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Discontinuous observer-based convex control of the Furuta pendulum ⋆

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Abstract: Discontinuous observer-based convex control of an underactuated rotatory system, better known as the Furuta pendulum, is the subject of this report. It is shown that a slight generalization of the traditional discontinuous observer design allows applying it to the referred plant. Since the observer gains are obtained via linear matrix inequalities and convex rewriting of nonlinear terms, convex control comes at hand as a better fit to drive the system to the upright position. Simulation and real-time results are presented which prove the effectiveness of the proposal.

Keywords: Discontinuous observer, Convex control, Observer-based control, Lyapunov stability, Linear matrix inequalities.

1. INTRODUCTION

Effectiveness of nonlinear control schemes is better put at test on underactuated systems with rapid dynamics such as the inverted pendulum on a cart (Angeli, 2001), the Pendubot (Begovich et al., 2002), the 3-link SISO system (Farwig et al., 1990), the pendulum on an inclined rail (Furuta et al., 1980), and the inertia wheel link (Spong et al., 2001). None of these examples have a 3D workspace as the Furuta pendulum, which is an underactuated mechanism with 2 degrees of freedom (DOF), 2 beams, and 2 rotational joints. (Furuta et al., 1992); this report is focused on observer-based controller design for the latter.

As most robotic plants with Lagrange-Euler model, the physical setup of the Furuta pendulum is such that only positions are measured via encoders: a first one for the angle of the beam at the base with respect to a fixed reference; a second one for the angle of the vertical beam with respect to the upright position. It is not common to incorporate any device directly measuring the beam velocities due to its cost; thus the need of observer schemes to recover all the states for control purposes, more specifically the velocities. The underactuated characteristic of Furuta pendulum resides in the fact that the actuator only rotates the horizontal beam which should be enough to drive the vertical beam to its upright position. Expectedly, such dynamics require a quick response on the part of the controller, which in turn requires the observer to converge as soon as possible.

Finite-time observer design is preferred for the task just described for obvious reasons; one of the first approaches in such class was the discontinuous observer in Edwards and Spurgeon (1994), which employs sliding modes to drive a part of the observer error to 0 in finite time. As most sliding mode approaches, it requires constructing an error system with a linear nominal model; nonlinearities are usually considered as perturbations (Edwards and Spurgeon, 1998); if coupled, the observer thus designed will be insensitive to their presence; if not coupled, the approach will fail or require incorporating some sort of attenuation such as $\mathscr{H}_\infty$. Other traditional schemes have the same linear nominal model characteristics, for instance Walcott and Zak (1987); Walcott et al. (1987); Walcott and Zak (1988). In this work, nonlinear nominal models are employed to alleviate the requisites for discontinuous observer design in order to apply it to the Furuta pendulum, which otherwise does not meet the conditions for such approach. The novel observer will be used to estimate the velocities of the plant in order to use them for control purposes. Handling nonlinear nominal models is performed by means of convex modelling of nonlinearities (Taniguchi et al., 2001), which is a common practice in convex control (Bernal et al., 2022) where, when combined with the direct Lyapunov method, it allows for design conditions to be put in the form of linear matrix inequalities (LMIs) (Boyd et al., 1994), which can be solved in polynomial time by means of efficient algorithms already

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implemented in commercially available software (Gahinet et al., 1995).

Solving for observer and controller gains by means of numerical tools is increasingly appreciated in the control community as it represents a paradigm change from purely analytical ad-hoc solutions to systematic numerically computable techniques (Tanaka and Wang, 2001). This is the reason for adopting a convex control scheme along with the novel observer proposal in this paper; the corresponding control law is a generalization of the well-known parallel distributed compensation first appeared in Wang et al. (1996); it will drive the Furuta pendulum to its upright position based exclusively on the position (angles) and the velocity estimations (angular velocities).

The paper is organized as follows: the model is presented and rewritten in a convex form in section 2, along with a convex control law requiring the whole state to be available; section 3 shows how a nonlinear nominal model can be handled for discontinuous observer design by means of convex modelling and slight modifications of the scheme in Edwards and Spurgeon (1994); based on the discontinuous observer just designed, section 4 presents how the control law in section 2 performs when velocities are replaced by their estimates, both in simulation and real-time implementation; the paper gathers some conclusions in section 5.

2. A CONVEX CONTROL LAW FOR THE FURUTA PENDULUM

Fig.1 shows the Furuta pendulum manufactured by Quanser as part of the mechatronics kit (Quanser, 2006): it has a horizontal beam at the base which rotates when activated by the sole actuator of the system (a DC motor), and a vertical beam which is linked to the horizontal one by a rotary joint with no actuator. This section proposes a convex control law to drive the vertical beam to its upright position within a controllable range (no swing-up is involved). In this section, it is assumed that the whole state is available.

