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Adaptive AEM Controller For A Wide Class of Nonlinear Discrete-Time Stochastic Systems Using On-Line State Estimation

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Abstract: This paper describes the Attractive Ellipsoid Method (AEM) application, which uses the state estimates obtained by a sliding mode observer (SMO) for a wide class of quasi-Lipschitz nonlinear stochastic discrete-time systems. For the extended vector, containing state estimation and tracking errors as its components, we prove the mean square convergence to an attractive ellipsoid, which "size" is done as small as possible by the corresponding optimal selection of the gain matrices in both the SMO and in the linear feedback, using obtained current state estimates. It is shown that the procedure of the gain matrices optimization consists of the numerical solution of a corresponding matrix optimization problem subject to a set of bilinear matrix inequalities (BMIs), which by a special transformation procedure can be converted to a set of linear matrix inequalities (LMIs). An illustrative example shows the effectiveness of the suggested approach.

Keywords: Control de Sistemas No Lineales, Sistemas Discretos, Sistemas Estocásticos.

1. INTRODUCTION

The *Attractive Ellipsoid Method* (AEM) provides researchers a special tool for designing linear feedback for a wide class of nonlinear systems containing both uncertainties in the description of the model and possible external bounded perturbations Poznyak et al. (2014). Usually, the application of this method requires the exact knowledge (availability) of all current states and control actions in the use. When the required variables or their part are not available online, a possible approach consists in the realization of a state estimation process with the direct usage of them in the applied control actions. In some sense, such construction may be treated as an adaptive controller, which in our case is referred as to the adaptive AEM. As an example of such approach, in the deterministic case, we can mention the recent paper Hernandez-Gonzalez et al. (2019), presents a method to identify an unknown discrete-time nonlinear system, using high-order neural networks and high-order sliding mode algorithms, which are subject to internal and external disturbances. A SMO have also been applied in systems with deterministic bounded perturbations Bejarano et al. (2007); Moreno and Osorio (2008); Davila et al. (2006). In Oliveira et al. (2018), an adaptive sliding mode control strategy, based on the extended equivalent control, is developed. The adaptation rule combines the qualities of monotonically increasing gains and the equivalent control. Here we consider the wide class of quasi-Lipschitz nonlinear *stochastic discrete time systems* where the state estimates are obtained by the special version of SMO, providing an acceptable mean square level of state space estimation accuracy. Specific feature of stochastic systems consists in the consideration of unbounded random external perturbation that makes impossible the direct application of AEM and SMO approaches: some special constraints and extensions are required. So, in Chen et al. (2019) the network-based SMO is investigated for a class of discrete nonlinear time-delay systems with stochastic communication protocol. The stochastic communication protocol is governed by a Markov chain, which converts the protocol-constrained system into a Markovian jump system. The purpose is to design a SMO such that the trajectories of the estimation error system are driven into a band of the sliding surface and, in subsequent time, the sliding motion is mean-square asymptotically stable. In Qiao et al. (2008) an adaptive SMO is designed to reconstruct the states of non-linear stochastic continuous time systems with uncertainties, from the measurable system output and the reconstructed states are employed to construct a sliding mode controller for the stabilization. It takes the advan-

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tages of the sliding mode schemes to design both the observer and the controller. The convergence of the observer and the globally asymptotic stability of the controller are analyzed in terms of stochastic Lyapunov stability, and the effectiveness of the control strategy is verified with numerical simulation studies. The number of works, in which the methodology of sliding modes is applied to observe or control the Discrete Time Stochastic Systems, is in fact very limited Wu et al. (2010); Abidi et al. (2007); L. Wu (2013); Kailath (1980), Alcorta-GarcIa et al. (2009), Basin and Rodríguez-Ramírez (2013), basically deals with linear models (see, for example, S. Janardhanan (2017)). However, in Velázquez and Poznyak (2021) you can see the application of SMOs to nonlinear stochastic systems using AEM as convergence analysis. The stability analysis of nonlinear discrete time stochastic systems can be found in Bensoubaya et al. (1999). The recent and most advanced studies, concerning the SMOs design for discrete time systems, can be found in Janardhanan and Bandyopadhyay (2007); Bandyopadhyay and Janardhanan (2005) and Alazki and Poznyak (2010). The merit of this paper is to propose the exact mechanism for designing an adaptive version of AEM and SMO, which, working simultaneously, provide a good behavior in some probabilistic sense for a wide class of uncertain nonlinear stochastic systems.

