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On the Trajectory Tracking Control of Hamiltonian Systems

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Abstract: In this work, the solution of the trajectory tracking control problem for a class of port-Hamiltonian systems is presented. The considered class is characterized by the fact that the interconnection matrix exhibits a particular structure in which the internal interconnection of the components of the vector state is modulated by the state itself. An additional feature of the presented result is the possibility to deal with under-actuated systems. Regarding the stability analysis, the structure of the approached class allows the formulation of the control error dynamic so that, using well-known results from the perturbed systems theory, it is possible to obtain asymptotic stability properties for the closed-loop system. The usefulness of the contribution is illustrated by solving the speed tracking control problem of the Permanent Magnet Synchronous Motor.

Keywords: Nonlinear control, Passivity Based Control, Tracking control, Hamiltonian Systems.

1. INTRODUCTION

The well-established nonlinear control technique called Passivity Based Control (PBC) has a framework that achieves stabilization via passivization and focuses on the structure and the energy of the systems to develop procedures to construct control schemes. Since the first work where the term PBC was introduced in Ortega and Spong (1989) and explained in Ortega et al. (2013), there have been numerous important results like the IDA-PBC in Ortega and Spong (2000), in which the authors develop a methodology for control regulation by injecting damping and designing the interconnection of the system with the control, resulting in a closed-loop system with a structure corresponding to a port-Hamiltonian (pH) system.

For a long time, pH systems have been investigated from different points of view by different areas. This class of systems has gained more attention in recent years because of the extensive framework that it provides; from physical modeling and its particular structure throughout theory and analysis of physical systems, as explained in Van Der Schaft and Jeltsema (2014). The results published over the years, starting with Van Der Schaft (1986), have solved several problems; for example, the stabilization of fully actuated and underactuated pH systems as published in Ortega et al. (1999) and in Castaños et al. (2009), as well as trajectory tracking of fully actuated systems as shown in Reyes-Báeza et al. (2018).

However, the lack of a systematic methodology that solves the control problem of tracking for a class of underactuated pH systems has not been developed, becoming the motivation of the present work. In order to contextualize the contribution, it must be considered that there are an important amount of results that have solved the tracking problem for specific systems, like in Turnwald et al. (2018) where they develop a trajectory tracking control for an autonomous bicycle which is considered an underactuated system. Another result is presented in Nguyen et al. (2019) where they develop a tracking-error control for a polymerization reactor. In a more general setting, in Yaghmaei and Yazdanpanah (2017) the contractive theory is used to achieve trajectory tracking of that class of pH systems, relying on the contractivness of the trajectories of the system.

In this work, we contribute to the solution of the trajectory tracking control problem by characterizing a particular class of pH systems for which it is possible to achieve convergence of control error to zero. The main feature of this particular class of nonlinear system is focused on the interconnection matrix, which can be represented as the

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sum of the products between constant skew-symmetric matrices and states components that appear in the $J(x)$ matrix. Another relevant quality of this class of Hamiltonian system is founded in the input matrix since it is possible to cover the case of underactuated systems.

The stability properties achieved by the closed-loop system come from the fact that under the assumed structure of the class of pH systems, it is possible to formulate a dynamic error equation with a structure that makes feasible the application of well-known results from the perturbed systems theory. Thus, a control law is designed to guarantee asymptotic stability properties of the equilibrium point that corresponds to the operation when the actual state equals the behavior of an admissible trajectory, being the latter a behavior that the original system can achieve.

In order to illustrate both the transcendence of the considered class of systems and the usefulness of the proposed controller, the speed tracking control problem of Permanent Magnet Synchronous Motors (PMSM) is analytically solved and numerically verified. For this, the two-phase equivalent motor model in dq coordinates is considered and, for the sake of ease of the presentation, it is presented the case when the load torque is zero.

The remaining of this paper is organized as follows: Section 2 shows the particular class of pH systems that is considered in this work. In Section 3, the problem formulation is stated and the error coordinates dynamics are obtained, while in Section 4, it is explained the control design together with its stability analysis which is based on identifying the vanishing perturbation term and proving the asymptotic stability properties of a perturbed system. In Section 5 it is introduced the case of study and its numerical evaluation to close the paper with Section 6, where the conclusions are formulated.