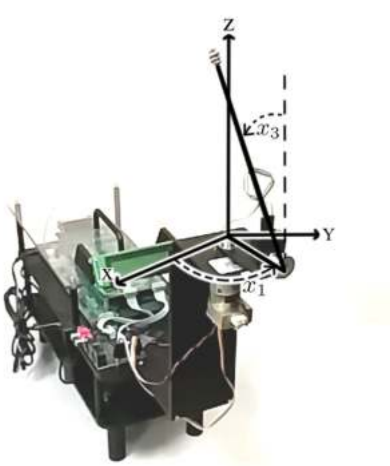

Fig. 1. Furuta pendulum

The Furuta pendulum has the following state space model:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{(\beta+\gamma)\left(\delta x_4^2 \sin x_3 - 2\beta x_2 x_4 \cos x_3 \sin x_3 + u\right)}{\left((\beta+\gamma)\beta+\delta^2\right)\sin^2 x_3 + (\beta+\gamma)\alpha - \delta^2} \\
&\quad - \frac{\delta \cos x_3 \left(\beta x_2^2 \cos x_3 \sin x_3 + \sigma g \sin x_3\right)}{\left((\beta+\gamma)\beta+\delta^2\right)\sin^2 x_3 + (\beta+\gamma)\alpha - \delta^2} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\left(\beta \sin^2 x_3 + \alpha\right)\left(\beta x_2^2 \cos x_3 \sin x_3 + \sigma g \sin x_3\right)}{\left((\beta+\gamma)\beta+\delta^2\right)\sin^2 x_3 + (\beta+\gamma)\alpha - \delta^2} \\
&\quad - \frac{\delta \cos x_3 \left(\delta x_4^2 \sin x_3 - 2\beta x_2 x_4 \cos x_3 \sin x_3 + u\right)}{\left((\beta+\gamma)\beta+\delta^2\right)\sin^2 x_3 + (\beta+\gamma)\alpha - \delta^2}
\end{aligned} \tag{1}$$

where x_1 and x_3 are the angles of the horizontal and vertical beams with respect to a fixed position at the base and the upright position, respectively, x_2 and x_4 are the corresponding angular velocities, and the system parameters are $\alpha = \left(J_0 + m_1 L_0^2\right)/T_c$, $\beta = \left(m_1 l_1^2\right)/T_c$, $\gamma = J_1/T_c$, $\delta = \left(m_1 L_0 l_1\right)/T_c$, $\sigma = \left(m_1 l_1\right)/T_c$, with $L_0 = 0.068$m and $J_0 = 6.9885e^{-5}$ being the length and total moment of inertia of the horizontal link, respectively, $m_1 = 0.02366$kg, $l_1 = 0.08$, and $J_1 = 1.7590e^{-4}$ being the mass, center of mass, and total moment of inertia of the vertical beam, respectively, $g = 9.81$ and $T_c = 0.0049431$ being the gravitational constant and the torque constant, respectively. The task of keeping the second beam at the upright position coincides with driving the system to the origin $x_i = 0$, $i \in \{1, 2, 3, 4\}$,

The state equations (1) can be written as in Vázquez et al. (2016), i.e.:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & z_1 z_2 z_5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & z_2 z_3 z_5 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ (\beta+\gamma) z_5 \\ 0 \\ -\delta z_4 z_5 \end{bmatrix} u(t), \tag{2}$$

where $z_1 = (\beta+\gamma)\delta x_4^2 - 2\beta(\beta+\gamma)x_2 x_4 \cos x_3 - \beta\delta x_2^2 \cos^2 x_3 - \delta\sigma g \cos x_3$, $z_2 = (\sin x_3)/x_3$, $z_3 = (\beta \sin^2 x_3 + \alpha)(\beta x_2^2 \cos x_3 + \sigma g) - \delta \cos x_3 \left(\delta x_4^2 - 2\beta x_2 x_4 \cos x_3\right)$, $z_4 = \cos x_3$, and $z_5 = 1/((\beta^2 + \gamma\beta + \delta^2)\sin^2 x_3 + \alpha\beta + \alpha\gamma - \delta^2)$.

Convex control is based on the fact that every nonlinearity in (1) can be rewritten as a convex sum within a region of interest of the state space (Bernal et al., 2022); let us call this region $\Omega \subset \mathbb{R}^4$ with the property of including the origin, i.e., $0 \in \Omega$. Physical bounds can be taken into account to define such region. In this work we will take the specifications in Vázquez et al. (2016), namely $\Omega = \{x : |x_2| \leq 6, |x_3| \leq 0.2618, |x_4| \leq 3\}$. Thus, z_i, $i \in \{1, 2, \ldots, 5\}$ are bounded in Ω as shown in Table 1.

It can be verified that $z_i = w_0^i(x) z_i^0 + w_1^i(x) z_i^1$, where $w_0^i(x) = (z_i^1 - z_i)/(z_i^1 - z_i^0)$ and $w_1^i(x) = (z_i - z_i^0)/(z_i^1 - z_i^0)$ are functions that hold the convex sum property within Ω, i.e., $w_0^i(x) + w_1^i(x) = 1$, $0 \leq w_j^i \leq 1$, $i \in \{1, 2, \ldots, 5\}$.

