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Adaptive Observer Design for the Fractional-Order Hindmarsh-Rose Neuron Model ⋆

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Abstract: In this work it is presented an adaptive observer design methodology for the Fractional-Order Hindmarsh-Rose Neuron Model. Using an analysis based on quadratic Lyapunov functions and an extension of Barbalat's theorem to the fractional-order case, the asymptotic convergence of the observed states to the real ones is proven, as well as the boundedness of the parameter reconstruction. Numeric examples are presented to show the effectiveness of the proposed design.

Keywords: Adaptive Observer, Parametric Identification, Hindmarsh-Rose neuron model, fractional-order system

1. INTRODUCTION

Fractional order systems can be seen as a generalization of the well-known integer-order systems, where the differential operator that describes their dynamics is, instead of the usual derivative $D = \frac{d}{dt}$ or the integral operator $I(\cdot) \triangleq D^{-1}(\cdot) = \int \cdot dt$, it is now expressed as D^{α}, where $\alpha \in \Re$. It is also known that there have been proposed different definitions of this operator, that in general give different results depending on the kernel functions, as well as the time horizon. Although the reported variety of fractional differential operators, the most used in control theory is the Caputo fractional operator (Gorenflo and Mainardi, 1997), given that it handles initial conditions in a similar way as in the integer counterpart. It has been noted through literature (Duarte Ortigueira and Tenreiro Machado, 2019) that special care has to be taken when selecting or using a given fractional-differential operator, as well as the role of the initial conditions (Sabatier et al., 2010).

The role of the fractional-order operators has been proven valuable to model systems that show a long-term memory behavior, and several biological systems have been modeled accordingly (Rihan, 2013). One of the main features of this model is that it includes fractional-order differential operators, instead of the integer order ones, is that the latter can neglect some of the intrinsic memory dynamics that the fractional-order operators inherently describe.

Several models for biological neural systems have been proposed using integer-order differential operators, such as the now historical Hodgkin-Huxley model (Hodgkin and Huxley, 1952), as well as the Fitzhugh-Nagumo (FitzHugh, 1961). The Hindmarsh-Rose neural model (HRNM) (Hindmarsh and Rose, 1982) was first introduced as a simplified version of the Hodgkin-Huxley model, and later the same authors expanded the model so it took into account slow adaptation in the exchange of ions through the neuron's ionic channel (Hindmarsh and Rose, 1984).

The Hindmarsh-Rose neuron model has proven to be an useful model of relative simplicity that helps to understand some of the underlying dynamics in a neuron. Among those, it is of special interest the generation of action potentials within the neuron's membrane (spiking) and the change of dynamics from relative resting to repetitive firing states (bursting). Recently, the role of fractional-order operators in the Hindmarsh-Rose neuron model has been explored (Xiao, 2012), analyzing the model and comparing the results obtained from the model to experimental data (Kaslik, 2017). It has been found that given different parameters on the fractional-order HRNM describe different types of dynamics, where for some values no bursting or spiking is achieved, as well as

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finding the parameter conditions where bifurcation is to be expected, as well as regions for sustained oscillations (Xiao, 2013).

However, not always all signals are available for measurement, and the reconstruction of the internal variables is one of the goals in order to model and simulate the interconnection of not only one but several neurons inside a network. In previous works, a methodology to design an adaptive observer design for a class of Nonlinear-Fractional-Order Systems (NFOS), based on quadratic Lyapunov functions, was presented (González-Olvera and Tang, 2018; Flores-Pérez et al., 2018).

Therefore, in this work we present a methodology to achieve the state estimation of the three-variable fractional-order HRNM, as well as a bounded parametric reconstruction error. In this paper, in Section 2 the theoretical antecedents are presented, showing a brief explanation on fractional-order operators and systems, as well as a description of the fractional-order HMNM. In Section 3, the design methodology for the adaptive observer with bounded parametric reconstruction error and convergent state observation is presented; and in Section 4 the simulation results are shown. Finally, conclusions and future work are depicted in Section 5.

