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# Synchronization of memristor based bidirectionally coupled Hindmarsh-Rose neurons ⋆

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**Abstract:** An ideal memristor is a device whose resistive memory value is determine by its initial conditions and the voltage that has been applied across its terminals. As such, it is a good candidate to model the *synaptic plasticity* of neural systems. When memristors are included in neural models, they are called memristive neural networks. In this contribution, we investigate the emergence of synchronization in an array of two identical Hindmarsh-Rose neurons bidirectionally coupled through their voltage variables via memristors. We show that, for a sufficiently large positive memductance, synchronization emerges between neurons while the memristors converge to constant synaptic weight values. We illustrate our results with numerical simulations.

*Keywords:* Memristor, Synapse, Hindmarsh-Rose Neuron, Synchronization

## 1. INTRODUCTION

Neurons are the basic processing units of neural systems. In [Hodgkin & Huxley (1949)] (HH) an electric-physiological model of its behavior was proposed where voltage-dependent conductances were used to approximate the effects of ionic currents and contraregulatory effects of their concentrations on the neuron's membrane potential. The main dynamical feature of these model is the emergence of an action potential. Latter, in [Hindmarsh & Rose (1984)] (HR) a simplified model was proposed to capture the dynamical features of HH model. In particular, the bursting of spikes (action potentials) observed in real-world neurons. Under an appropriate choice of parameters the HR model can produce diverse firing patterns including single spiking, square bursting, chaotic bursting, and periodic firing [Innocenti (2007)].

A synapse is the extracellular space between neurons where electro-chemical transmission takes place [Kandel (2013)]. A transmitting neuron is called the *presynaptic* neuron while the receiving neuron is called *postsynaptic*. The action potential associated with the transmission of information is caused by an electrical current and the release of specialized molecules (neurotransmitters) by the dendrites on the synaptic space. Next, they bind to receptors on the postsynaptic neuron, that allow the opening of ion channels and therefore modify the electrical response in the postsynaptic neuron. One property of synapses is *plasticity* [Serrat (2011)], which consists in the variation of synaptic conductance, as a result of this property the inhibition or excitation of postsynaptic neuron can be achieved.

An alternative way to have an electrical representation of neurons and synapses is using circuits with memristors [Amirsoleimani (2016)]. The memristor [Chua (1971)] is a theoretical electronic device with resistive memory, characterized by a function that relates its electric charge with its magnetic flux. It is called an *ideal*-memristor because the current and voltage in the device correspond exactly with the derivatives of its charge and magnetic flux, respectively. From these relationships, the resistivity value of a flux-controlled memristor depends on the history of its voltage. Furthermore, once its voltage becomes zero, the resistance value of the memristor remains fixed. As a result, memristors have potential applications [Sanchez-Lopez (2019)][Carro-Perez (2018)] as non-volatile memories. When memristors are used in models of neurons and synapses, they are called memristive neural networks (MNN).

There are several applications of MNN such as pattern classification [Amirsoleimani (2016)], experimental demonstration of associative memory PershinVentra (2009), supervised learning [Nishitani (2015)] and secure communication[Li (2021)]. Among the different dynamical behaviors that neuron models coupled by memristive synapse can present we are interested in the synchronization of their firing patterns, our approach is analytical and does not involve a physical implementation. In this

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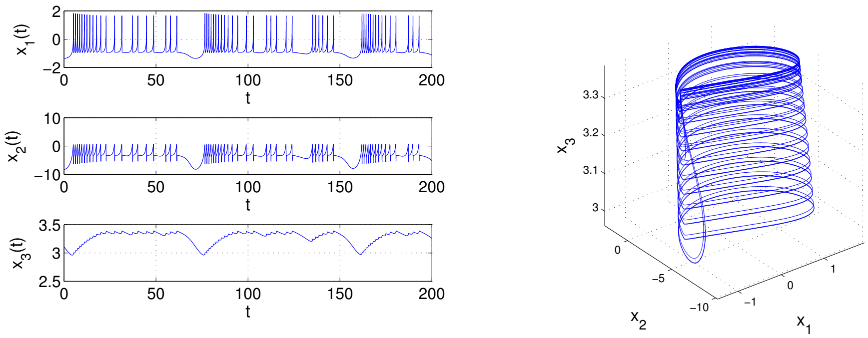


(a) (b)

Fig. 1. HR model with chaotic bursting behavior where (a) states vs time (b) chaotic spiking attractor

paper, we focus on the synchronization in two identical HR neurons bidirectionally coupled by ideal memristors, we determine that for a sufficient large memristance value identical synchronization is achieved even though the memory states of the synapses may not be identical but fixed.