The fact that each z_i in (2) can be written as a convex sum of its bounds in Ω allows writing (2) as:

$$\dot{x}(t) = \sum_{\mathbf{i}\in\mathbb{B}^5} \mathbf{w_i}(x)\left(A_\mathbf{i}x(t) + B_\mathbf{i}u(t)\right), \tag{3}$$

since convex sums can be stacked together at the leftmost side of expressions, where $\mathbb{B} = \{0,1\}$, $\mathbf{i} = (i_1, i_2, \ldots, i_5)$, $\mathbf{w_i}(x) = w^1_{i_1}(x)w^2_{i_2}(x)w^3_{i_3}(x)w^4_{i_4}(x)w^5_{i_5}(x)$,

$$A_\mathbf{i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & z_1^{i_1}z_2^{i_2}z_5^{i_5} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & z_2^{i_2}z_3^{i_3}z_5^{i_5} & 0 \end{bmatrix}, B_\mathbf{i} = \begin{bmatrix} 0 \\ (\beta+\gamma)z_5^{i_5} \\ 0 \\ -\delta z_4^{i_4}z_5^{i_5} \end{bmatrix}.$$

The model (3) is known as a tensor product equivalent of the nonlinear model (1): it is not an approximation. A generalization of the well-known parallel distributed compensation can be used to drive such model to 0, namely,

$$u(t) = \sum_{\mathbf{j}\in\mathbb{B}^5} \mathbf{w_j}(x)F_\mathbf{j}x(t), \tag{4}$$

where $F_\mathbf{j}$ are gains to be obtained from the solution of the LMIs:

$$\sum_{(\mathbf{i},\mathbf{j})\in\mathcal{P}(\mathbf{k},\mathbf{l})} \left(A_\mathbf{i}X + B_\mathbf{i}M_\mathbf{j} + XA_\mathbf{i}^T + M_\mathbf{j}^T B_\mathbf{i}^T\right) < 0, \tag{5}$$

where $\mathbf{k},\mathbf{l} \in \mathbb{B}^5$, $\mathcal{P}(\mathbf{k},\mathbf{l})$ is the set of indexes $(\mathbf{i},\mathbf{j})$ such that $\mathbf{w_i}\mathbf{w_j} = \mathbf{w_k}\mathbf{w_l}$, X and $M_\mathbf{j}$ are decision variables, and $F_\mathbf{j} = M_\mathbf{j}X^{-1}$. The LMI Toolbox (Gahinet et al., 1995) or SeDuMi (Sturm, 1999) can be used to numerically solve these conditions; they are obtained by means of the direct Lyapunov method. The interested reader is referred to (Bernal et al., 2022).

The nonlinear control (4) relies on the availability of the whole state, which is unrealistic. Therefore, a novel discontinuous observer is proposed in the next section to obtain the states that the physical setup of Quanser does not provide, i.e., the angular velocities x_2, x_4.

3. AN IMPROVED DISCONTINUOUS OBSERVER DESIGN FOR THE FURUTA PENDULUM

Our goal is to estimate the non-measurable states x_2 and x_4 from the knowledge of the output $y = [x_1\ x_3]^T$ and input $u(t)$. To this end, consider the change of coordinates

$$\begin{bmatrix} \chi_1 \\ \chi_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

with $\chi = [\chi_1\ \chi_2]^T$, and $y = [y_1\ y_2]^T$. Thus, system (1) can be rewritten in the new coordinates as

$$\begin{aligned} \dot{\chi}(t) &= f_\chi(\chi, y) + g_\chi(y)u(t), \\ \dot{y}(t) &= f_y(\chi, y), \end{aligned}$$

Table 1. Bounds for $z_i \in [z_i^0, z_i^1]$ in (2)

Nonlinearity	z_1	z_2	z_3	z_4	z_5
Lower bound	−0.184	0.9886	0.1251	0.9659	524.91
Upper bound	−0.0352	1	0.2057	1	580.12

with $f_\chi(\chi,y) = [f_\chi^1(\chi,y)\ f_\chi^2(\chi,y)]^T$, $g_\chi(y) = [g_\chi^1(y)\ g_\chi^2(y)]^T$, $f_y(\chi,y) = [f_y^1(\chi,y)\ f_y^2(\chi,y)]^T$, and

$$\begin{aligned} f_\chi^1(\chi,y) = & \frac{(\beta+\gamma)\left(\delta\chi_2^2\sin y_2 - 2\beta\chi_1\chi_2\cos y_2\sin y_2\right)}{((\beta+\gamma)\beta+\delta^2)\sin^2 y_2 + (\beta+\gamma)\alpha - \delta^2} \\ & - \frac{\delta\cos y_2\left(\beta\chi_1^2\cos y_2\sin y_2 + \sigma g\sin y_2\right)}{((\beta+\gamma)\beta+\delta^2)\sin^2 y_2 + (\beta+\gamma)\alpha - \delta^2}, \\ f_\chi^2(\chi,y) = & \frac{\left(\beta\sin^2 y_2 + \alpha\right)\left(\beta\chi_1^2\cos y_2\sin y_2 + \sigma g\sin y_2\right)}{((\beta+\gamma)\beta+\delta^2)\sin^2 y_2 + (\beta+\gamma)\alpha - \delta^2} \\ & - \frac{\delta\cos y_2\left(\delta\chi_2^2\sin y_2 - 2\beta\chi_1\chi_2\cos y_2\sin y_2\right)}{((\beta+\gamma)\beta+\delta^2)\sin^2 y_2 + (\beta+\gamma)\alpha - \delta^2}, \\ g_\chi^1(y) = & (\beta+\gamma)/(((\beta+\gamma)\beta+\delta^2)\sin^2 y_2 + (\beta+\gamma)\alpha - \delta^2), \\ g_\chi^2(y) = & -\delta\cos y_2/(((\beta+\gamma)\beta+\delta^2)\sin^2 y_2 + (\beta+\gamma)\alpha - \delta^2), \\ f_y^1(\chi) = & \chi_1, \quad f_y^2(\chi) = \chi_2. \end{aligned}$$