2. A CLASS OF PORT HAMILTONIAN SYSTEMS

In this section, the class of pH systems considered in this paper is presented. It is first considered a general structure to identify the particular features assumed in this work. Several remarks are included explaining the transcendence of the obtained class of systems.

The Hamiltonian system considered in this paper is of the form

$$\dot{x} = [J(x) - R(x)]\,\nabla H(x) + g(x)u \tag{1}$$

$$y = g(x)^T \nabla H(x) \tag{2}$$

where $x \in \mathbb{R}^n$, $J(x) = -J(x)^T$, $R(x) = R(x)^T \geq 0$, $H(x) : \mathbb{R}^n \to \mathbb{R}$, $\nabla H(x) = \begin{bmatrix} \frac{\partial H(x)}{\partial x_1} & \dots & \frac{\partial H(x)}{\partial x_n} \end{bmatrix}^T$, $u \in \mathbb{R}^m$ and $g(x) \in \mathbb{R}^{n\times m}$.

From the structure presented above, it is clear that when $m \leq n$ the system qualifies as underactuated while if $m = n$ the system is fully-actuated. Another well-known fact is that if $u \in \mathbb{R}^m$ is considered as input and $y \in \mathbb{R}^m$ as output, then it is defined a passive map.

The particular class of systems considered in this paper is obtained by considering the following assumptions:

A.1 The Hamiltonian function is a quadratic-type and is given by

$$H(x) = \frac{1}{2}x^T Qx;\ Q = Q^T > 0$$

leading to the fact that $\nabla H(x) = Qx$.

A.2 The input matrix is a constant full-column rank matrix $g(x) = g$.

A.3 The dissipation matrix is a symmetric positive semi-definite constant matrix $R = R^T \geq 0$.

A.4 The interconnection matrix has a structure such that can be written as

$$J(x) = J_0 + J_1x_1 + ... + J_px_p;\ \ n \geq p$$

where J_i are constant skew-symmetric matrices while x_i are the states that appear in the interconnection matrix. If $p = n$, then all the states appear in the interconnection matrix.

Considering the assumptions **A.1-A.4** the class of Hamiltonian system approached in this paper takes the form

$$\dot{x} = [\ J(x) - R]\,Qx + gu \tag{3}$$

$$y = g^T Qx \tag{4}$$

where $x \in \mathbb{R}^n$, $J(x) = -J(x)^T$, $R = R^T \geq 0$, $Q = Q^T > 0 \in \mathbb{R}^{n\times n}$, $u \in \mathbb{R}^m$, $g \in \mathbb{R}^{n\times m}$, $m \leq n$, $y \in \mathbb{R}^m$.

The following remarks, that explain the transcendence of the class of systems, are in order:

Remark 1: It is clear that assumptions **A.1-A.3** correspond to the fact that the class of systems is composed, in part, by a linear structure. As will be clear below, this feature is fundamental to obtain an error dynamic equation suitable to carry out a stability analysis.

Remark 2: Even though assumption **A.2** restricts to accept pH systems that have a constant input matrix, it does not restrict the systems to be fully actuated because it considers full column rank input matrix that allows us to consider underactuated systems.

Remark 3: The importance of assumption **A.4** is two-fold: On the one hand, it defines the nonlinear nature of the class approached in this paper. In this sense, the nonlinearities that can appear are given by products of components of the state vector. On the other hand, its structure is also fundamental to obtain a suitable error equation. However and besides this advantage, this structure is usually found in practice since it corresponds to the situation when the internal interconnection between components of the vector state are modulated by a component of the same vector.

3. PROBLEM FORMULATION

This section is devoted to the formal formulation of the control problem solved in this paper. This formulation begins by recognizing the kind of (time-varying) behaviors that the system can achieve, denoted as admissible trajectories, to later formulate the dynamic behavior of the error variable defined as the difference between the actual and the admissible trajectories. Once this equation is stated, the control problem is established.

The admissible trajectories of the system (3) is defined as the set of trajectories that this system is able to reach. In this sense, these behaviors must be compatible with the structure of (3) and therefore must be solutions of the set of differential equations given by

$$\dot{x}_\star = [J(x_\star) - R]\, Q x_\star + g u_\star \tag{5}$$

which is a copy of the original system. It must be notice that for this system it must be assumed that for a given trajectory $x_\star$ there exists a corresponding input $u_\star$ that generates it.

