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A fractional-order integral transformation for the diffusion model

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Abstract: In this paper, the problem of model equivalence between integer-order and fractional-order systems with Caputo derivative is considered. Motivated by diverse studies that show a connection between the diffusion model and its equivalent fractional diffusion models, the construction of the integral transformation is presented that maps the solution of the one-dimensional integer diffusion model to the equivalent solution of the fractional diffusion model. It is demonstrated that the unique integral transformation between both models corresponds to the integral of half-order. The derivation includes the inverse integral transformation, which allows the validation of the equivalence and well-posedness of the models. Moreover, the explicit analytic solution for the equivalent fractional partial differential equation is given through the proposed transformation.

Keywords: Integer-order diffusion model, fractional calculus, integral transformation, fractional-order diffusion model

1. INTRODUCTION

Current demands in science and technology require mathematical models for describing complex phenomena that include many parameters and variables. Thus, the accuracy of the results and studies could be degraded whenever the effect of some of them are neglected or omitted. The new trends in modelling indicate that the extra parameter design of fractional-order models can improve the system's characterization [Kopka, 2015]. This is a consequence of the non locality and memory effects associated with the derivatives and integrals of non-integer-order; i.e. fractional-order operators (FOOs) Podlubny [1998]; Ortigueira [2011]. Therefore, fractional calculus (FC) has become an intensive field of study in areas such as model-based fault detection [Azimi and Shandiz, 2020; He et al., 2021], the modelling of materials [Bonfanti et al., 2020; Biswas et al., 2017; Ortigueira, 2011], the modelling of Susceptible-Infected-Recovered-Deceased (SIRD) of COVID19 Nisar et al. [2021]; Jahanshahi et al. [2021] and automatic control [Monje et al., 2010; Caponetto et al., 2010; Modiri and Mobayen, 2020; Oustaloup, 2014; Jajarmi and Baleanu, 2021]. On the other hand, partial differential equations (PDEs) played an important role in the early stages of FC. While some members of the scientific community concentrated on justifying the use of FOOs in applied sciences Oldham and Spanier [1974], another part of the community developed important results by using physical models described by PDEs. The latter with the contributions of [Oldham and Spanier, 1974; Podlubny, 1998; Kilbas et al., 2006] laid groundwork in the FC area. Inspired by the mathematical interest, some research began looking for mathematical models in the frequency domain that contained rational exponents of the complex variable [Carlson and Halijak, 1964; Nigmatullin, 1986; Machado, 2001; Radwan and Salama, 2012; Valsa and Vlach, 2013]. Because the exponents in this framework did not correspond to those described by derivatives and integrals with an integer-order, the mathematical models were analytically manipulated to obtain equations involving functions of FOOs. The first contribution that introduced the fractional diffusion model (FDM) was published in [Oldham and Spanier, 1974], and it was obtained from the solution of the one-dimensional (1D) diffusion model $v_t(x,t) = \frac{1}{b}v_{xx}(x,t)$ via the Laplace transform method with b as the diffusion's constant. Thus, one could interpret that the fractional partial equation $D_t^{0.5}v(x,t) = -\frac{1}{\sqrt{b}}v_x(x,t)$ with $D_t^{0.5}$ as the half-order derivative is enclosed in the standard 1D diffusion model (1DDM), see Section 5.1 for more details. Motivated by this result, diverse works were published by considering variations of the idea published by Oldham and Spanier and looked for the fractional diffusion models enclosed in the standard one. Many authors [Metzler et al., 1994; Zhang and Xue, 2007; Kulish and Lage, 2002; Sierociuk et al., 2013] proposed a systematic procedure, however, to obtain the respective fractional model enclosed in the diffusion model, some conditions between both models were missing. One can identify the following deficiencies and disadvantages below.