2. STOCHASTIC DISCRETE - TIME NONLINEAR PLANT

2.1 Model of the process

Consider the stochastic discrete-time system

$$\left.\begin{array}{r} x\left(k+1\right)=f(k,x(k))+Bu(k)+\xi(k+1)\in\mathbb{R}^{n} \\ y(k)=Cx(k)+\zeta(k)\in\mathbb{R}^{m} \\ u(k)\in\mathbb{R}^{l},\quad k=0,1,2... \end{array}\right\} \quad (1)$$

Random sequence $\{y(k)\}_{k\geq 0}$ is available during the process, but $\{x(k)\}_{k\geq 0}$ not. Measurable input is $\{u(k)\}_{k\geq 0}$. $\xi(k+1)$ and $\zeta(k)$ are the input and output stochastic noises, respectively. This sequences are defined on the probability space $\left(\Omega,\{\mathcal{F}_k\}_{k\geq 0},P\right)$, where $\{\mathcal{F}_k\}_{k\geq 0}$ is a flow of the σ -algebras $\mathcal{F}_k$, which for each $k=0,1,...$ is a minimal sigma-algebra, generated by the prehistory of the process, i.e.,

$$\mathcal{F}_k=\sigma\left\{x\left(0\right),u\left(0\right),\xi_y\left(0\right);...;x\left(k\right),u\left(k\right),\xi_x\left(k\right),\xi_y\left(k\right)\right\}. \quad (2)$$

2.2 Main assumptions

Suppose that

A1) Random variables $\xi_x\left(k+1\right)$ and $\xi_y\left(k\right)$ are independent martingale-differences, namely,

$$\left.\begin{array}{c} \mathrm{E}\left\{\xi_x\left(k+1\right)\mid\mathcal{F}_k\right\}\overset{a.s.}{=}0,\ \mathrm{E}\left\{\xi_y\left(k\right)\mid\mathcal{F}_k\right\}\overset{a.s.}{=}0, \\ \mathrm{E}\left\{\xi_x\left(k+1\right)\xi_y^{\intercal}\left(k\right)\mid\mathcal{F}_k\right\}\overset{a.s.}{=}0 \end{array}\right\} \quad (3)$$

with bounded conditional covariation matrices

$$\left.\begin{array}{r} \mathrm{E}\left\{\xi_x\left(k+1\right)\xi_x^{\intercal}\left(k+1\right)\mid\mathcal{F}_k\right\}\overset{a.s.}{\leq}\Xi_x, \\ \mathrm{E}\left\{\xi_y\left(k\right)\xi_y^{\intercal}\left(k\right)\mid\mathcal{F}_k\right\}\overset{a.s.}{\leq}\Xi_y; \end{array}\right\} \quad (4)$$

A2) the nonlinear mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ is supposed to be *a* priory *unknown* but belonging to the class $C\left(A,f_0,f_1\right)$ of *quasi-Lipschitz functions* (see Poznyak et al. (2014)), which means

$$\left\|f\left(x\left(k\right),k\right)-Ax\left(k\right)\right\|^2\leq f_0+f_1\left\|x\left(k\right)\right\|^2 \quad (5)$$

globally on $\mathbb{R}^n$;

A3) The matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times l}$ and $C\in\mathbb{R}^{m\times n}$ are assumed to be known such that the pair (C,A) is observable, and (A,B) is controllable.

Here $\mathrm{E}\left\{\cdot/\mathcal{F}_k\right\}$ and $\mathrm{E}\{\cdot\}$ represent the operators of conditional and complete mathematical expectation.

3. PROBLEM STATEMENT

Before the formulation problem we need to describe the class of observer and controller which will be considered.