Once the admissible trajectories have been defined, the next step in the problem formulation corresponds to obtain the dynamic behavior of the control error variable defined as $\tilde{x} = x - x_\star$. This result is included in the next proposition.

Proposition 1: Consider the system (3) and the admissible trajectories defined by (5), then the error coordinates dynamics are defined as:

$$\dot{\tilde{x}} = [J(x) - R]\, Q\tilde{x} + B(x_\star) Q_p \tilde{x}_p + g\tilde{u} \tag{6}$$

where $\tilde{u} = u - u_\star \in \mathbb{R}^m$, $\tilde{x}_p \in \mathbb{R}^p$, $p \leq n$, $Q_p = Q_p^T > 0 \in \mathbb{R}^{p\times p}$, and $B(x_\star) \in \mathbb{R}^{n\times p}$ is a matrix given by

$$B(x_\star) = [J_1 x_\star \;\; J_2 x_\star \;\; \ldots \;\; J_p x_\star]$$

Proof 1: The dynamic in the error coordinates can be obtained as follows

$$\dot{\tilde{x}} = J_0 Q\tilde{x} - RQ\tilde{x} + g\tilde{u} + \bar{J}(x)Qx - \bar{J}(x - \tilde{x})Qx_\star \tag{7}$$

where $\bar{J}(x) = J_1 x_1 + \cdots + J_p x_p$.

Using **A4.** we can separate $\bar{J}(x - \tilde{x}) = \bar{J}(x) - \bar{J}(\tilde{x})$ so that

$$\dot{\tilde{x}} = [J(x) - R]\, Q\tilde{x} + g\tilde{u} + \bar{J}(\tilde{x})Qx_\star \tag{8}$$

With the property of the **A4.** the next equivalence is correct

$$\bar{J}(\tilde{x})Qx_\star = \bar{B}(x_\star)\tilde{x} \tag{9}$$

□

Given the structure of the error dynamic presented in (6) it is possible to formulate the control problem solved in this paper in the following way:

Considering the dynamic system given by (6), design a control law $\tilde{u}$ such that

$$\lim_{t\to\infty} \tilde{x} = 0$$

guaranteeing internal stability.

The proposed solution to this problem is presented in the next section.

4. CONTROL DESIGN

The main result of this paper is presented in this section, namely, the proposition of a control law for system (3) such that the control problem formulated in Section 3 is solved.

Due to the structure exhibited by the error Equation (6), the methodology design is divided into two steps. First, considering the term $B(x_\star)Q_p\tilde{x}_p$ as a state-dependent perturbation, a control law that asymptotically stabilizes the equilibrium point $\tilde{x} = 0$ of the nominal system (when $B(x_\star)Q_p\tilde{x}_p \equiv 0$) is designed. Second, noticing that the perturbation term actually defines a vanishing perturbation, a well-known result from the perturbed systems theory is applied in order to obtain the desired result.

4.1 Stability of the nominal system

In the next proposition, stabilization of the equilibrium point $\tilde{x} = 0$ for the nominal system, obtained by considering $B(x_\star)Q_p\tilde{x}_p \equiv 0$ in system (6), is achieved.

Proposition 2: Consider the error coordinate dynamic (6) without the perturbation term

$$\dot{\tilde{x}} = [J(x) - R]\, Q\tilde{x} + g\tilde{u} \tag{10}$$

with an equilibrium point $\tilde{x} = 0$. The system (10) in closed loop with the proportional control of the passive output

$$\tilde{u} = -Ky = -Kg^T Q\tilde{x} \tag{11}$$

where $K \in \mathbb{R}^{m\times m}$ such that the symmetric part of the matrix $\left[R + gKg^T\right] > 0$, is asymptotically stable.