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- The equivalence between boundary conditions for the FDM cannot be computed by these methods.
- The FOOs are determined by the original model assuming zero initial conditions, which is not the case in a practical application with disturbances.
- The link between both models is unidirectional. This means the inverse transformation between the solutions of the FDM and the 1DDM is missing. This limitation is outlined in Fig.1 where the operator is represented by the solid line with only one direction.
- To return to the original model, additional partial derivatives of the functions with respect to the spatial variable are required, in addition to the Laplace inversion of the functions.
- The Laplace solution of the integer-order system must be known.

Therefore, to have an strong impact on the FC in the analysis of PDE, it is necessary to have general methodologies to transform an integer-order PDE to its equivalent fractional partial differential equation (FPDE), which guarantees its existence and that can be applied for any initial and boundary conditions of the system. Moreover, one should have the possibility of transforming the solution of the FPDE to the PDE solution. This requirement motivated this work in which as a first case study the whole equivalent operator between the integer diffusion model and its fractional description is introduced via an integral transformation with a specific kernel.
The principal characteristic of this insight is a precise formulation of the fractional-order transformation problem, which is inspired by the works published by Colton [1977] and Seidman [1984] for integer partial differential equations by using integer-order operators defined in terms of an unknown kernel. The demonstration of the equivalence between both models is achieved via Caputo's fractional-order derivative maintaining the same initial conditions as in the integer order case. Specifically, the definition of a fractional-order integral transformation (FOIT) and its inverse fractional-order transformation (IFOIT) for the 1DDM is proposed. Moreover, it is found that the unique FOIT corresponds to a half-order one, so the equivalence between both models is valid only whenever the fractional-order derivative corresponds to one-half. The bidirectional property of this operator is indicated by the dotted arrows of the drawing of Fig. 1. Since the well-posedness of the integer 1DDM with general boundary and initial conditions is guaranteed [Evans, 1997], the existence and uniqueness of the solution for any boundary and initial conditions for the FDM is guaranteed as well.

The paper is organized as follows: Preliminaries and properties of FOO are given in Section 2. The main result related to the model equivalence for the 1DDM and FDM is obtained in Section 3. A numerical example of the FDM solution is presented in Section 4. Finally, the concluding remarks are presented in Section 5.

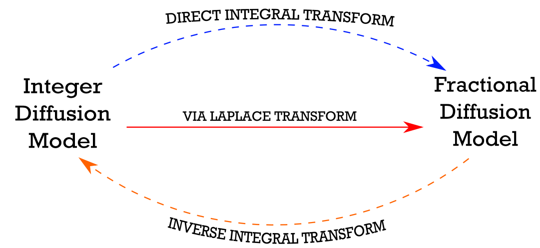

Fig. 1. Commutative diagram for the diffusion model transformation.

2. PRELIMINARIES OF FRACTIONAL CALCULUS

Notation. $\mathbf{N}$, $\mathbb{R}$, $\mathbb{R}_+$ denote the set of natural, real and positive real numbers, respectively. Let $[a,b] \subset \mathbb{R}$ and $[c,d] \subset \mathbb{R}$ denote compact intervals on the real line, and $g : [a,b] \to \mathbb{R}$ and $f : [a,b] \times [c,d] \to \mathbb{R}$ are real-valued functions of a single variable or two variables, respectively. $C^1[a,b]$ denotes the space of continuous differentiable functions on $[a,b]$ such that if $g \in C^1[a,b]$, then its derivative g' is continuous on $[a,b]$; $L_1[a,b]$ denotes the space of integrable functions over $[a,b]$. $AC[a,b]$ denotes the space of absolutely continuous functions. Note that if $g \in AC[a,b]$, then g has a derivative g' almost everywhere and $g' \in L_1[a,b]$. Let $f(x,t)$ be a differentiable function on $(x,t) \in [a,b] \times [c,d]$, *i.e.*, $f(\cdot,t) \in C^1[a,b]$ for all $x \in [a,b]$ and each $t \in [c,d]$ as well as $f(x,\cdot) \in C^1[a,b]$ for all $t \in [c,d]$ and each $x \in [a,b]$. The integer-order partial derivative of the function $f(x,t)$ with respect to the time t and the spatial variable x are denoted as $f_t(x,t) := \frac{\partial f(x,t)}{\partial t}$ and $f_x(x,t) := \frac{\partial f(x,t)}{\partial x}$, respectively. The symbol $\circ$ denotes the function composition. The following definitions and auxiliary results are immediate extensions of the results found in Kilbas et al. [2006]; Diethelm [2010]; Ishteva [2005].