3.1 Sliding mode observer

The on-line state estimates $\{\hat{x}(k)\}_{k\geq 0}$ of $\{x(k)\}_{k\geq 0}$ is generated by the SMO:

$$\left.\begin{array}{c} \hat{x}(k+1)=A\hat{x}(k)+Bu(k)+L\sigma(k)+L_a\mathrm{Sign}\left(\sigma(k)\right), \\ \sigma(k)=y(k)-C\hat{x}(k) \end{array}\right\} \quad (6)$$

3.2 Robust controller

The control actions will be deigned as a linear feedback

$$\left.\begin{array}{c} u(k):=K\hat{x}(k)+v(k), \\ v(k):=-Kx_k^*-B^{+}[Ax_k^*-\varphi(k+1,x_k^*)], \\ BB^{+}B=B,\ B^{+}BB^{+}=B^{+} \end{array}\right\} \quad (7)$$

depending on the desired dynamics given by $x^*(k)=\varphi(k,x^*(k-1))\in\mathbb{R}^n$.

3.3 Problem formulation

The main problem of this paper can be formulated as follows.

Problem 1. For the extended vector

$$z(k)=\left(\ \delta^{\intercal}(k)\ \ e^{\intercal}(k)\ \right)^{\intercal}\in\mathbb{R}^{2n} \quad (8)$$

with the components, defined as

$$\delta(k):=x(k)-x^*(k),\ e(k):=x(k)-\hat{x}(k), \quad (9)$$

where $\delta(k)$ is the tracking error and $e(k)$ the state estimation error, to find the gain matrices $K\in\mathbb{R}^{l\times n}$, $L\in\mathbb{R}^{n\times m}$ and $L_a\in\mathbb{R}^{n\times m}$ such that the joint mean square weighted error $\mathrm{E}\left\{z^{\intercal}(k)P_z z(k)\right\}$ belongs asymptotically to the **stochastic attractive ellipsoid**, fulfilling the inequality

$$\limsup_{k\to\infty}\mathrm{E}\left\{z^{\intercal}(k)P_z z(k)\right\}\leq 1 \quad (10)$$

for any admissible nonlinearity $f\in C\left(A,f_0,f_1\right)$.

4. ZONE-CONVERGENCE ANALYSIS

4.1 Tracking error

The tracking error dynamics in (9) with the control (7) results in:

$$\left.\begin{array}{r}\delta(k+1)=(A+BK)\,\delta(k)-BKe(k)+\vartheta(k)\\ \vartheta(k):=\hat{\xi}(k+1)-\hat{\delta}(k),\\ \hat{\delta}(k):=\varphi(k+1,x^{*}(k))-Ax^{*}(k).\end{array}\right\} \quad (11)$$

4.2 Observation error

For the observation error $e(k)$ in (9) it follows

$$\left.\begin{array}{r}e(k+1)=(A-LC)\,e(k)-L_{a}\mathrm{Sign}\,(\sigma(k))+\omega(k),\\ \omega(k):=\hat{\xi}(k+1)-L\zeta(k),\\ \mathrm{Sign}\,(\sigma):=(\mathrm{sign}\,(\sigma_{1})\,,...,\mathrm{sign}\,(\sigma_{n}))^{\intercal}\,,\\ \mathrm{sign}\,(\sigma_{i}):=\left\{\begin{array}{cl}1 & \text{if } \sigma_{i}>0\\ -1 & \text{if } \sigma_{i}<0\\ [-1,1] & \text{if } \sigma_{i}=0\end{array}\right..\end{array}\right\} \quad (12)$$

4.3 Storage function analysis

Theorem 1. If matrices P,K,L,L_a and scalars α,β,γ are selected in such a way that

$$\left.\begin{array}{c}\tilde{W}\left(P,K,L,L_{a}\mid\alpha,\lambda,\gamma\right)=\\ \begin{pmatrix}3\tilde{A}^{\intercal}P\tilde{A}+\tilde{\Lambda}_{\delta}-\alpha P & 0 & 0\\ 0 & 2Q^{\intercal}PQ-\lambda I & Q^{\intercal}P\\ 0 & PQ & 2P-\gamma I\end{pmatrix}\leq 0,\end{array}\right\} \quad (13)$$