Proof 2: Define the Lyapunov function candidate as system energy function or the Hamiltonian of the system

$$H(\tilde{x}) = \tilde{x}^T Q\tilde{x} \tag{12}$$

The closed-loop system (10) and (11) may be written in the pH form

$$\dot{\tilde{x}} = [J(x) - R]\, Q\tilde{x} - gKg^T Q\tilde{x} \tag{13}$$

The time derivative along the systems trajectories is

$$\dot{H}(\tilde{x}) = \tilde{x}^T Q\dot{\tilde{x}} \tag{14}$$
$$= \tilde{x}^T Q \left[J(x) - R - gKg^T\right] Q\tilde{x} \tag{15}$$
$$= -\tilde{x}^T Q \left[R + gKg^T\right] Q\tilde{x} < 0 \tag{16}$$

□

Remark 4: It is important to underscore that asymptotic stability can be guaranteed if there is natural damping in the non-actuated coordinates of the system.

Remark 5: Using assumption **A.1**, the first condition of exponential stability defined in Khalil (2002) can be established.

$$\lambda_{\min} \|\tilde{x}\|^2 \leq \tilde{x}^T Q\tilde{x} \leq \lambda_{\max} \|\tilde{x}\|^2 \tag{17}$$

where $\lambda_{\min}, \lambda_{\max}$ are the minimum and maximum eigenvalues of matrix Q, respectively. And with the negative proportional feedback of the passive output (11) the second condition defined in Khalil (2002) can be established:

$$\dot{H}(\tilde{x}) \leq -K \left\|\tilde{x}\right\|^2 \tag{18}$$

4.2 Stability of the perturbed system

Considering the nonlineal system given by (6), the term $B(x_\star)Q_p\tilde{x}_p$ can be considered as a vanishing perturbation because if $\tilde{x} = 0 \rightarrow \tilde{x}_p = 0$ and therefore $B(x_\star)Q_p\tilde{x}_p = 0$. This property allows us to consider the term $B(x_\star)Q_p\tilde{x}_p$ as a vanishing perturbation term, let us use classical perturbation theorems to analyze the perturbed system. This analysis is developed to investigate the stability of the origin as an equilibrium point of the perturbed system (6) and is based on the definitions included in Chapter 9 of Khalil (2002). For this purpose in the sequel it is considered the definition $b(x_\star, \tilde{x}_p) = B(x_\star)Q_p\tilde{x}_p$.

Proposition 3: Consider that $x_\star$ is bounded and that the perturbation term $b(x_\star, \tilde{x}_p)$ satisfies the bound

$$\|b(x_\star, \tilde{x}_p)\| \leq \gamma\phi(\tilde{x}_p), \gamma < \frac{c_3}{c_4} \tag{19}$$

for all $t \geq 0$ and for all $x \in \mathbb{R}^n$, where c_3, c_4 are some positive constants and $\phi(\tilde{x}_p) : \mathbb{R}^p \rightarrow \mathbb{R}$ is positive definite and continuous, then the origin of the perturbed system is asymptotically stable.

Proof 3: Define the energy function $H(\tilde{x})$ as a Lyapunov function candidate

$$\dot{H}(\tilde{x}) = \frac{\partial H}{\partial \tilde{x}_p}\left[-R - gKg^T\right]Q\tilde{x}_p + \frac{\partial H}{\partial \tilde{x}_p}B(x_\star)Q\tilde{x}_p \tag{20}$$

$$\leq -W_3(\tilde{x}_p)\left\|\tilde{x}_p\right\|^2 + \left\|\frac{\partial H}{\partial \tilde{x}_p}B(x_\star)Q\tilde{x}_p\right\| \tag{21}$$

where $W_3(\tilde{x})$ is positive definite and continuous.

When the energy function is positive definite, decreasing and satisfies

$$\frac{\partial H(\tilde{x})}{\partial t} + \frac{\partial H(\tilde{x})}{\partial \tilde{x}}\dot{\tilde{x}} \leq -c_3\phi^2(\tilde{x}) \tag{22}$$

$$\left\|\frac{\partial H(\tilde{x})}{\partial \tilde{x}}\right\| \leq c_4\phi(\tilde{x}) \tag{23}$$

A quadratic Lyapunov function (**A.1**) satisfies (22) and (23), then the derivative along the trajectories of the system satisfies

$$\dot{H}(\tilde{x}) \leq -c_3\phi^2(\tilde{x}_p) + c_4\phi(\tilde{x}_p)\left\|b(x_\star, \tilde{x}_p)\right\| \tag{24}$$

$$\dot{H}(\tilde{x}) \leq -(c_3 - c_4\gamma)\phi^2(\tilde{x}_p) \tag{25}$$

Therefore, if γ is small enough to satisfy the bound then $\dot{H}(x)$ is negative definite and therefore the origin of the perturbed system is asymptotically stable. □

Remark 6: If the equilibrium point is exponentially stable, the analysis can be simplified and the bound of the perturbation is given by $\|b(x_\star, \tilde{x})\| \leq \gamma \left\|\tilde{x}\right\|$.