In Section 2, we present the neuron and memristor models used to construct our proposed MNN model. In Section 3, we state the synchronization problem and present our main result. While in Section 4 we illustrate our results with numerical simulations and close the contribution with some final comments and remarks.

## 2. PRELIMINARIES

The HR neuron model is described by:

$$\begin{aligned} \dot{x}_1(t) &= -ax_1^3(t) + bx_1^2(t) + x_2(t) - x_3(t) + I(t) \\ \dot{x}_2(t) &= c - dx_1^2(t) - x_2(t) \\ \dot{x}_3(t) &= \epsilon\left[\omega\left(x_1(t) - x_0\right) - x_3(t)\right] \end{aligned} \tag{1}$$

where $x_1(t)$ is related to the neuron voltage, $x_2(t)$ to the recuperation variable, $x_3(t)$ to the adaptation variable, and $I(t)$ is the excitation current. With the following parameters $a = 1, b = 3, c = 1, d = 5, \omega = 4, x_0 = -1.6, I(t) = 5, \epsilon = 0.0021$ a chaotic bursting behavior as observed as shown in Figures 1(a)-1(b).

Rewriting (1) is in vector form one gets:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \tag{2}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^\top$ and $f(\cdot)$ is the vector field described by equation (1), $f : \mathbb{R}^3 \to \mathbb{R}^3$, where $f(\cdot)$ is locally Lipschitz in $\mathbb{R}^3$.

Biological neural systems can be characterized through memristors. An ideal memristor is defined in [Chua (1971)] as theoretically being a basic electronic passive two terminal device that relates electric charge to magnetic flux, such that the following relationship is found:

$$q_w(t) = g(\varphi_w(t)) \tag{3}$$

where $q_w(t) \in \mathbb{R}$ is the electric charge, and $\varphi_w(t) \in \mathbb{R}$ is the magnetic flux, $g : \mathbb{R} \to \mathbb{R}$ is its characteristic function, that satisfies the conditions: (i) $g(0) = 0$, $g(\cdot) \in C^1$; and (ii) $g(\cdot)$ is strictly monotonic increasing. The electrical representation of such device is shown in Figure 2.

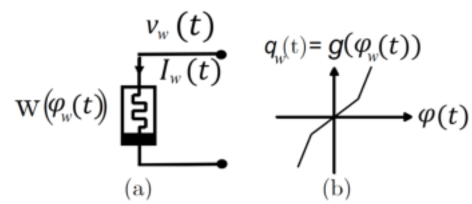


Fig. 2. (a) electric representation (b) memristive characteristic function [Itoh (2008)]

For the ideal memristor a current-voltage relation is given by:

$$i_w(t) = w(\varphi_w)v_w(t) \tag{4}$$

where $v_w(t) = \dot{\varphi}_w(t)$ and $i_w(t) = \dot{q}_w(t)$ are the voltage and current of the memristor, respectively; with its memductance given by:

$$w(\varphi_w) = \frac{dg(\varphi_w)}{d\varphi_w} \tag{5}$$

where $w(\varphi_w) > 0, \forall\varphi_w$, $w(\cdot)$ is a bounded function.

By integrating the voltage variable with respect to time, the magnetic flux $\varphi_w(t)$ is found to be:

$$\varphi_w(t) = \int_0^t v_w(\tau)d\tau + \varphi_w(0) \tag{6}$$

where $\varphi_w(t_0)$ is the initial magnetic flux. Therefore, the magnetic flux described by (6) depends on the *history* of the memristor voltage $v_w(t)$.

## 3. PROBLEM STATEMENT

Consider a MNN consisting of two identical HR neurons bidirectionally coupled by two ideal memristors, let $\mathbf{x}_1(t)$ the state of neuron 1 and $\mathbf{x}_2(t)$ the state of neuron 2. One says that the MNN achieves identical synchronization when the states of each nodes move at unison, *i.e.*,

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = s(t).$$

Where $s(t)$ is called synchronization solution of the network.