with

$$\tilde{\Lambda}_{\delta}=\begin{bmatrix}6\gamma f_{1}I_{n\times n} & 0\\ 0 & 0\end{bmatrix}, \quad (14)$$

then for the storage (Lyapunov-like) function

$$V(k)=z^{\intercal}(k)Pz(k),\ 0<P=\begin{pmatrix}P_{1} & 0\\ 0 & P_{2}\end{pmatrix}, \quad (15)$$

we may guarantee that

$$\mathrm{E}\left\{V_{k+1}\right\}\leq\alpha\mathrm{E}\left\{V(k)\right\}+\tilde{\beta}_{k}(L). \quad (16)$$

where

$$\left.\begin{array}{r}\tilde{\beta}_{k}(L):=m\lambda+\gamma\left(2\mathrm{tr}\left\{\Sigma_{x}\right\}+3f_{0}+6f_{1}\left\|x_{k}^{*}\right\|^{2}\right)\\ +\gamma\left(\mathrm{tr}\left\{L\Sigma_{y}L^{\intercal}\right\}+2\|\tilde{\delta}_{k}\|^{2}\right),\ \tilde{\delta}_{k}=\left(I-BB^{+}\right)\hat{\delta}(k).\end{array}\right\} \quad (17)$$

Proofs of this and the following next statement can be found in Appendix.

4.4 Analytical representation of attractive ellipsoid

Taking θ as upper bound of $\mathrm{tr}\left\{L^{\intercal}\Sigma_{y}L\right\}$ in (17), we can write:

$$\mathrm{tr}\left\{\frac{\theta}{n}I_{n\times n}-L\Sigma_{y}L^{\intercal}\right\}\geq 0, \quad (18)$$

then, $\limsup_{k\to\infty}$ of (16), is:

$$\limsup_{k\to\infty}\mathrm{E}\left\{V_{k}\right\}\leq\frac{\tilde{\beta}_{k}(L)}{1-\alpha}\leq\frac{\psi(\beta,\gamma,\theta)}{1-\alpha},$$
$$\psi(\cdot):=m\beta+\gamma\left(2\mathrm{tr}\left\{\Sigma_{x}\right\}+3f_{0}+6f_{1}X_{+}^{*}+2\Delta_{+}^{*}+\theta\right),$$
$$\|x_{k}^{*}\|^{2}\leq X_{+}^{*},\ \text{ and }\ \left\|\tilde{\delta}_{k}\right\|^{2}\leq\Delta_{+}^{*},$$

for convenience $\psi(\cdot)=\psi(\beta,\gamma,\theta)$. This may be represented as

$$\limsup_{k\to\infty}\mathrm{E}\left\{z_{k}^{\intercal}\ P_{z}\ z_{k}\right\}\leq 1,\ \ P_{z}:=\frac{1-\alpha}{\psi(\cdot)}\begin{pmatrix}P_{1} & 0\\ 0 & P_{2}\end{pmatrix},$$

which, according to the definition (10), defines the stochastic attractive ellipsoid with matrix P_z.

4.5 Gain matrix optimization

To minimize the errors e_k and δ_{k+1}, we need to maximize P_z with respect to the matrices L_a, L, K and the scalar positive parameters α,β,γ. The optimal matrix gains L_a^*, L^*, K^* are suggested to be found as the solution of the following optimization problem

$$\mathrm{tr}\left\{\frac{1-\alpha}{\psi(\cdot)}\begin{pmatrix}P_{1} & 0\\ 0 & P_{2}\end{pmatrix}\right\}\to\sup_{\substack{P_{1}>0,P_{2}>0,L_{a},L,K;\\ \alpha>0,\beta>0,\gamma>0}}$$

under constraints (13) and (18). These constraints are bilinear ones, to apply Matlab Package, we need to transform them into linear ones. See the next theorem.

