 IEEE Aerospace Conference*, 1–8.

Liang, P., Blasch, E., and Ling, H. (2015). Encoding color information for visual tracking: Algorithms and benchmark. *IEEE Transactions on Image Processing*, 24(12), 5630–5644.

Liu, W. (2015). Optimal estimation for discrete-time linear systems in the presence of multiplicative and time-correlated additive measurement noises. *IEEE Transactions on Signal Processing*, 63(17), 4583–4593.

Petovello, M.G., O'Keefe, K., Lachapelle, G., and Cannon, M.E. (2009). Consideration of time-correlated errors in a Kalman filter applicable to GNSS. *Journal of Geodesy*, 83(1), 51–56.

Shmaliy, Y.S. (2011). An iterative kalman-like algorithm ignoring noise and initial conditions. *IEEE Transactions on Signal Processing*, 59(6), 2465–2473.

Shmaliy, Y.S., Zhao, S., and Ahn, C.K. (2017). Unbiased finite impluse response filtering: An iterative alternative to kalman filtering ignoring noise and initial conditions. *IEEE Control Systems Magazine*, 37(5), 70–89.

Shmaliy, Y., Zhao, S., and Ahn, C.K. (2019). Optimal and unbiased filtering with colored process noise using state differencing. *IEEE Signal Processing Letters*, PP.

Sieranoja, S. and Kinnunen, T. (2016). Gps trajectory biometrics: From where you were to how you move. volume 10029.

Simon, D. (2006). Optimal state estimation: Kalman,h_∞, and nonlinear approaches.

Tong, R. and Ye, Z. (2017). Supplementations to the higher order subspace algorithm for suppression of spatially colored noise. *IEEE Signal Processing Letters*, PP, 1–1.

You, C., Lee, K.A., and Leung, C.C. (2011). Speech enhancement with masking properties in eigen-domain for colored noise. 4748 – 4751.

Zao, L. and Coelho, R. (2011). Colored noise based multicondition training technique for robust speaker identification. *IEEE Signal Process. Lett.*, 18, 675–678.

Zheng, Y., Fu, H., Ma, W.Y., and Li, Q. (2012). Geolife: Building social networks using human location history. `https://www.microsoft.com/en-us/research/project/geolife`.

Zheng, Y., Zhang, L., Xie, X., and Ma, W.Y. (2009). Mining interesting locations and travel sequences from gps trajectories. 791–800.