Based on the structure of the transformed system above, the following discontinuous observer is proposed

$$\begin{aligned} \dot{\hat{\chi}}(t) = & f_\chi(\hat{\chi}, y) + g_\chi(y)u(t) + L_\chi(\hat{y} - y) \\ & -KL_y(\hat{y} - y) - K\nu, \\ \dot{\hat{y}}(t) = & f_y(\hat{\chi}, y) + L_y(\hat{y} - y) + \nu, \end{aligned} \tag{6}$$

where $[\hat{\chi}^T \hat{y}^T]^T$ is the observer state; $K \in \mathbb{R}^{2\times 2}$, $L_\chi \in \mathbb{R}^{2\times 2}$, and $L_y \in \mathbb{R}^{2\times 2}$ are constant observer gains to be designed; ν is a discontinuous term to be defined later.

Now, taking into account the error signals $e_\chi = \hat{\chi} - \chi$, $e_y = \hat{y} - y$, and the factorization given in (Quintana et al., 2020), the nonlinear error system

$$\begin{aligned} \begin{bmatrix} \dot{e}_\chi \\ \dot{e}_y \end{bmatrix} = & \begin{bmatrix} \bar{A}^{11}(\chi,\hat{\chi},y) & \bar{A}^{12}(\chi,\hat{\chi},y) \\ \bar{A}^{21}(\chi,\hat{\chi},y) & \bar{A}^{22}(\chi,\hat{\chi},y) \end{bmatrix} \begin{bmatrix} e_\chi \\ e_y \end{bmatrix} \\ & + \begin{bmatrix} L_\chi e_y - KL_y e_y - K\nu \\ L_y e_y + \nu \end{bmatrix}, \end{aligned} \tag{7}$$

is obtained, where

$$\bar{A}^{11}(\chi,\hat{\chi},y) = \begin{bmatrix} \zeta_1(-2(\beta+\gamma)\beta\zeta_6\zeta_2\zeta_4 - \delta\beta(\zeta_5+\zeta_7)\zeta_3\zeta_4) & \zeta_1((\beta+\gamma)(\delta\zeta_4(\zeta_6+\zeta_8) - 2\beta\zeta_2\zeta_4\zeta_7)) \\ \zeta_1((\beta(1-\zeta_3)+\alpha)\beta\zeta_2\zeta_4(\zeta_5+\zeta_7) + 2\delta\zeta_3\beta\zeta_4\zeta_6) & \zeta_1(-\delta(\delta\zeta_2\zeta_4(\zeta_6+\zeta_8) - 2\beta\zeta_3\zeta_4\zeta_7)) \end{bmatrix},$$

$$\bar{A}^{21}(\chi,\hat{\chi},y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{A}^{12}(\chi,\hat{\chi},y) = \bar{A}^{22}(\chi,\hat{\chi},y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with $\zeta_1 = 1/(((\beta+\gamma)\beta+\delta^2)\sin^2 y_2 + (\beta+\gamma)\alpha - \delta^2)$, $\zeta_2 = \cos y_2$, $\zeta_3 = \cos^2 y_2$, $\zeta_4 = \sin y_2$, $\zeta_5 = \hat{\chi}_1$, $\zeta_6 = \hat{\chi}_2$, $\zeta_7 = \chi_1$, and $\zeta_8 = \chi_2$.

Applying the change of coordinates

$$\begin{bmatrix} \tilde{e}_\chi \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} e_\chi + Ke_y \\ e_y \end{bmatrix}, \tag{8}$$

to system (7), yields

$$\begin{bmatrix} \dot{\tilde{e}}_\chi \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} \tilde{A}^{11}(\chi,\hat{\chi},y) & \tilde{A}^{12}(\chi,\hat{\chi},y) \\ \tilde{A}^{21}(\chi,\hat{\chi},y) & \tilde{A}^{22}(\chi,\hat{\chi},y) \end{bmatrix} \begin{bmatrix} e_\chi \\ e_y \end{bmatrix} + \begin{bmatrix} L_\chi e_y \\ L_y e_y + \nu \end{bmatrix}, \tag{9}$$

with $\tilde{A}^{11}(\chi,\hat{\chi},y) = \bar{A}^{11}(\chi,\hat{\chi},y) + K\bar{A}^{21}(\chi,\hat{\chi},y)$, $\tilde{A}^{12}(\chi,\hat{\chi},y) = \bar{A}^{12}(\chi,\hat{\chi},y) + K\bar{A}^{22}(\chi,\hat{\chi},y) - (\bar{A}^{11}(\chi,\hat{\chi},y) + K\bar{A}^{21}(\chi,\hat{\chi},y))K$, $\tilde{A}^{21}(\chi,\hat{\chi},y) = \bar{A}^{21}(\chi,\hat{\chi},y)$, and $\tilde{A}^{22}(\chi,\hat{\chi},y) = \bar{A}^{22}(\chi,\hat{\chi},y) - \bar{A}^{21}(\chi,\hat{\chi},y)K$.