Theorem 2. Taking $P_1^*=X_1$, $P_2=X_2$, $K=G$, $P_2L=Y_1$, and $P_2L_a=Y_2$. Inequalities (13) and (18) are fulfilled if the following LMIs hold:

$$\bar{W}^{+}\leq 0,\ W_{Q_{1}}\geq 0,\ W_{Q_{2}}\geq 0,\ Q_{1}\geq 0,\ Q_{2}\geq 0,W_{\theta}\geq 0 \quad (19)$$

here

$$\bar{W}^{+}:=\begin{pmatrix}-Q_{1} & 0 & 0\\ 0 & -Q_{2} & W_{Y_{2}}\\ 0 & W_{Y_{2}} & 2X-\gamma I\end{pmatrix},W_{Y_{2}}:=\begin{bmatrix}0 & 0\\ 0 & Y_{2}\end{bmatrix}$$

$$W_{Q_{1}}:=\begin{pmatrix}\frac{1}{3}\left(\alpha X-\tilde{\Lambda}_{\delta}-Q_{1}\right) & W_{Q_{1}}(1,2)\\ W_{Q_{1}}(2,1) & \begin{bmatrix}\frac{\gamma}{2}I & 0\\ 0 & X_{2}\end{bmatrix}\end{pmatrix},$$

$$W_{Q_{1}}(1,2)=W_{Q_{1}}^{\intercal}(2,1):=\begin{pmatrix}(A+BG)^{\intercal} & 0\\ -G^{\intercal}B^{\intercal} & A^{\intercal}X_{2}-C^{\intercal}Y_{1}\end{pmatrix}$$

$$W_{Q_{2}}:=\begin{pmatrix}\frac{1}{2}\left(\lambda I-Q_{2}\right) & W_{Y_{2}}\\ W_{Y_{2}} & X\end{pmatrix},W_{\theta}:=\begin{bmatrix}\frac{\theta}{n}I_{n\times n} & \frac{\gamma}{2}Y_{1}\\ \frac{\gamma}{2}Y_{1}^{\intercal} & \Sigma_{y}^{-1}\end{bmatrix}.$$

Notice that, with Theorem 2, for the parameters α, λ, γ and θ after some transformations the matrix inequalities (13) and (18) become LMIs. They can be solved using the LMItoolbox, SeDuMi and Yalmip. Our optimization problem can be also solved following the next two-steps:

1 we fix the scalar parameters α, λ, γ and θ, and solve the LMIs with respect to the matrix variables.

2 for the found matrix variables X_1, Y_1, X_2, Y_2 and G, solve our optimization problem only with respect to scalar parameters α, λ, γ and θ.

Finally, iterating this process we find the optimal solution $K^* = G^*$, $L^* = (X_2^*)^{-1} Y_1^*$, $L_a^* = (X_2^*)^{-1} Y_2^*$.

5. ILLUSTRATIVE NUMERICAL ACADEMIC EXAMPLE

Consider the system

$$x(k+1) = f(x(k)) + Bu + \xi(k+1),$$
$$y(k) = x_1(k) + \zeta(k),$$

which in the quasi-linear format is presented as

$$x(k+1) = Ax(k) + Bu(k) + \hat{\xi}(k+1),$$
$$y(k) = Cx(k) + \zeta(k),$$

where $x(k) = [\, x_1(k) \;\, x_2(k) \,]^\intercal$ and

$$\hat{\xi}(k+1) = \xi(k+1) + f(x(k)) - Ax(k),$$
$$f(x(k)) = \begin{bmatrix} x_2(k)\sin(x_1(k)) \\ -0.1(x_1(k) + x_2(k)) \end{bmatrix}$$
$$A = \begin{bmatrix} 0 & 1 \\ 0.1 & 0.1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C = [\, 1 \;\, 0 \,].$$

For (5), $f_0 = 0$ and $f_1 = 1$. Notice that (A, B) is controllable and (C, A) is observable. Here, $\xi(k+1) \leq \Xi_x = 0.01 I_{2\times 2}$ and $\zeta(k) \leq \Xi_y = 0.01$. The desired dynamics is given by

$$x^*(k) = \begin{bmatrix} C_1^* \\ C_2^* \sin(k) \end{bmatrix}, \; C_1^* = 1, \; C_2^* = 0.5.$$