The dynamics of MNN are described by:

$$\dot{\mathbf{x}}_1(t) = f(\mathbf{x}_1(t)) + w_{21}(\varphi_{21}(t))\Gamma(\mathbf{x}_2(t) - \mathbf{x}_1(t)) \tag{7a}$$
$$\dot{\mathbf{x}}_2(t) = f(\mathbf{x}_2(t)) + w_{12}(\varphi_{12}(t))\Gamma(\mathbf{x}_1(t) - \mathbf{x}_2(t)) \tag{7b}$$
$$\dot{\varphi}_{21}(t) = v_2(t) - v_1(t) \tag{7c}$$
$$\dot{\varphi}_{12}(t) = v_1(t) - v_2(t) \tag{7d}$$

where $\mathbf{x}_1(t) = [x_{11}(t), x_{21}(t), x_{31}(t)]^\top$ is the state of neuron one, and $\mathbf{x}_2(t) = [x_{12}(t), x_{22}(t), x_{32}(t)]^\top$ is that of neuron two, $\Gamma = \mathrm{diag}(1,0,0) \in \mathbb{R}^{3\times 3}$ is the internal coupling matrix, and the voltage of neurons are $v_1(t) = \gamma\mathbf{x}_1(t), v_2(t) = \gamma\mathbf{x}_2(t)$, where $\gamma = [1,0,0]$. In this example is considered a perturbation signal in the first neuron. The memristor that connects neuron 1 with neuron 2 is $M_{12}$ with $\varphi_{12}(t)$ its magnetic flux (8a). While $M_{21}$ is the memristor that connects neuron 2 with neuron 1, with its magnetic flux $\varphi_{21}(t)$ given by (8b).

$$\varphi_{12}(t) = \int_0^t (v_1(\tau) - v_2(\tau))d\tau + \varphi_{12}(0) \tag{8a}$$
$$\varphi_{21}(t) = \int_0^t (v_2(\tau) - v_1(\tau))d\tau + \varphi_{21}(0) \tag{8b}$$

the memristive characteristic function of $M_{12}$ is:

$$\begin{aligned} g_{12}(\varphi_{12}) &= a_{12}\varphi_{12} + \frac{1}{2}(b_{12} - a_{12})(|\varphi_{12} + l_{12}|) \\ &\quad -\frac{1}{2}(b_{12} - a_{12})(|\varphi_{12} - l_{12}|) \end{aligned} \tag{9}$$

where $a_{12}$,$b_{12}, l_{12} > 0$ are constants, $b_{12} < a_{12}$, taking the derivative of (9) is obtained its memristance function:

$$w_{12}(\varphi_{12}) = \frac{dg_{12}(\varphi_{12})}{d\varphi_{12}} = \begin{cases} a_{12}\ , & \varphi_{12} < -l_{12} \\ b_{12}\ , & -l_{12} \le \varphi_{12} \le l_{12} \\ a_{12}\ , & l_{12} < \varphi_{12} \end{cases} \tag{10}$$

On the other hand, the memristive characteristic function of $M_{21}$ is:

$$\begin{aligned} g_{21}(\varphi_{21}) &= a_{21}\varphi_{21} + \frac{1}{2}(b_{21} - a_{21})(|\varphi_{21} + l_{21}|) \\ &\quad -\frac{1}{2}(b_{21} - a_{21})(|\varphi_{21} - l_{21}|) \end{aligned} \tag{11}$$

where $a_{21}$,$b_{21}, l_{21} > 0$ are constants, $b_{21} < a_{21}$, taking the derivative of (11) is obtained its memristance function:

$$w_{21}(\varphi_{21}) = \frac{dg_{21}(\varphi_{21})}{d\varphi_{21}} = \begin{cases} a_{21}\ , & \varphi_{21} < -l_{21} \\ b_{21}\ , & -l_{21} \le \varphi_{21} \le l_{21} \\ a_{21}\ , & l_{21} < \varphi_{21} \end{cases} \tag{12}$$

as consequence of identical synchronization, $\varphi_{12}(t) = \bar{\varphi}_{12}$ and $\varphi_{21}(t) = \bar{\varphi}_{21}$, where:

$$\bar{\varphi}_{12} = \lim_{t\to\infty} \int_0^t (v_1(\tau) - v_2(\tau))d\tau + \varphi_{12}(0)$$
$$\bar{\varphi}_{21} = \lim_{t\to\infty} \int_0^t (v_2(\tau) - v_1(\tau))d\tau + \varphi_{21}(0)$$

At the synchronized state $s(t)$ the coupling term in the MNN goes to zero, therefore one has that its behavior is that of an isolated node:

$$\dot{s}(t) = f(s(t)) \tag{13}$$

Lets define the synchronization error as

$$\mathbf{e}_1(t) = \mathbf{x}_1(t) - s(t) \tag{14a}$$
$$\mathbf{e}_2(t) = \mathbf{x}_2(t) - s(t) \tag{14b}$$

where $\mathbf{e}_1(t) = [e_{11}(t), e_{21}(t), e_{31}(t)] \in \mathbb{R}^3$, and $\mathbf{e}_2(t) = [e_{12}(t), e_{22}(t), e_{32}(t)] \in \mathbb{R}^3$. Therefore, identical synchronization in the MNN is equivalent to the stability of the zero solution of the synchronization error dynamics.