Since the goal is to design a discontinuous observer, the following sliding surface

$$\mathcal{S} = \{[\tilde{e}_\chi^T \; e_y^T]^T : e_y = 0\}. \tag{10}$$

is proposed. As with any other sliding-mode-based approach, we need to guarantee: stability of the reduced-order error system once the sliding motion takes place; asymptotic stability of the origin of (9); an ideal sliding motion on $\mathcal{S}$.

Assuming an ideal sliding motion is already taking place on $\mathcal{S}$, the reduced-order error system is

$$\dot{\tilde{e}}_\chi(t) = \left(\bar{A}^{11}(\chi,\hat{\chi},y) + K\bar{A}^{21}(\chi,\hat{\chi},y)\right)\tilde{e}_\chi, \tag{11}$$

where $\tilde{e}_\chi = 0$ needs to be asymptotically stable to guarantee the stability of the sliding motion. To do that, convex modelling and LMI-based design comes at hand to find an adequate observer gain K. As before, consider the region of interest $\Omega = \{[\chi^T \; y^T] : |\chi_1| \le 10, |\chi_2| \le 3, |y_2| \le 0.2618\}$ and let the observer states mimic them, i.e., $|\hat{\chi}_1| \le 10, |\hat{\chi}_2| \le 3$. Thus, using the bounds in Table 2, the reduced-order nonlinear system (11) can be rewritten as

$$\dot{\tilde{e}}_\chi(t) = \sum_{\mathbf{i}\in\mathbb{B}^8} \boldsymbol{\omega}_\mathbf{i}(\chi,\hat{\chi},y)\left(\bar{A}^{11}_\mathbf{i} + K\bar{A}^{21}_\mathbf{i}\right)\tilde{e}_\chi, \tag{12}$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_8) \in \mathbb{B}^8$, $\boldsymbol{\omega}_\mathbf{i}(\zeta) = \omega^1_{i_1}(\zeta_1)\omega^2_{i_2}(\zeta_2)\omega^3_{i_3}(\zeta_3)\omega^4_{i_4}(\zeta_4)\omega^5_{i_5}(\zeta_5)\omega^6_{i_6}(\zeta_6)\omega^7_{i_7}(\zeta_7)\omega^8_{i_8}(\zeta_8)$, $\omega^i_0(\zeta_i) = (\zeta^1_i - \zeta_i)/(\zeta^1_i - \zeta^0_i)$, $\omega^i_1(\zeta_i) = 1 - \omega^i_0(\zeta_i)$, $i \in \{1,2,\ldots,8\}$; $\bar{A}^{i1}_\mathbf{i} = \bar{A}^{i1}(\chi,\hat{\chi},y)|_{\boldsymbol{\omega}_\mathbf{i}=1}$, $i \in \{1,2\}$.

Theorem 1. The origin of the nonlinear reduced-order system (11), which is algebraically equivalent to (12), is asymptotically stable if there exist $P \in \mathbb{R}^{2\times2}$ and $K \in \mathbb{R}^{2\times2}$ such that the LMIs

$$P > 0, \quad He\left\{P\bar{A}^{11}_\mathbf{i} + \bar{N}\bar{A}^{21}_\mathbf{i}\right\} < 0 \tag{13}$$

hold for all $\mathbf{i} \in \mathbb{B}^8$, with $K = P^{-1}\bar{N}$.

Proof. Consider $V_1(\tilde{e}_\chi) = \tilde{e}_\chi^T P\tilde{e}_\chi > 0$, $P > 0$ as a Lyapunov function candidate; its time derivative is

$$\begin{aligned}\dot{V}(\tilde{e}_\chi) =& \tilde{e}_\chi^T\left(\text{He}\left\{P\bar{A}^{11}(\chi,\hat{\chi},y) + PK\bar{A}^{21}(\chi,\hat{\chi},y)\right\}\right)\tilde{e}_\chi \\ =& \sum_{\mathbf{i}\in\mathbb{B}^8}\boldsymbol{\omega}_\mathbf{i}(\chi,\hat{\chi},y)\left(\tilde{e}_\chi^T\left(\text{He}\left\{P\bar{A}^{11}_\mathbf{i} + PK\bar{A}^{21}_\mathbf{i}\right\}\right)\tilde{e}_\chi\right).\end{aligned}$$

Thus, taking into account the fact $\omega^i_0(\chi,\hat{\chi},y), \omega^i_1(\chi,\hat{\chi},y) \in [0,1]$, $\dot{V}(\tilde{e}_\chi) < 0$ is guaranteed if

$$\text{He}\left\{P\bar{A}^{11}_\mathbf{i} + \bar{N}\bar{A}^{21}_\mathbf{i}\right\} < 0$$

for all $\mathbf{i} \in \mathbb{B}^8$, with $\bar{N} = PK$; which corresponds to the design conditions (13), which concludes the proof. □