5. CASE STUDY

In this section, it is shown that the proposed methodology ensures asymptotic stability for the trajectory tracking of the PMSM system.

Consider the PMSM dq model given by

$$\dot{x} = \begin{bmatrix} -R_1 & 0 & n_p x_2 \\ 0 & -R_2 & -n_p(x_1 + \phi) \\ -n_p x_2 & n_p(x_1 + \phi) & -r_m \end{bmatrix} \begin{bmatrix} \frac{x_1}{L_1} \\ \frac{x_2}{L_2} \\ \frac{x_3}{J_m} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u \tag{26}$$

where the state space vector $x = [i_d L_1 \; i_q L_2 \; \omega J_m]^T$ is formed by the coils magnetic fluxes in the d and q coordinates and the rotor angular momentum, the input vector $u = [V_d \; V_q]^T$ is assembled by the coil terminal voltages in the d and q coordinates and the parameters of the PMSM are $R_1 = R_2 = 0.225\,[\Omega]$, $r_m = 0.00063\,[\Omega]$, $L_1 = L_2 = 3.8\,[\text{mH}]$, $J_m = 0.012\,[\text{ Kg m}^2]$, $\phi = 0.17\,[\text{ Wb}]$ with 3 poles, 1 [KW] rated power and 1 [Nm] rated torque, obtained from Shah et al. (2014).

The PMSM model given by (26) satisfies the properties (**A.1-A.4**) of the characterized class system (3) and considering that the torque of the motor is $\tau_L = 0$ the system is enclosed in the underactuated system class. Beyond the practical implications of considering the torque $\tau_L = 0$, it is important to clear out that this example is given to prove that the proposed methodology works with underactuated systems. The unknown perturbation case is already solved but as it is not the objective of this work, it is left out for future publication.

The velocity profile followed starts with a desired velocity (ω^*) of 0 [rad/s] and increases for 1 [s] until reaching a speed of 10 [rad/s] that stays for 2 [s]. Following the velocity starts increasing again until getting to 80 [rad/s] for 2 [s] and maintain the velocity for 3 [s] to later start decreasing for 2 [s] till getting to −10 [rad/s] for 2 [s], then it decreases more for 2 [s] to reach −80 [rad/s] for 3 [s] to finally return to the start velocity of 0 [rad/s], as illustrated in Figure 1.

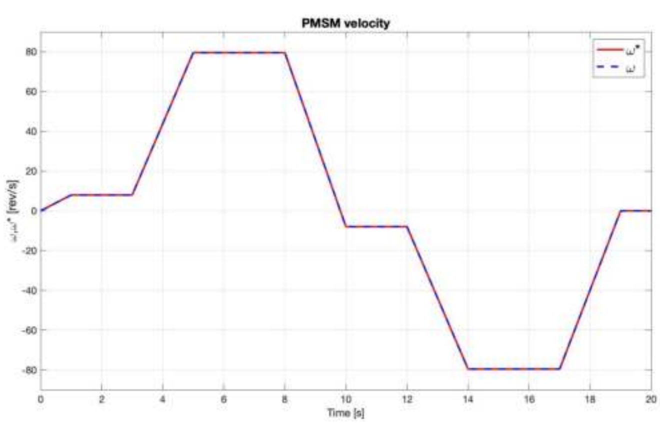

Figure 1. Trajectory Tracking of the PMSM Velocity

The motor velocity tracking results of the MATLAB - Simulink ® simulations are exposed in Figure 1, where we can observe the comparison between the desired velocity (ω^*) and the velocity followed by the PMSM (ω). This tracking behavior is achieved through the input control signals that are illustrated in Figure 2.