Definition 1. Diethelm [2010] Euler's Gamma function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt.$$

Definition 2. [Kilbas et al., 2006] The Beta function is defined by the Euler integral of the first kind:

$$B(y,p) = \int_0^1 \xi^{y-1}(1-\xi)^{p-1} d\xi, \tag{1}$$

with $y, p \in \mathbb{R}_+$.

Property 1. [Kilbas et al., 2006] The Beta function satisfies

$$B(y,p) = \frac{\Gamma(p)\Gamma(y)}{\Gamma(y+p)}. \tag{2}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 3. [Kilbas et al., 2006] Given $\ell, T \in \mathbb{R}_+$ and with $f : [0,\ell] \times [0,T] \to \mathbb{R}$, $f(x,\cdot) \in L_1[0,T]$ for each $x \in [0,\ell]$, the Riemann-Liouville fractional integral (RLFI) of order α with respect to t is defined by

$$I_t^\alpha f(x,t) := \frac{1}{\Gamma(\alpha)} \int_0^t f(x,\tau)(t-\tau)^{\alpha-1} d\tau, \tag{3}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $\alpha \in (0,1)$. $\triangle$

Definition 4. [Kilbas et al., 2006] Given $\ell, T \in \mathbb{R}_+$ and with $f : [0,\ell] \times [0,T] \to \mathbb{R}$, $f(x,\cdot) \in AC[0,T]$ for each $x \in [0,\ell]$, the Caputo fractional partial derivative (CFPD) of order α with respect to t is defined by

$$D_t^\alpha f(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f_\tau(x,\tau)}{(t-\tau)^\alpha} d\tau, \tag{4}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $\alpha \in (0,1)$. $\triangle$

Property 2. Ishteva [2005] Let $\ell, T \in \mathbb{R}_+$, if $f(x,\cdot) \in AC[0,T]$ for each $x \in [0,\ell]$. Thus,

$$\lim_{\alpha \to 1} D_t^\alpha f(x,t) = f_t(x,t). \tag{5}$$

Property 3. According to [Diethelm, 2010], the CFPD satisfies the following semigroup property. Let $\ell, T \in \mathbb{R}_+$, consider $f(x,\cdot) \in C^1[0,T]$ for almost all $x \in [0,\ell]$, and let $\alpha, \beta > 0$ be such that $[\alpha, \alpha+\beta] \in [0,1]$, then

$$D_t^\alpha D_t^\beta f(x,t) = D_t^{\beta+\alpha} f(x,t). \tag{6}$$

Property 4. Theorem 3.7 [Diethelm, 2010]. Let $\ell, T \in \mathbb{R}_+$ and $(x,t) \in [0,\ell] \times [0,T]$. If $f(x,\cdot) \in C[0,T]$, and $\alpha \geq 0$ for each $x \in [0,\ell]$, then

$$D_t^\alpha I_t^\alpha f(x,t) = f(x,t). \tag{7}$$

By a generalization of the fundamental theorem of calculus, the CFPD can be understood as the left inverse of the RLFI operator. The CFPD, however, is not the right inverse for the RLFI as the following property indicates.