2. ANTECEDENTS

2.1 Fractional-order operators

Commonly, the Riemann-Liouville fractional-order integral operator is presented as a generalization from the Cauchy's formula for the n-fold repeated integration for $n \in \mathbb{N}$ as:

$$J^n f(t) \triangleq \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau) d\tau \cdots d\tau_2 d\tau_1 = \frac{1}{(n-1)!} \int_a^t f(\tau)(t-\tau)^{n-1} d\tau \quad (1)$$

When n is allowed to change from an integer value to any real value $n \to \alpha \in \Re$, then the definition is extrapolated to the *Riemman-Liouville fractional integral*, given by

$${}_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t f(\tau)(t-\tau)^{\alpha-1} d\tau. \quad (2)$$

where $\Gamma(w)$ is the Gamma function of $w \in \mathbb{C}$. Finally, the Riemann-Liouville fractional-integral operator is givven by

$${}_a^{\mathcal{RL}}D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \\ \qquad \alpha \in (n-1, n),\ n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} f(t), \\ \qquad \alpha = n \in \mathbb{N}, \end{cases} \quad (3)$$

This operator can act as left-inverse for (2) (Caponetto, 2010), i.e., $D^\alpha(J^\alpha f(t)) = f(t)$.

However, from a system theory point-of-view, the previous definition as analysis challenges, given that it requires the definition of initial conditions for $t \in (-\infty, a)$. In order to overcome those limitations, a different definition is given by (Caputo, 1967) and is known as the *Caputo Fractional Differential Operator*:

$${}_a^{\mathcal{C}}D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{\frac{d^n f(\tau)}{dt^n}}{(t-\tau)^{\alpha+1-n}} d\tau, \\ \qquad \alpha \in (n-1, n),\ n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} f(t), \\ \qquad \alpha = n \in \mathbb{N}. \end{cases} \quad (4)$$

In this work, the Caputo's definition is preferred used for the fractional derivative, and the initial time is taken as $a = t_0$. In the following, the simplified notation ${}_a^{\mathcal{C}}D_t^\alpha f(t) = D^{(\alpha)} = f^{(\alpha)}(t)$ is used.

2.2 Hindmarsh-Rose fractional-order model

Through the literature on neural dynamics analysis, the description of the *spiking* and *bursting* phenomena of the membrane potential, resulting from the exchange of sodium and potassium in the ion channels in single neurons, have been a subject of interest (Shilnikov et al., 2005). One of the most used models, due to its successful and relatively simple description the aforementioned phenomena, is the Hindmarsh-Rose neuron model, that is given by:

$$\dot{\xi} = \begin{pmatrix} a\xi_1^2 - \xi_1^3 - \xi_2 - \xi_3 + u(t) \\ (a+\beta)\xi_1^2 - \xi_2 \\ \mu(b\xi_1 + c - \xi_3) \end{pmatrix} \quad (5)$$

$$\mathbf{y} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \quad (6)$$

In this model, ξ_1 is the membrane potential in the axon of a neuron, ξ_2 is the transport rate of sodium and potassium ions through fast ion channels, and ξ_3 models the ion exchange in slow ion channels. Usually, ξ_2 is referred as the *spiking* variable, and ξ_3 is called the *bursting* variable. The *spiking* refers to the rapid change in the potentials almost in an impulsive fashion, while *bursting* refers to the periods of successive repetitive spikes followed by relatively *resting* conditions.

Neural models have been proposed to have fractional-order dynamics (Lundstrom et al., 2008), given that it can better model some of the information processing and dynamics related to dielectric processes and memory characteristics in the membrane. Works have focused in the two-pseudo-state fractional-order HRNM. Some studies have once again expanded the model to have a three-pseudo-state representation (Jun et al., 2014; Kaslik, 2017), analyzing its dynamics properties with different fractional-order values of the derivative, as well as the bifurcation phenomena linked to parameter variation.

Following previous works, in this paper it is used the Hindmarsh-Rose Fractional-Order model given by the pseudo-state equations:

$$\xi^{(\alpha)} = \begin{pmatrix} a\xi_1^2 - \xi_1^3 - \xi_2 - \xi_3 + u(t) \\ (a+\beta)\xi_1^2 - \xi_2 \\ \mu(b\xi_1 + c - \xi_3) \end{pmatrix} \quad (7)$$

$$\mathbf{y} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \quad (8)$$

with $\alpha \in (0.9, 1)$. Based on this model, an adaptive observer is proposed to estimate the states of the system by measuring the output signals $\mathbf{y}$ with bounded parametric error.