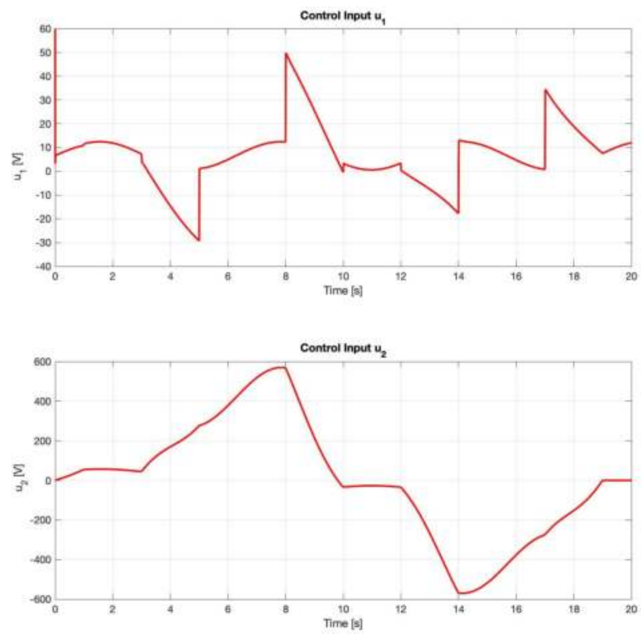

Figure 2. Input Control Signals

In order to have a more precise view of the trajectory of the state tracking, the graphics of the tracking errors of the three states are shown in Figure 3. The tracking states errors are of the order of 10^{-2} in the state x_3, which is the biggest error in all states, as shown in Figure 3. This figure gives a clearer view of the differences between the desired trajectories and the ones followed by the PMSM with the designed controller.

6. CONCLUSIONS

A new trajectory tracking methodology has been established for a class of Hamiltonian systems characterized along with this paper. The properties of the system class allowed the development of the dynamics in the error coordinates with a specific structure. This determined structure of the error let us establish that the term $B(x_\star)Q_p x_p$ can be considered as a vanishing perturbation and enable us to use the vanishing perturbation analysis to design the control law. The control scheme maintains the pH structure in closed-loop by considering a term as a vanishing perturbation. The results obtained in the example of the PMSM system illustrate the trajectory tracking of the desired velocity profile with a small error when the trajectory suffers from a sudden change. However, the most relevant result of the case study is that the PMSM is an underactuated system and it shows that this methodology can be applied in underactuated pH systems that fulfill the properties of the pH system class characterized.

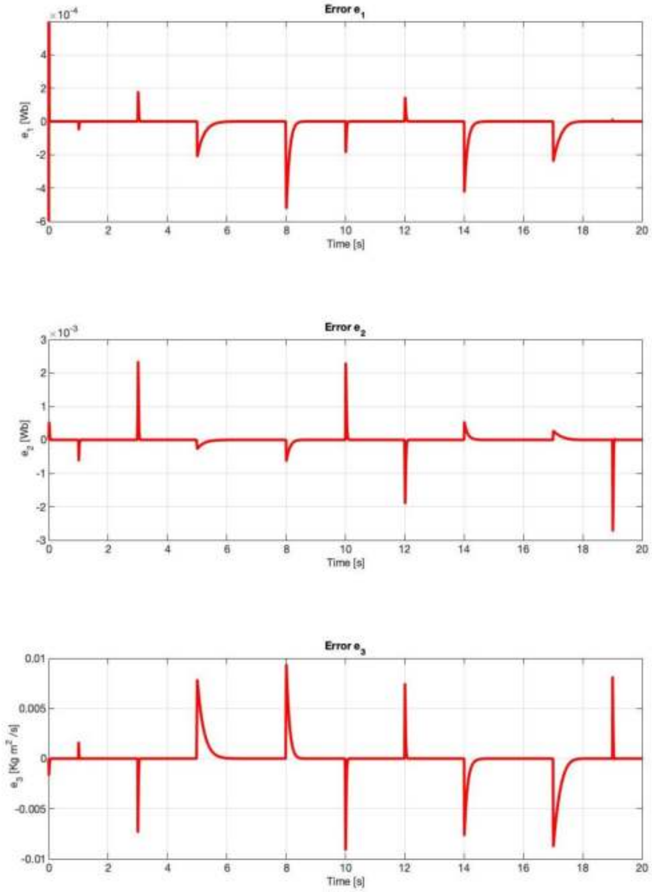

Figure 3. Tracking Error Coordinate States

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