The gain optimization procedure (4.5) leads to

$$\alpha^* = 0.1, \; \beta = 0.1, \; \gamma = 0.9,$$
$$P^* = \begin{bmatrix} 0.0503 & 0 & 0 & 0 \\ 0 & 0.0503 & 0 & 0 \\ 0 & 0 & 0.4499 & -0.0012 \\ 0 & 0 & -0.0012 & 0.0021 \end{bmatrix},$$
$$K^* = \begin{bmatrix} 0 & -0.2596 \\ -0.2596 & 0.0499 \end{bmatrix},$$
$$L^* = \begin{bmatrix} -0.3 \\ -0.6 \end{bmatrix}, \; L_a^* = \begin{bmatrix} 0.015 \\ 0.01 \end{bmatrix}.$$

Trajectories of the real, observed and desired of states are showed on Fig.1 and Fig.2, respectively. Fig.3 and Fig.4 shows the convergence errors to the attractive ellipsoid with simulation step time of $0.01s$.

6. CONCLUSIONS

In this paper we prove the *mean square convergence* of state estimation and tracking errors by the application of AEM for the robust control design of a large class of quasi-Lipschitz nonlinear stochastic discrete-time systems using the on-line state estimates, obtained by the corresponding SMO; the size of the convergence zone (trace of P_z) is minimized by the optimal selection of the gain matrices in the SMO and the linear feedback; the optimal selection is achieved transformation of BMIs into LMIs and using standard MATLAB packages. A numerical example illustrate the effectiveness of the suggested technique.

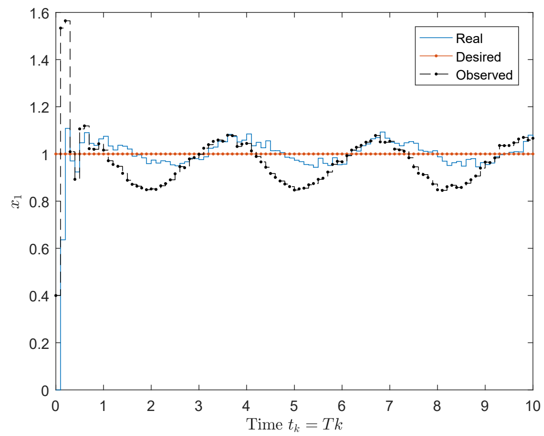

Fig. 1. Real, desired and observed trajectories of the state $x_1(k)$.

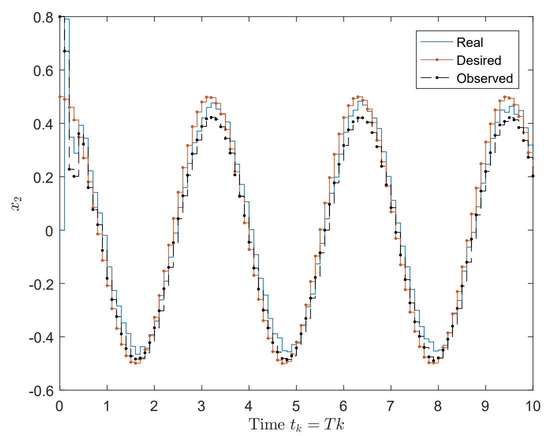

Fig. 2. Real, desired and observed trajectories of state $x_2(k)$.

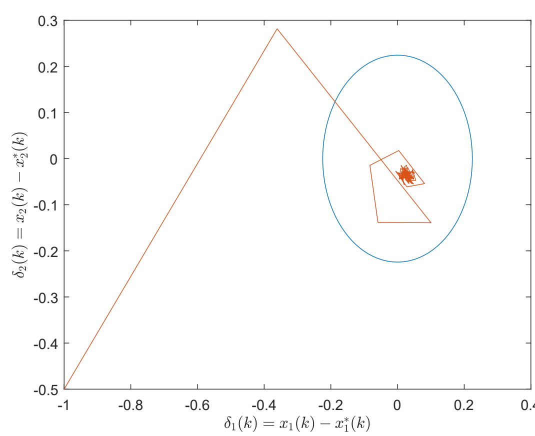

Fig. 3. Traking error $\delta(k)$ convergence.