Once K has been designed, it can be substituted in (9); the resulting expression will have the same non-constant terms in (11). Thus, using the same bounds for such non-constant terms, (9) is rewritten as

Table 2. Bounds for $\zeta_i \in [\zeta^0_i, \zeta^1_i]$ in $\bar{A}^{11}(\chi,\hat{\chi},y)$

ζ_i	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6	ζ_7	ζ_8
ζ^0_i	524	0.9659	0.9330	−0.2588	−10	−3	−10	−3
ζ^1_i	581	1	1	0.2588	10	3	10	3

$$\begin{bmatrix}\dot{\tilde{e}}_\chi \\ \dot{e}_y\end{bmatrix} = \sum_{\mathbf{i}\in\mathbb{B}^8}\boldsymbol{\omega}_\mathbf{i}(\chi,\hat{\chi},y)\left(\begin{bmatrix}\tilde{A}^{11}_\mathbf{i} & \tilde{A}^{12}_\mathbf{i} \\ \tilde{A}^{21}_\mathbf{i} & \tilde{A}^{22}_\mathbf{i}\end{bmatrix}\begin{bmatrix}\tilde{e}_\chi \\ e_y\end{bmatrix} + \begin{bmatrix}L_\chi e_y \\ L_y e_y + \nu\end{bmatrix}\right), \tag{14}$$

where $\tilde{A}^{ij}_\mathbf{i} = \tilde{A}^{ij}(\chi,\hat{\chi},y)|_{\boldsymbol{\omega}_\mathbf{i}=1}$, $i,j \in \{1,2\}$.

Theorem 2. The origin $[\tilde{e}_\chi^T \; e_y^T]^T = 0$ of the error system (9), is asymptotically stable if conditions in Theorem 1 hold for $P \in \mathbb{R}^{2\times2}$ and $K \in \mathbb{R}^{2\times2}$, ν being defined as

$$\nu = \begin{cases} -\eta\frac{Qe_y}{\|Qe_y\|}, & \text{if } e_y \ne 0 \\ 0, & \text{otherwise,}\end{cases} \tag{15}$$

with $\eta > 0$, and if there exist $Q = Q^T \in \mathbb{R}^{2\times2}$, $L_\chi \in \mathbb{R}^{2\times2}$, and $N \in \mathbb{R}^{2\times2}$, such that the LMIs

$$Q > 0, \begin{bmatrix}\text{He}\left\{P\tilde{A}^{11}_\mathbf{i}\right\} & P\tilde{A}^{12}_\mathbf{i} + PL_\chi + (\tilde{A}^{21})^T_\mathbf{i}Q \\ (*) & \text{He}\left\{Q\tilde{A}^{22}_\mathbf{i} + N\right\}\end{bmatrix} < 0, \tag{16}$$

hold for all $\mathbf{i} \in \mathbb{B}^8$, with $L_y = Q^{-1}N$.

Proof. Let $V(\tilde{e}_\chi, e_y) = [\tilde{e}_\chi^T \; e_y^T]\text{block-diag}(P,Q)[\tilde{e}_\chi^T \; e_y^T]^T$, with $P > 0$, $Q > 0$, be a Lyapunov function candidate; we have that

$$\begin{aligned}\dot{V} \le& 2\begin{bmatrix}\tilde{e}_\chi \\ e_y\end{bmatrix}^T\begin{bmatrix}P & 0 \\ 0 & Q\end{bmatrix}\left(\begin{bmatrix}\tilde{A}^{11}(\chi,\hat{\chi},y) & \tilde{A}^{12}(\chi,\hat{\chi},y)+L_\chi \\ \tilde{A}^{21}(\chi,\hat{\chi},y) & \tilde{A}^{22}(\chi,\hat{\chi},y)+L_y\end{bmatrix}\right)\begin{bmatrix}\tilde{e}_\chi \\ e_y\end{bmatrix} \\ =& \sum_{\mathbf{i}\in\mathbb{B}^8}\boldsymbol{\omega}_\mathbf{i}\begin{bmatrix}\tilde{e}_\chi \\ e_y\end{bmatrix}^T\begin{bmatrix}\text{He}\{P\tilde{A}^{11}_\mathbf{i}\} & P\tilde{A}^{12}_\mathbf{i} + PL_\chi + (\tilde{A}^{21}_\mathbf{i})^TQ \\ (*) & \text{He}\{Q\tilde{A}^{22}_\mathbf{i} + QL_y\}\end{bmatrix}\begin{bmatrix}\tilde{e}_\chi \\ e_y\end{bmatrix},\end{aligned}$$

which means that a sufficient condition to guarantee $\dot{V}(\tilde{e}_\chi, e_y) < 0$ is

$$\begin{bmatrix}\text{He}\{P\tilde{A}^{11}_\mathbf{i}\} & P\tilde{A}^{12}_\mathbf{i} + PL_\chi + (\tilde{A}^{21}_\mathbf{i})^TQ \\ (*) & \text{He}\{Q\tilde{A}^{22}_\mathbf{i} + N\}\end{bmatrix} < 0$$

for all $\mathbf{i} \in \mathbb{B}^8$, with $N = QL_y$, since $\omega^i_0(\chi,\hat{\chi},y)$, $\omega^i_1(\chi,\hat{\chi},y) \in [0,1]$, $\nu \le 0$. These conditions correspond to (16), which concludes the proof.