A simulation of this model considering the same parameters as the ones reported in Section 4 is shown in Fig. 1, where it can be seen the spiking and bursting phenomena, as well as the change in the dynamics after a change in the parameter a is considered. Depending on the value of a it can be seen that the model changes its behaviour from a spiking-bursting dynamics, to only sustained spiking, and once again returns to spiking-bursting, denoting the role that this parameter has in the internal dynamics of the neuron.

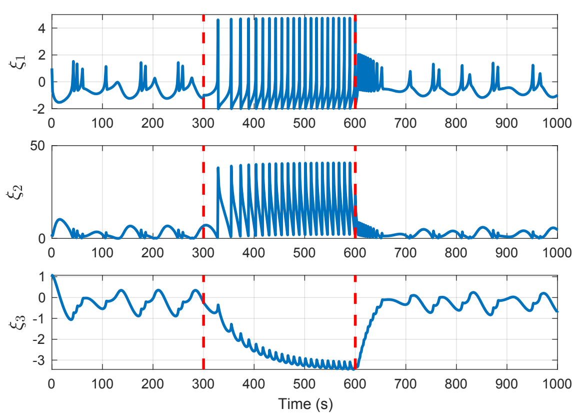

Fig. 1. In this figure all states are shown when beginning in $t = 0$ with the initial condition $\xi(0) = (1\ 1\ 1)^T$. In $t = 300$ an abrupt change in the parameter a is forced, from $a = 2.8$ for $t \in [0, 300]$ to $a = 5.6$ for $t \in (300, 600]$, and returned to $a = 2.8$ for $t > 600$. It can be seen how the amplitude, as well as the frequency of the oscillations, change depending on the value of a.

3. ADAPTIVE OBSERVER DESIGN FOR THR FRACTIONAL-ORDER HRNM

Considering that a is an unknown parameter and that $\mathbf{y} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is the measurable output of the system, (8) can be rewritten as

$$\xi^{(\alpha)} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ \mu b & 0 & -\mu \end{pmatrix} \xi + \begin{pmatrix} -y^3 + u(t) \\ \beta y_1^2 \\ \mu c \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (a y_1^3) \quad (9)$$

Following the results presented in González-Olvera and Tang (2018), note that (8) belongs to the class of single-input-multiple-output fractional-order nonlinear systems given by

$$\begin{aligned} \xi^{(\alpha)}(t) &= \mathbf{A}\xi(t) + \mathbf{f}_0(\mathbf{y}(t), u(t)) + \mathbf{b}\left(a g(\mathbf{y}(t), u(t))\right) \\ \mathbf{y} &= \mathbf{C}\xi \end{aligned} \quad (10)$$

where $\xi(t) \in \Re^n$ is the pseudo-state vector, $0 < \alpha < 1$ the derivative order, $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{f}_0 : \Re \times \Re \to \Re^n$, $\mathbf{b} \in \Re^n$, $g : \Re \times \Re \to \Re$, $\mathbf{C} \in \Re^{2\times n}$. In this case:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ \mu b & 0 & -\mu \end{pmatrix}, \quad (11)$$

$$\mathbf{f}_0(\mathbf{y}(t), u(t)) = \begin{pmatrix} -y_1^3 + u(t) \\ \beta y_1^2 \\ \mu c \end{pmatrix}, \quad (12)$$

$$\mathbf{b} = \left(1\ 1\ 0\right)^T,\ g(\mathbf{y}, a) = y_1^3,\ \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (13)$$

The objective is to obtain an adaptive observer with the structure:

$$\hat{\xi}^{(\alpha)}(t) = h_\xi(\hat{\xi}, u, \mathbf{y}, \hat{a}) \quad (14)$$

$$\hat{a}^{(\alpha)} = h_a(u, \mathbf{y}, \hat{\xi}) \quad (15)$$

where it can be achieved that $\lim_{t\to\infty}(\xi - \hat{\xi}) = 0$ and $\lim_{t\to\infty}(a - \hat{a}) = 0$.