Property 5. Theorem 3.8 [Diethelm, 2010]. Let $(x,t) \in [0,\ell] \times [0,T]$. If $f(x,\cdot) \in AC[0,T]$, $\alpha \in (0,1)$ for each $x \in [0,\ell]$ then

$$I_t^\alpha D_t^\alpha f(x,t) = f(x,t) - f(x,0). \tag{8}$$

Property 6. Lemma 2.24 [Kilbas et al., 2006]. Let $\ell, T \in \mathbb{R}_+$, $\alpha \in (0,1)$, $f : [0,\ell] \times [0,T] \to \mathbb{R}$, if $f(x,\cdot) \in L_1[0,T]$ and $f(x,\cdot) \in AC[0,T]$ for each $x \in [0,\ell)$, then the Laplace transform of the CFPD s given by

$$\mathcal{L}\{D_t^\alpha f(x,t)\} = s^\alpha F(x,s) - s^{\alpha-1} f(x,0), \tag{9}$$

where $F(x,s) = \int_0^\infty f(x,t) \mathrm{e}^{-st} dt$ is the Laplace transform with respect to time of $f(x,t)$ for all $x \in [0,\ell]$.

3. PROBLEM FORMULATION AND MAIN RESULT

This section focuses on the search for the integral transformation that establishes the equivalence between the FDM and the integer-order 1DDM. Specifically, the necessary and sufficient conditions for the construction of the integral transformations are given.

3.1 Problem formulation

Consider without a loss of generality the non dimensional 1DDM

$$v_{xx}(x,t) = v_t(x,t) \tag{10}$$

and the functions $\phi : [0,\ell] \to \mathbb{R}$ and $\rho : [0,T] \to \mathbb{R}$ associated with the boundary and initial conditions

$$v(x,0) = \phi(x); v(0,t) = \rho(t); v(\ell,t) < \infty. \tag{11}$$

such that (10) is well-posed, *i.e.*, the solution $v(x,t)$ exists and is unique for all $(x,t) \in [0,\ell] \times [0,T]$.

It is claimed that the solution $w(x,t)$ of the FDM of order $0 < \alpha < 1$

$$D_t^\alpha w(x,t) = -w_x(x,t), \tag{12}$$

on the same bounded domain of (10), is one-time continuously differentiable with respect to x and t, respectively, and it can be expressed as the FOIT

$$w(x,t) = v(x,t) - \int_0^t k(t-\tau) v_x(x,\tau) d\tau, \tag{13}$$

where the fraction α and the kernel function $k(t,\tau) := k(t-\tau)$ are unknown and must be determined together with the initial and boundary conditions of the equivalent FDM.

To establish the existence of (13) and that its integral converges, the functions $k(t,\tau)$ and $v_x(x,t)$ are assumed $\mathcal{L}_1[0,T]$ for each fixed x. Note that the integral in (13) corresponds to the convolution of $k(t)$ with $v_x(x,t)$. As a consequence of the above guess, if the function $k(t)$ and the rational α are determined, one can look for the IFOIT that allows obtaining the solution $v(x,t)$ from $w(x,t)$. Hence, the transformation and its inverse validate the equivalence between the models; see the commutative diagram of Fig. 1.

3.2 Main result

The following two propositions establish the conditions of the above claims. This means the direct and inverse integral transformations for the integer and fractional diffusion models are assigned. Firstly the parameter α and the function $k(t)$ are determined, and later on, by using this characterization, the inverse integral transformation is obtained.

Direct integral transformation

Proposition 1. Let $v(x,t)$ be the solution of the 1DDM (10) on the bounded domain $(x,t) \in [0,\ell] \times [0,T]$ with initial and boundary conditions given by (11), and consider that kernel function $k(t)$ and $v_x(x,t)$ are $\mathcal{L}_1[0,T]$ functions for each x. Therefore, the FOIT (13) is the unique solution of (12) on the same bounded domain of (10) with its boundary and initial conditions given by

$$w(x,0) = \phi(x); w(0,t) = \rho(t) - I_t^\alpha v_x(0,t); w(\ell,t) < \infty, \tag{14}$$

if and only if the kernel function is $k(t-\tau) = \frac{1}{\sqrt{\pi}} \frac{1}{(t-\tau)^\alpha}$ and the fractional order is $\alpha = \frac{1}{2}$.