Once the stability of $\tilde{e}_1 = 0$ in the reduced-order error system and the attractiveness of the nonlinear sliding surface are ensured, an ideal sliding motion needs to be guaranteed on $\mathcal{S}$. To this end, consider the Lyapunov function candidate $V(e_y) = e_y^TQe_y$ along with the definition of ν with $\eta > 0$, and $\text{He}\left\{Q(\tilde{A}^{22}(\chi,\hat{\chi},y) + L_y)e_y\right\} < 0$. The time derivative of $V(e_y)$ satisfies

$$\begin{aligned}\dot{V}(e_y) =& 2e_y^TQ(\tilde{A}^{22}(\chi,\hat{\chi},y)+L_y)e_y + 2e_y^TQ(\tilde{A}^{21}(\chi,\hat{\chi},y)\tilde{e}_1 + \nu) \\ \le& 2e_y^TQ\tilde{A}^{21}(\chi,\hat{\chi},y)\tilde{e}_1 - 2\eta\|Qe_y\| \\ \le& 2\|Qe_y\|(\|\tilde{A}^{21}(\chi,\hat{\chi},y)\tilde{e}_1\| - \eta).\end{aligned}$$

In the compact set $\Omega_V = \{(\tilde{e}_1, e_y) : \|\tilde{A}^{21}(\chi,\hat{\chi},y)\tilde{e}_1\| < \eta - \gamma\}$, with $\gamma > 0$ sufficiently small, it is clear that

$$\dot{V}(e_y) = -2\gamma\|Qe_y\| \le -2\gamma\sqrt{\lambda_{min}(Q)}\sqrt{V(e_y)},$$

which along with Theorem 2 guarantees e_y enters Ω_V in finite time and remains there (Edwards and Spurgeon, 1998).

4. SIMULATION AND REAL-TIME RESULTS

The proposed observer-based control scheme has been implemented both in simulation and real-time: the former within a MATLAB R2015a platform where LMIs were solved using the LMI Toolbox Gahinet et al. (1995) and the latter using Quanser software which is a Simulink/MATLAB-based platform for real-time implementation in the Mechatronics Kit along with Wincon 5.2 (Quanser, 2006).

Figure 2 shows the positions of the horizontal and vertical beams which are driven to 0 from $x_1(0) = -0.565$ and $x_3(0) = 0.125$ radians, by the observer-based control scheme; their estimates $\hat{x}_1$ and $\hat{x}_3$ are shown too for completeness, though they play no roll. For better appreciation of the observer performance, small figures zooming around the initial conditions are included. Figure 3 shows the corresponding angular velocities x_2 and x_4 and their estimates $\hat{x}_2$ and $\hat{x}_4$, which are used in the control law instead of the real states. On the left side of Figure 6 the observer-based control law in the previous simulations is displayed.

Figure 4 shows the angles x_1 and x_3 and their estimates in real-time implementation; the slight oscillations in x_1 needed to maintain the vertical beam at the upright position can be appreciated. Figure 5 shows the angular

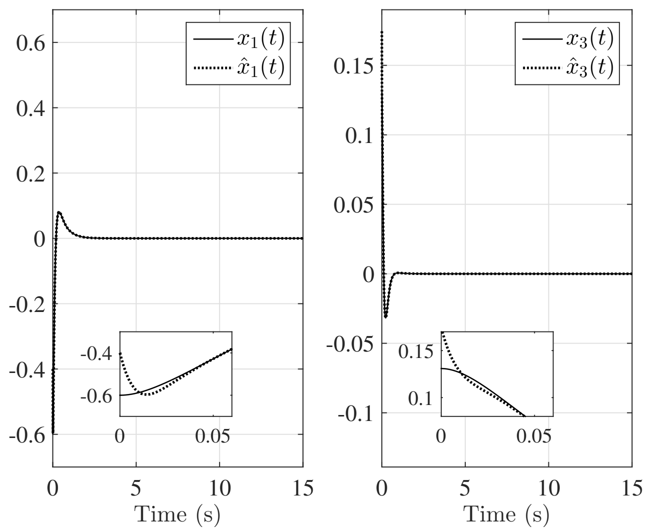

Fig. 2. Simulation results: positions x_1 and x_3 (solid-line) and their estimations $\hat{x}_1$ and $\hat{x}_3$ (dotted-line) in radians.