Defining $\tilde{\xi} = \xi - \hat{\xi}$ as the observation error and $\tilde{a} = a - \hat{a}$ as the parametric error, the associated fractional dynamics is

$$\begin{aligned} \tilde{\xi}^{(\alpha)}(t) &= \mathbf{A}\xi(t) + \mathbf{f}_0(\mathbf{y}(t), u(t)) \\ &+ \mathbf{b}a g(\mathbf{y}(t), u(t)) - h_\xi(\hat{\xi}, u, \mathbf{y}, \hat{a}). \end{aligned} \quad (16)$$

Selecting

$$\begin{aligned} h_\xi(\hat{\xi}, u, y, \hat{a}) &= \mathbf{A}\hat{\xi}(t) + \mathbf{f}_0(\mathbf{y}(t), u(t)) \\ &+ \mathbf{b}\hat{a} g(\mathbf{y}(t), u(t)) - \mathbf{K}\tilde{\mathbf{y}}, \end{aligned} \quad (17)$$

where $\mathbf{K} \in \Re^{n\times 1}$ is a design matrix, $\hat{y} = C\hat{\xi}$, and $\tilde{\mathbf{y}} = \mathbf{y} - \hat{\mathbf{y}}$, it is obtained

$$\tilde{\xi}^{(\alpha)}(t) = \mathbf{A}\tilde{\xi}(t) + \mathbf{b}\tilde{a} g(\mathbf{y}(t), u(t) - \mathbf{K}\tilde{\mathbf{y}} \quad (18)$$

In order to design the adaptive observer dynamics, the following Lyapunov candidate function is proposed:

$$V(\tilde{\xi}, \tilde{\theta}) = \frac{1}{2}\tilde{\xi}(t)^T \mathbf{P}\tilde{\xi}(t) + \tilde{a}^2(t)\gamma_i^{-1}, \quad (19)$$

where $\gamma_i > 0$ and $\mathbf{P} = \mathbf{P}^T \in \Re^{n\times n}$, $\mathbf{P} > 0$ is a design matrix. From Duarte-Mermoud et al. (2015), it is obtained that the fractional derivative of order α of $V(\tilde{\xi}, \tilde{\theta})$ is

$$V^{(\alpha)}(\tilde{\xi},\tilde{\theta}) \leq \tilde{\xi}^T\mathbf{P}\left((\mathbf{A}-\mathbf{KC})\,\tilde{\xi}+\mathbf{b}\tilde{a}g(y(t),u(t))\right) + \gamma_i^{-1}\tilde{a}\tilde{a}^{(\alpha)} = \tilde{\xi}^T(\mathbf{P}(\mathbf{A}-\mathbf{KC}))\tilde{\xi} + \tilde{a}\left(\tilde{\xi}^T\mathbf{Pb}g(\mathbf{y}(t),u(t))+\gamma_i^{-1}\tilde{a}^{(\alpha)}\right)$$

Defining

$$V_1 = \tilde{\xi}^T(\mathbf{P}(\mathbf{A}-\mathbf{KC}))\tilde{\xi} = \tilde{\xi}^T\mathbf{Q}_s\tilde{\xi}+\tilde{\xi}^T\mathbf{Q}_A\tilde{\xi},$$

where $\mathbf{Q}_s = \frac{1}{2}\left(\mathbf{P}(\mathbf{A}-\mathbf{KC})+(\mathbf{A}-\mathbf{KC})\mathbf{P}\right)^T$ is the symmetric part of $\mathbf{P}(\mathbf{A}-\mathbf{KC})$ and $\mathbf{Q}_A$ is the anti-symmetric component.

$$V_1 = \tilde{\xi}^T\mathbf{Q}_s\tilde{\xi}.$$

and $V^{(\alpha)}(\tilde{\xi},\tilde{a})$, we can choose

$$\tilde{a}^{(\alpha)} = \hat{a}^{(\alpha)} = -\gamma_i\left(\tilde{\xi}^T\mathbf{Pb}g(y(t),u(t))\right), \qquad (20)$$

so

$$V^{(\alpha)}(\tilde{\xi},\tilde{\theta}) \leq \tilde{\xi}^T\mathbf{Q}_s\tilde{\xi}$$