Proof 1. **Necessary condition:** To determine the unique transformation, it is necessary to replace (13) in the target system (12) and find $k(t,\tau)$ and α as follows.

By taking the derivative of the transformation (13) with respect to x and by using the model (10) with the integration by parts of the last term, one can write

$$w_x(x,t) = v_x(x,t) - \int_0^t k(t,\tau) v_{xx}(x,\tau) d\tau, \qquad (15)$$
$$= v_x(x,t) - \Big[k(t,\tau) v(x,\tau) \Big]_{\tau=0}^{t} + \int_0^t k_\tau(t,\tau) v(x,\tau) d\tau.$$

By using Property 4, the first term of (15) can be written by

$$v_x(x,t) = D_t^\alpha I_t^\alpha v_x(x,t) \qquad (16)$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial}{\partial \tau} \int_0^\tau \frac{v_x(x,\xi)}{(\tau-\xi)^{1-\alpha}} d\xi d\tau$$

Now by differentiating the term into the integral, one gets

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{v_x(x,\xi)}{(\tau-\xi)^{1-\alpha}} \Big|_{\xi=\tau} d\tau +$$
$$+ \frac{\alpha-1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^\tau \frac{v_x(x,\xi)}{(\tau-\xi)^{2-\alpha}} d\xi d\tau \qquad (17)$$

Therefore, the right-hand side of the target system (12) is given by

$$w_x(x,t) = -\Big[k(t,\tau) v(x,\tau) \Big]_{\tau=0}^{t} + \int_0^t k_\tau(t,\tau) v(x,\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{v_x(x,\xi)}{(\tau-\xi)^{1-\alpha}} \Big|_{\xi=\tau} d\tau \qquad (18)$$
$$+ \frac{\alpha-1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^\tau \frac{v_x(x,\xi)}{(\tau-\xi)^{2-\alpha}} d\xi d\tau$$

On the other hand, according to Definition 4, the CFPD (4) of $w(x,\tau)$ is a function of

$$w_\tau(x,\tau) = \frac{\partial}{\partial \tau} \left(v(x,\tau) - \int_0^\tau k(\tau,\xi) v_x(x,\xi) d\xi \right) \qquad (19)$$
$$= v_\tau(x,\tau) - k(\tau,\tau) v_x(x,\tau) - \int_0^\tau k_\tau(\tau,\xi) v_x(x,\xi) d\xi.$$

Thus, by (19), the CFPD of $w(x,t)$ can be written

$$D_t^\alpha w(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} v_\tau(x,\tau) d\tau$$
$$- \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} k(\tau,\tau) v_x(x,\tau) d\tau \qquad (20)$$
$$- \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^\tau k_\tau(t,\xi) v_x(x,\xi) d\xi d\tau.$$

By integrating by parts the first term in (20), by adding (18) and by factorizing similar terms one gest

$$w_x(x,t) + D_t^\alpha w(x,t) = \frac{1}{\Gamma(1-\alpha)} \Bigg($$
$$\int_0^t (t-\tau)^{-\alpha} \left[\frac{1}{\Gamma(\alpha)(\tau-\xi)^{1-\alpha}} - k(\tau,\xi) \right]_{\xi=\tau} v_x(x,\tau) d\tau \Bigg)$$
$$+ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \Bigg($$
$$\int_0^\tau \left[\frac{\alpha-1}{\Gamma(\alpha)(\tau-\xi)^{2-\alpha}} - k_\tau(\tau,\xi) \right] v_x(x,\xi) d\xi d\tau \Bigg)$$
$$- \left[\left(k(t,\tau) - \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \right) v(x,\tau) \right]_{\tau=0}^{t} \qquad (21)$$
$$+ \int_0^t \left[k_\tau(t,\tau) - \frac{\alpha(t-\tau)^{-\alpha-1}}{\Gamma(1-\alpha)} \right] v(x,\tau) d\tau$$

Then $w(x,t)$ is a solution of (12) only if the right-hand side of (21) is equal to 0 for all $v(x,t)$ on its bounded domain. As a consequence, to satisfy the summation equal to zero for all $v(x,t)$ on the domain $[0,\ell] \times [0,T]$, the four conditions

Condition 1 : $k(\tau,\xi) = \dfrac{1}{\Gamma(\alpha)(\tau-\xi)^{1-\alpha}}$.