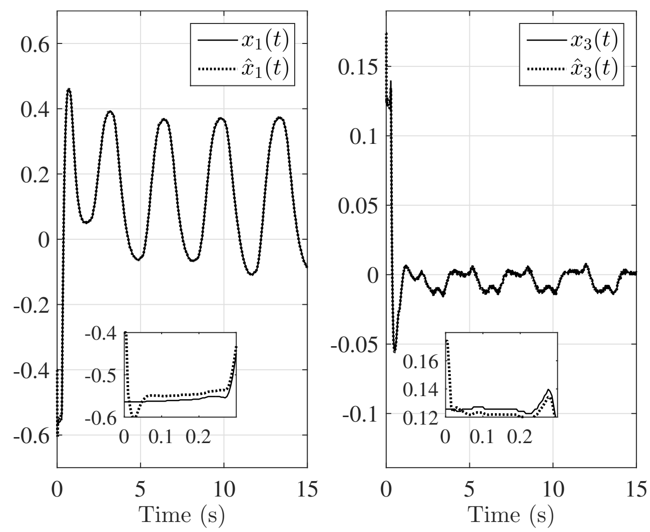

Fig. 4. Real-time results: positions x_1 and x_3 (solid-line) and their estimations $\hat{x}_1$ and $\hat{x}_3$ (dotted-line) in radians.

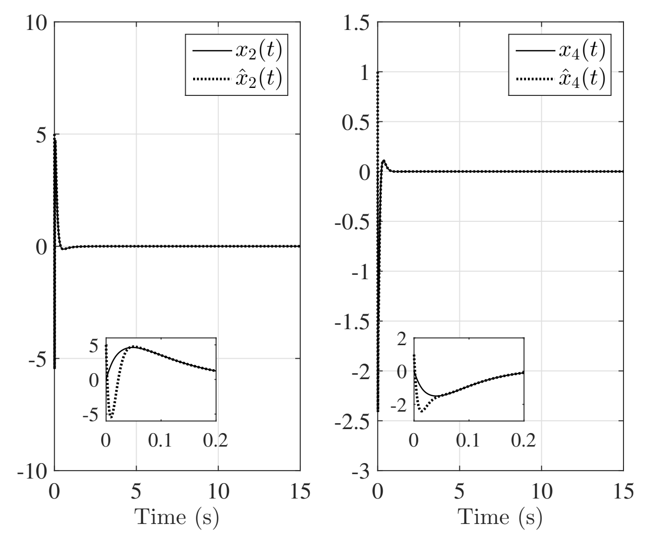

Fig. 3. Simulation results: velocities x_2 and x_4 (solid-line) and their estimations $\hat{x}_2$ and $\hat{x}_4$ (dotted-line) in radians per second.

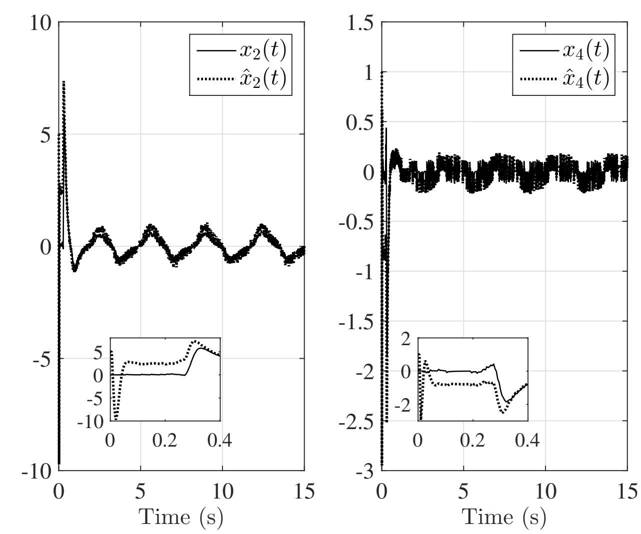

Fig. 5. Real-time results: velocities x_2 and x_4 (solid-line) and their estimations $\hat{x}_2$ and $\hat{x}_4$ (dotted-line) in radians per second.

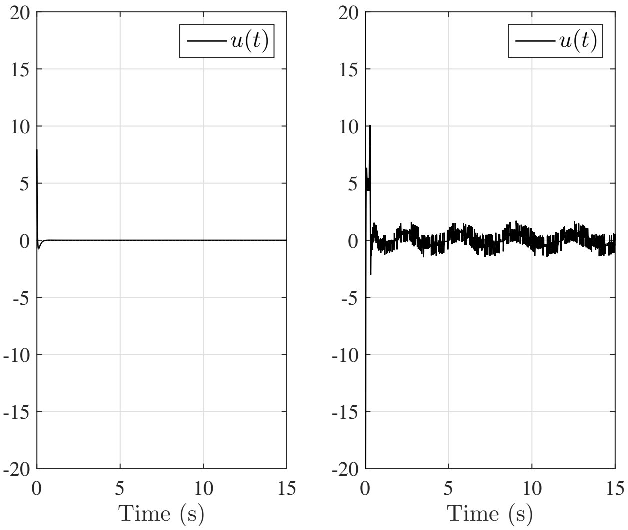

Fig. 6. Real-time results: observer-based control signal u in volts.

velocities (from the provider) and their estimates (from the observer) which are actually used in the observer-based control law shown on the right side of Figure 6; both results are real-time.

Note that observation takes place fast enough to guarantee the entire implementation to be stable and the corresponding analysis –control and observation– to be independent.

5. CONCLUSIONS

A novel discontinuous observer-based convex control scheme has been presented. The proposal has been put at test on the Furuta pendulum, both in simulation and real-time. It has been shown that traditional discontinuous observer design can be adapted to the referred plant by means of convex modelling and linear matrix inequalities. Future work will generalize the proposal to a wider family of systems.

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