As $\tilde{\xi}$ is not available, if there exists $\mathbf{P}>0$ and $\mathbf{K}$ such that $\mathbf{Q}_s<0$ under the restriction $\mathbf{Pb}=\mathbf{C}^T$, then it can be selected

$$h_a(u,y,\hat{\xi}) = -\gamma g\left(y(t),u(t)\right)(y-\mathbf{C}\hat{\xi}) \qquad (21)$$

With the previous discussion, the following theorem is proposed (González-Olvera and Tang, 2018):

Theorem 1. Given the fractional-commensurate-order nonlinear fractional-order system (10) and the adaptive observer given by (17) and (21), if there exists a constant matrix $\mathbf{P}>0$ and a gain $\mathbf{K}\in\Re^{n\times 1}$ such that $\mathbf{P}(\mathbf{A}-\mathbf{KC})^T+(\mathbf{A}-\mathbf{KC})\mathbf{P}<0$ and $\mathbf{Pb}=\mathbf{C}^T$, then $\lim_{t\to\infty}\tilde{\xi}=0$ and the parameter error $\tilde{a}$ remains bounded.

Proof 1. Given that there exist $\mathbf{P}>0$ and $\mathbf{K}$ such that $\mathbf{Q}_s=\frac{1}{2}\left(\mathbf{P}(\mathbf{A}-\mathbf{KC})+(\mathbf{A}-\mathbf{KC})^T\mathbf{P}\right)<0$ and $\mathbf{Pb}=\mathbf{C}^T$ [1], then from the quadratic Lyapunov function (19) we obtain, using (17) and (21), that

$$V^{(\alpha)}(\tilde{\xi},\tilde{a}) \leq \tilde{\xi}^T\mathbf{Q}_s\tilde{\xi}.$$

Then, there exist some $\lambda_P,\lambda_{Q_s}>0$ such that $\tilde{\xi}^T\mathbf{P}\xi>\lambda_P||\tilde{\xi}||^2$ and $\tilde{\xi}^T\mathbf{Q}_s\xi<-\lambda_{Q_s}||\tilde{\xi}(t)||^2$. So,

$$V^{(\alpha)}(\tilde{\xi},\tilde{a}) \leq -\lambda_{Q_s}||\tilde{\xi}(t)||^2 \qquad (22)$$

In consequence, from Duarte-Mermoud et al. (2015) as $V(\tilde{\xi},\tilde{a})$ is non-negative, therefore $\tilde{\xi}$ and $\tilde{a}$ remain bounded, and necessarily $0\leq V(\tilde{\xi},\tilde{a})<\bar{V}<\infty$. Following Navarro-Guerrero and Tang (2017), applying the Riemann-Liouville fractional integral of order α to both sides if the previous equation, we get

$${}_{t_0}I_t^{\alpha}\left(V^{(\alpha)}\right) \leq -\lambda_{Q_s}\,{}_{t_0}I_t^{\alpha}||\tilde{\xi}(t)||^2. \qquad (23)$$

By the Newton-Leibniz formula generalization we know that ${}_{t_0}I_t^{\alpha}\left(V^{(\alpha)}\right)=V(\tilde{\xi}(t),\tilde{a}(t))-V(\tilde{\xi}(t_0),\tilde{a}(t_0))$, so

$${}_{t_0}I_t^{\alpha}||\tilde{\xi}(t)||^2 \leq -\frac{1}{\lambda_{Q_s}}\left(V(\tilde{\xi}(t),\tilde{a}(t))-V(\tilde{\xi}(t_0),\tilde{a}(t_0))\right) \qquad (24)$$

Then, ${}_{t_0}I_t^{\alpha}||\tilde{\xi}(t)||^2<M<\infty$. Using the Barbalat's Lemma extension (Duarte-Mermoud et al., 2015; Navarro-Guerrero and Tang, 2017), $\lim_{t\to\infty}\tilde{\xi}=0$, as the asymptotic convergence of $\hat{\xi}(t)$ to $\xi(t)$ is proven, then the parametric error $\tilde{\theta}$ remains bounded.