Condition 2 : $k_\tau(\tau,\xi) = \dfrac{\alpha-1}{\Gamma(\alpha)(\tau-\xi)^{2-\alpha}}$.

Condition 3 : $k(t,\tau) = \dfrac{1}{\Gamma(1-\alpha)(t-\tau)^{\alpha}}$. and

Condition 4 : $k_\tau(t,\tau) = \dfrac{\alpha}{\Gamma(1-\alpha)(t-\tau)^{\alpha+1}}$.

must be satisfied.

One can see from the above conditions that

- Conditions 1 and 3 are the anti derivative of Conditions 2 and 4 respectively;
- Conditions 2 and 4 are the derivative of Conditions 1 and 3 respectively.

Thus, it remains to obtain the function $k(t-\tau)$ and the parameter α such that Condition 1 implies Condition 3. A dummy change of variables in Condition 1 and Condition 3 yields

$$k(t,\tau) = \frac{1}{\Gamma(\alpha)(t-\tau)^{1-\alpha}} = \frac{1}{\Gamma(1-\alpha)(t-\tau)^{\alpha}},$$

which can be rewritten as

$$\frac{\Gamma(\alpha)}{(t-\tau)^{\alpha}} = \frac{\Gamma(1-\alpha)}{(t-\tau)^{1-\alpha}}. \qquad (22)$$

Thus, this equality holds when $\alpha = 1-\alpha$, i.e., $\alpha = 1/2$.

In summary, if $\alpha = 1/2$, the kernel function

$$k(t) = \frac{1}{\Gamma(\frac{1}{2})(t)^{\frac{1}{2}}} = \frac{1}{\sqrt{\pi t}}$$

is assigned into the direct integral transformation

$$w(x,t) = v(x,t) - \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}} v_x(x,\tau)d\tau, \quad (23)$$

then the fractional diffusion model

$$D_t^{1/2} w(x,t) = -w_x(x,t) \quad (24)$$

is obtained.

Since, $v_x(x,t)$ and $k(t,\tau)$ [1] are $\mathcal{L}_1[0,T]$. functions, the integral term in (23) is well defined for all fixed $x \in [0,\ell]$, and the initial and boundary conditions of (24) can be fixed by evaluating the integral transformation (23). Thus, one gets the conditions

- For $t = 0$, $w(x,0) = v(x,0) = \phi(x)$,
- For $x = 0$, $w(0,t) = \rho(t) - \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}} v_x(x,\tau)|_{x=0} d\tau$,
- For $x = \ell < \infty$, $w(0,t) = v(\ell,t) - \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}} v_x(x,\tau)|_{x=\ell} d\tau < \infty$.

Sufficient condition: This condition is trivial, and it follows by substituting the FOIT (13) with $\alpha = 1/2$ and $k(t,\tau) = \dfrac{1}{\sqrt{\pi}(t-\tau)^{1/2}}$ into the 1DDM in (10), which yields to FDM (12). □

Inverse integral transformation To guarantee the equivalent relation between (10) and (24), the transformation such that the solution of (10) can be expressed by the inverse of (23) is presented. For simplicity, the following sufficient condition is given as an IFOIT.

Proposition 2. Let $w(x,t)$ be the solution of the FDM (24) on the bounded domain $(x,t) \in [0,\ell] \times [0,T]$ with initial and boundary conditions given by (14) and the $w_x(x,t)$ be $\mathcal{L}_1[0,T]$ function for all x. If the inverse integral transformation is given by

$$v(x,t) = w(x,t) + \frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}} w_x(x,\tau)d\tau, \quad (25)$$

then the IFOIT (25) is the right inverse of the FOIT (23). Moreover, $v(x,t)$ is the solution of the 1DDM (10) on the same bounded domain and conditions given by (11).