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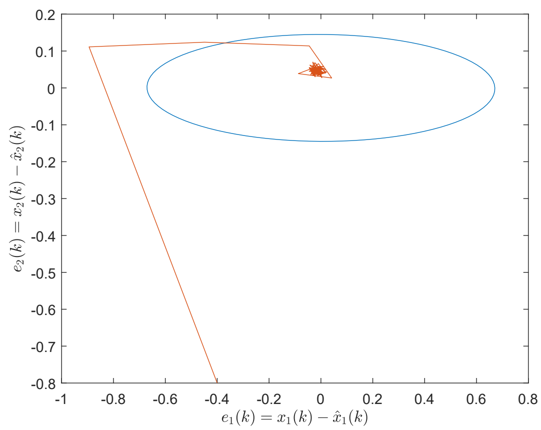

Fig. 4. The observation error $e(k)$ convergence.

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Appendix A. PROOF: THEOREM 1

a) For the vector $z(k) \in \mathbb{R}^{2n}$ (8) with the control (7), we have

$$z(k+1) = \tilde{A}\,(K, L)\, z(k) + Q\,(L_a)\, s(k) + \eta(k, k+1) \quad \text{(A.1)}$$

where

$$\left.\begin{aligned} \tilde{A}\,(K,L) = \begin{pmatrix} (A+BK) & -BK \\ 0_{n\times n} & (A-LC) \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \\ Q\,(L_a) = \begin{pmatrix} 0_{n\times m} \\ -L_a \end{pmatrix} \in \mathbb{R}^{2n\times m}, \\ s(k) = \mathrm{Sign}\,(\sigma\,(k)) \in \mathbb{R}^m,\ \eta(k) = \begin{pmatrix} \vartheta(k) \\ \omega(k) \end{pmatrix} \in \mathbb{R}^{2n}. \end{aligned}\right\} \quad \text{(A.2)}$$

Now, for the storage function $V(k+1)$ it follows

$$\left.\begin{aligned} V(k+1) = z^{\intercal}(k+1) P z(k+1) \\ = \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} \begin{pmatrix} \tilde{A}^{\intercal} P \tilde{A} & \tilde{A}^{\intercal} P Q & \tilde{A}^{\intercal} P \\ Q^{\intercal} P \tilde{A} & Q^{\intercal} P Q & Q^{\intercal} P \\ P\tilde{A} & PQ & P \end{pmatrix} \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} \end{aligned}\right\} \quad \text{(A.3)}$$

Applying Λ-matrix inequality $W^{\intercal}Z + Z^{\intercal}H \leq H^{\intercal}\Lambda H + Z^{\intercal}\Lambda^{-1}Z$, valid for $H, Z \in \mathbb{R}^{K\times M}$ and $\Lambda = \Lambda^{\intercal} > 0$, to $2z^{\intercal}(k)\tilde{A}^{\intercal}PQs(k)$ and $2z^{\intercal}(k)\tilde{A}^{\intercal}P\eta(k,k+1)$, we get

$$\left.\begin{array}{c} V(k+1) \leq \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} W \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} + \\ \alpha z^{\intercal}(k)Pz(k) + \lambda \left\| s(k) \right\|^2 + \gamma \left\| \eta(k) \right\|^2, \ \ |\alpha| < 1, \end{array}\right\} \tag{A.4}$$

where

$$W := \begin{pmatrix} 3\tilde{A}^{\intercal}P\tilde{A} - \alpha P & 0 & 0 \\ 0 & 2Q^{\intercal}PQ - \lambda I & Q^{\intercal}P \\ 0 & PQ & P - \gamma I \end{pmatrix} \tag{A.5}$$

Taking $\mathrm{E}\left\{\cdot|\mathcal{F}_k\right\}$, in both sides of (A.4), we obtain

$$\begin{array}{c} \mathrm{E}\left\{V(k+1)|\mathcal{F}_k\right\} \overset{a.s.}{\leq} \mathrm{E}\left\{ \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} W \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} |\mathcal{F}_k \right\} \\ + \alpha V(k) + \lambda \left\| s(k) \right\|^2 + \gamma \mathrm{E}\left\{ \left\| \eta(k) \right\|^2 |\mathcal{F}_k \right\}. \end{array} \tag{A.6}$$