Remark 1. Note that the condition of existence of a constant matrix $\mathbf{P}>0$ and a gain $\mathbf{K}\in\Re^{n\times 1}$ such that $\mathbf{P}(\mathbf{A}-\mathbf{KC})^T+(\mathbf{A}-\mathbf{KC})\mathbf{P}<0$ and $\mathbf{Pb}=\mathbf{C}^T$ can be related to the Kalman-Yakubovich-Popov (KYP) lemma. In order to determine the gain matrix $\mathbf{K}$ and $\mathbf{P}$ jointly, the problem can be reformulated as a Linear Matrix Inequality problem (LMI).

From the previous discussion, the adaptive observer for joint state estimation and reconstruction of the parameter a by $\hat{a}$ can be described by the set of equations:

$$\Sigma_M:\begin{cases}\xi^{(\alpha)}=\begin{pmatrix}a\xi_1^2-\xi_1^3-\xi_2-\xi_3+u(t)\\(a+\beta)\xi_1^2-\xi_2\\\mu(b\xi_1+c-\xi_3)\end{pmatrix}\\ \mathbf{y}=\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}\end{cases} \qquad (25)$$

$$\Sigma_S:\begin{cases}\hat{\xi}^{(\alpha)}=\begin{pmatrix}0&-1&-1\\0&-1&0\\\mu b&0&-\mu\end{pmatrix}\hat{\xi}+\begin{pmatrix}-y_1^3+u(t)\\\beta y_1^2\\\mu c\end{pmatrix}\\ \qquad+\begin{pmatrix}1\\1\\0\end{pmatrix}\hat{a}g(\mathbf{y}(t),u(t))-\mathbf{K}\tilde{\mathbf{y}}\\ \hat{a}^{(\alpha)}=\gamma g\left(\mathbf{y}(t),u(t)\right)(\mathbf{y}-\hat{\mathbf{y}})\\ \hat{\mathbf{y}}=\begin{pmatrix}\hat{\xi}_1\\\hat{\xi}_2\end{pmatrix}\end{cases}$$

4. NUMERICAL RESULTS

The fractional-order HRNM was simulated using the parameters $\beta=1.6$, $c=5$, $b=9$, $I=0.2$, $\mu=0.01$ and the fractional-order $\alpha=0.95$, and considering a changing parameter a described as

$$a=\begin{cases}2.8,\ t\in[0,300)\\5.6,\ t\in[0,600)\\2.8,\ t\geq 600\end{cases} \qquad (26)$$

In order to solve the matrix inequality $\mathbf{P}(\mathbf{A}-\mathbf{KC})^T+(\mathbf{A}-\mathbf{KC})\mathbf{P}<0$ for $\mathbf{P}>0$ and $\mathbf{K}\in\Re^{n\times 1}$ such that and $\mathbf{Pb}=\mathbf{C}^T$, we used CVX, a package for specifying and solving convex programs (Grant and Boyd, 2014, 2008). This resulted in the solution for the observer gain matrix

$$\mathbf{K}=\begin{pmatrix}0.5104&-0.4577\\-1.036&0.04031\\-0.5243&0.5835\end{pmatrix}. \qquad (27)$$

The gain for the parameter adaptation γ was selected as $\gamma=0.05$. The results are shown in Fig 2,3 and 4, where it

[1] This is the same Kalman-Yakubovich condition required in the integer-order case

can be seen the dynamics of each state compared to their observed values, and how the observation error converges to zero even after the parameter change in $t = 300$ and $t = 600$.

In Fig. 5 it can be seen how the reconstructed parameter $\hat{a}$ tends to the real value even after the change. This indicates that some typical conditions related to the persistence of excitation in the integer-order case could be extrapolated to the fractional-order case. However, this is still an open research field, although some advances have been presented (Aguila-Camacho and Duarte-Mermoud, 2016).