Proof 2. The proof consists of two steps. First, by substituting (25) into (23), it is shown that the IFOIT (25) is the right inverse of the FOIT (23). Second, the IFOIT (25) is substituted into 1DDM (10) to obtain the FDM (24).

4. EXAMPLE FOR SPECIFIC BOUNDARY CONDITIONS

As mentioned before, the previous works on the 1DDM did not obtain the initial and boundary conditions of the FDM. The goal of this section is to exemplify that there is no need to impose arbitrary or un realistic boundary and initial conditions for obtaining the solution of the FDM. Instead, the results here determine that the solution of the FDM through the transformation based on a well-posed problem of the 1DDM that actually obeys the physical rules.

[1] Note that $\frac{1}{\sqrt{\pi}}\int_0^T |t^{-\frac{1}{2}}|dt = -\frac{2}{\sqrt{\pi}}\sqrt{T} < \infty$

4.1 Boundary condition evaluation

Consider the particular case of the 1DDM (10), *e.g.*, a leakage-free non-inductive cable with the following boundary and initial conditions [Cheng, 1959]:

$$\begin{aligned} v(x,0) &= 0 \\ v(0,t) &= u(t) \\ v(\ell,t) &< \infty \end{aligned} \quad (26)$$

By considering Proposition 1, one can say that the solution of the target system (24) exists and is unique. Moreover, the boundary and initial conditions

$$\begin{aligned} w(x,0) &= 0, \\ w(0,t) &= 2u(t), \\ w(\ell,t) &< \infty. \end{aligned} \quad (27)$$

can be obtained from the transformation (23).

To show the effectiveness of the result once the boundary and initial conditions (27) have been determined, the FDM solution through the direct FOIT is obtained with the knowledge of the solution of $v(x,t)$.

4.2 Solution of FDM

The solution of the FDM (24) can be found through the direct FOIT (23) by considering the solution of the 1DDM (10) with boundary and initial conditions (26), $v(x,t) = erfc\left(\dfrac{x}{2\sqrt{t}}\right)$, and its partial derivative with respect to x,

$$v_x(x,t) = \frac{-e^{-z^2}}{\sqrt{\pi t}} \quad (28)$$

where $z = \dfrac{x}{2\sqrt{t}}$. By substituting $v(x,t)$ and $v_x(x,t)$ into direct FOIT (23), one gets

$$w(x,t) = erfc\left(\frac{x}{2\sqrt{t}}\right) - \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}}\left(\frac{-e^{-z^2}}{\sqrt{\pi\tau}}\right)d\tau. \quad (29)$$

It is left to reduce the integral term in (29), which fulfills the following equality (as in [Bateman, 1954], on page 187, Eq. 18):

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}}\frac{-e^{-z^2}}{\sqrt{\pi\tau}}d\tau = \left(\frac{x^2}{4}\right)^{-1/4} t^{1/4} exp(-\frac{x^2}{8t})\mathcal{W}_{-1/4,1/4}\left(\frac{x^2}{4t}\right) \quad (30)$$

where $\mathcal{W}(\cdot)$ is the Whittaker function. From the relationship of the Whittaker function and the error function found on Bateman [1954] pag. 431, one can find that

$$\frac{1}{\sqrt{\pi}}\left(\triangle^2\right)^{-1/4} exp\left(-\frac{1}{2}\triangle^2\right) W_{-1/4,1/4}\left(\triangle^2\right) = Erfc(\triangle) \tag{31}$$

where $\triangle = \dfrac{x}{2\sqrt{t}}$. Finally, from (29), (30) and (31) the solution of the FDM is given by

$$w(x,t) = 2erfc\left(\frac{x}{2\sqrt{t}}\right). \tag{32}$$

Fig. 2 shows the behavior of the solution of the (24) with the boundary and initial conditions (27) given in (32). This result coincides with the solution that can be obtained by the Fourier transform method for (24) and (27).