Expanding $\mathrm{E}\left\{\left\|\eta(k)\right\|^2|\mathcal{F}_k\right\}$ and taking into account the relations (3), (4) and (5), we derive

$$\left.\begin{array}{c} \mathrm{E}\left\{V(k+1)|\mathcal{F}_k\right\} \overset{a.s.}{\leq} \mathrm{E}\left\{ \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} W \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} |\mathcal{F}_k \right\} \\ + \alpha \mathrm{E}\left\{V(k)|\mathcal{F}_k\right\} + \beta_k(L), \\ \beta_k(L) := m\lambda + \gamma\left(2\mathrm{tr}\left\{\Sigma_x\right\} + 3\left(f_0 + f_1 \left\|x(k)\right\|^2\right)\right) + \\ \gamma\left(2\left\|\tilde{\delta}(k)\right\|^2 + \mathrm{tr}\left\{L\Sigma_y L^{\intercal}\right\}\right) \end{array}\right\} \tag{A.7}$$

b) Since $x(k) = \delta(k) + x^*(k)$, it follows $\left\|x(k)\right\|^2 \leq 2\left\|\delta(k)\right\|^2 + 2\left\|x^*(k)\right\|^2$. The term $2\left\|\delta(k)\right\|^2$ can be included $\tilde{W}$ and $2\left\|x_k^*\right\|^2$ into $\beta_k(L)$, that leads to

$$\left.\begin{array}{c} \mathrm{E}\left\{V(k+1)|\mathcal{F}_k\right\} \overset{a.s.}{\leq} \mathrm{E}\left\{ \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} \tilde{W} \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} |\mathcal{F}_k \right\} \\ + \alpha \mathrm{E}\left\{V(k)|\mathcal{F}_k\right\} + \tilde{\beta}_k(L). \end{array}\right\} \tag{A.8}$$

If the matrices P, K, L, L_a and scalars α, β, γ are selected in such a way that $\tilde{W} \leq 0$, from (A.4) we get

$$\mathrm{E}\left\{V(k+1)|\mathcal{F}_k\right\} \leq \alpha \mathrm{E}\left\{V(k)|\mathcal{F}_k\right\} + \tilde{\beta}_k(L). \tag{A.9}$$

Taking the complete mathematical expectation of (A.9) we finally obtain (16).

Appendix B. PROOF: THEOREM 2

Let us introduce the change of variable $X_1 = P_1$, $X_2 = P_2$, $K = G$, $Y_1 = P_2 L$, $Y_2 = P_2 L_a$. Entering the matrices $Q_1 > 0$ and $Q_2 > 0$ such that:

$$3\tilde{A}^{\intercal}(K,L)\,P\tilde{A}(K,L) + \tilde{\Lambda}_\delta - \alpha P \leq -Q_1 \leq 0, \tag{B.1}$$

$$2Q^{\intercal}(L_a)\,PQ(L_a) - \lambda I \leq -Q_2 \leq 0. \tag{B.2}$$

Substituting this in (13) we get $\tilde{W} \leq \tilde{W}^{+}$. Applying the Schur's complement into (B.2), we get

$$\begin{bmatrix} \frac{1}{3}\left(\alpha P - \tilde{\Lambda}_\delta - Q_1\right) & \tilde{A}^{\intercal}\begin{bmatrix} I & 0 \\ 0 & P_2 \end{bmatrix} \\ \begin{bmatrix} I & 0 \\ 0 & P_2 \end{bmatrix}\tilde{A} & \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{bmatrix} \end{bmatrix} \geq 0. \tag{B.3}$$

Now, from $\bar{W}^{+} \leq 0$, (B.3) and taking into account $X_1 = P_1$, it's easy to see that $X_1^{-1} \geq \frac{2}{\gamma}I$, which implies $W_{Q_1} \geq 0$. In turn, by the same reasoning, applying the Schur's complement to inequality (B.2) and substitution $Y_2 = P_2 L_a$, results $W_{Q_2} \geq 0$. Finally, in view of the relation $Y_1 = LX_2$ and applying, again, the Schur's complement into (18), we have $W_\theta \geq 0$.