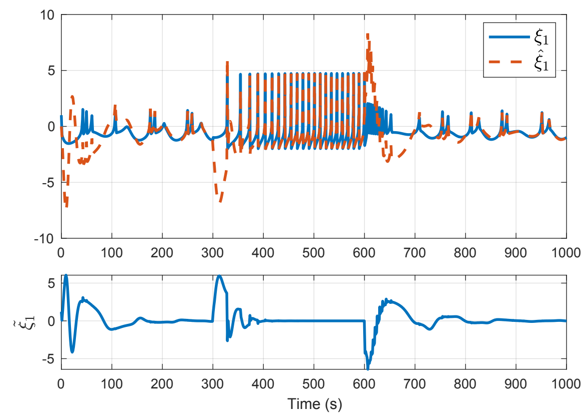

Fig. 2. State observation $\hat{\xi}_1$ for the proposed adaptive observer and parameter change in $t = 300$ and $t = 600$.

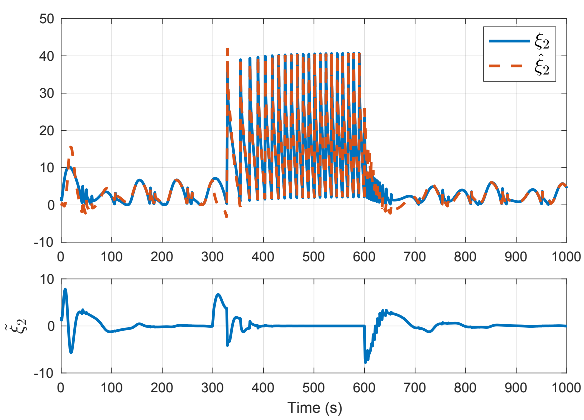

Fig. 3. State observation $\hat{\xi}_2$ for the proposed adaptive observer and parameter change in $t = 300$ and $t = 600$.

5. CONCLUSIONS

In this work it was presented an adaptive observer scheme for the fractional-order Hindmarsh-Rose neuron model, based on Lyapunov quadratic candidate functions, with

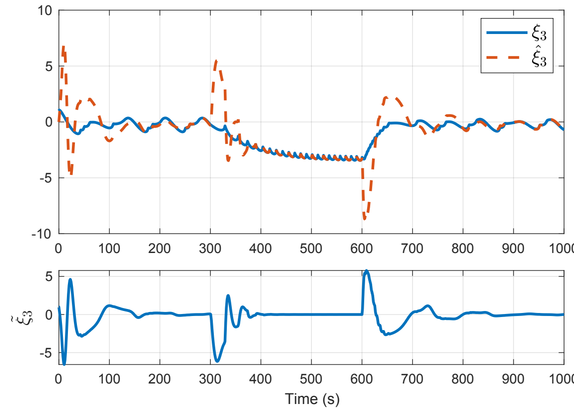

Fig. 4. State observation $\hat{\xi}_3$ for the proposed adaptive observer and parameter change in $t = 300$ and $t = 600$.

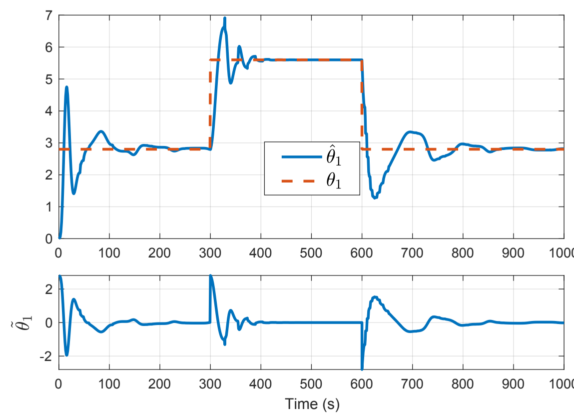

Fig. 5. Parameter reconstruction $\hat{a}$ for the proposed adaptive observer and parameter change in $t = 300$ and $t = 600$.

convergence in the pseudo-states and parameter boundedness. Numeric examples were shown in order to demonstrate the efectiveness of the proposed methodology. Future work involves the estimation of further parameters, as well as applications to synchronization techniques.

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