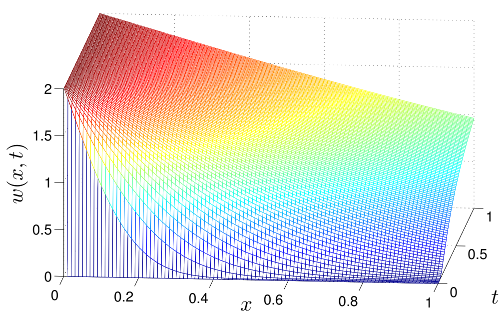

Fig. 2. Solution profile of $w(x,t)$

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5. DISCUSSION AND CONCLUSION

5.1 Discussion

The importance of the proposed transformations is that there exist a bidirectional link between the two integer and fractional order diffusion models (see right loop in Fig. 3). This is not the case in the methodology commonly found in the literature based on the Laplace transform (see unidirectional left path in Fig. 3). Even if the common methodology is simple (see dotted red line in Fig. 3), this requires the explicit solution of the diffusion model obtained from zero initial conditions, which is not the case in the proposed approach in this paper.

A limitation in this work is that the proposed transformation is valid for the diffusion equation. However, for $n \in \mathbb{N} \setminus \{0\}$, the transformation can be easily generalized for the equation

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^{2n} v(x,t)}{\partial x^{2n}}$$

that can be transformed to

$$D_t^{1/2} w(x,t) = \frac{\partial^n w(x,t)}{\partial x^n}$$

where only for the case $n = 1$ the diffusion equation has physical and practical importance.

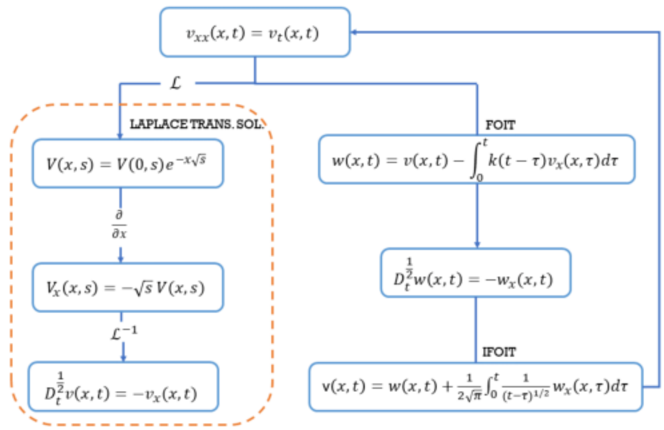

Fig. 3. Method comparison diagram.

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In the authors' opinion, the framework introduced here to deal with the equivalence between integer-order and fractional-order partial equations offers new opportunities for research:

- By means of the introduction of model equivalence, the application of fractional order models to real problems is possible.
- Model order reduction, specifically to reduce the space derivatives included in the models.
- To find some physical interpretation of fractional order models.

Future directions of this work consider the study of model equivalence of more classes of PDEs.

5.2 Conclusion

A new insight for studying the equivalence between first-order PDEs and FPDEs has been addressed. In particular, the equivalence between a 1DDM and a FDM through the use of fractional-order integral transformations is introduced. For the first time, the equivalence is formally obtained through an integral transformation and its inverse. Furthermore, it is possible to show that the FDM corresponds to the equivalent 1DDM for $\alpha = 1/2$ by using the proposed integral transformations based on the Caputo derivative. Moreover, the well-posedness of the FDM of order $\alpha = 1/2$ is trivially determined by this technique. It is worth mentioning that this transformation is not general. This means, the integral transformation is found according to the systems's order and structure, the specific definition of the fractional derivative and boundary conditions.

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