Notice that when identical synchronization occurs $\mathbf{e}_1(t) = \mathbf{e}_2(t) = 0$, therefore $s(t) = \mathbf{x}_1(t) = \mathbf{x}_2(t)$. The error dynamics is obtained by taking the derivative of (14a)-(14b)

$$\dot{\mathbf{e}}_1(t) = \dot{\mathbf{x}}_1(t) - \dot{s}(t) \tag{15a}$$
$$\dot{\mathbf{e}}_2(t) = \dot{\mathbf{x}}_2(t) - \dot{s}(t) \tag{15b}$$

substituting $\mathbf{x}_1(t) = \mathbf{e}_1(t) + s(t)$ in (15a), and $\mathbf{x}_2(t) = \mathbf{e}_2(t) + s(t)$ (15b) one obtains:

$$\begin{aligned} \dot{\mathbf{e}}_1(t) &= f(\mathbf{x}_1(t)) - \dot{s}(t) \\ &\quad + w_{21}(\varphi_{21}(t))\Gamma(\mathbf{e}_2(t) + s(t) - \mathbf{e}_1(t) - s(t)) \end{aligned} \tag{16a}$$
$$\begin{aligned} \dot{\mathbf{e}}_2(t) &= f(\mathbf{x}_2(t)) - \dot{s}(t) \\ &\quad + w_{12}(\varphi_{12}(t))\Gamma(\mathbf{e}_1(t) + s(t) - \mathbf{e}_2(t) - s(t)) \end{aligned} \tag{16b}$$

substituting (13) in (16a),(16b) and rearranging:

$$\begin{aligned} \dot{\mathbf{e}}_1(t) &= f(\mathbf{x}_1(t)) - f(s(t)) \\ &\quad + w_{21}(\varphi_{21}(t))\Gamma(\mathbf{e}_2(t) - \mathbf{e}_1(t)) \end{aligned} \tag{17a}$$
$$\begin{aligned} \dot{\mathbf{e}}_2(t) &= f(\mathbf{x}_2(t)) - f(s(t)) \\ &\quad + w_{12}(\varphi_{12}(t))\Gamma(\mathbf{e}_1(t) - \mathbf{e}_2(t)) \end{aligned} \tag{17b}$$

where $f(\cdot)$ is described by (2), expressing (17a) in vector form becomes:

$$\dot{\mathbf{e}}(t) = F(\mathbf{X}(t)) - F(\mathbf{S}(t)) + W(\varphi(t)) \otimes \Gamma\mathbf{e}(t) \tag{18a}$$
$$\dot{\varphi}(t) = G \otimes \gamma\mathbf{e}(t) \tag{18b}$$

where $\mathbf{e}(t) = [\mathbf{e}_1^\top(t), \mathbf{e}_2^\top(t)]^\top \in \mathbb{R}^6$, $F(\cdot) = [f^\top(\cdot), f^\top(\cdot)]^\top \in \mathbb{R}^6$, $\mathbf{X}(t) = [\mathbf{x}_1^\top(t), \mathbf{x}_2^\top(t)]^\top \in \mathbb{R}^6$, $\mathbf{S}(t) = [s^\top(t), s^\top(t)]^\top \in \mathbb{R}^6$ is the synchronous solution, $\varphi(t) = [\varphi_{12}(t), \varphi_{21}(t)]^\top \in \mathbb{R}^2$, $\Gamma = \mathrm{diag}(\gamma) \in \mathbb{R}^{3\times 3}$, $\gamma = [1,0,0]$, and $\otimes$ is the Kronecker product.

Assuming the existence of a bounded solution to magnetic flux equation (18b) on $\mathbb{R}^2$, the time dependent connection matrix is given by:

$$W(\varphi(t)) = \begin{bmatrix} -w_{21}(\varphi_{21}(t)) & w_{21}(\varphi_{21}(t)) \\ w_{12}(\varphi_{12}(t)) & -w_{12}(\varphi_{12}(t)) \end{bmatrix} \tag{19}$$

where $W(\varphi(t))$ is a continuous piecewise linear matrix

$$W(\varphi(t)) = \begin{cases} W_1(\varphi(t)), & \varphi_{12}(t) < l_{12}\ , \quad \varphi_{21}(t) < -l_{21} \\ W_2(\varphi(t)), & \varphi_{12}(t) < -l_{12}\ , \ -l_{21} \le \varphi_{21}(t) < l_{21} \\ W_3(\varphi(t)), & \varphi_{12}(t) < -l_{12}\ , \quad {}_{21} \le \varphi_{21}(t) \\ W_4(\varphi(t)), & -l_{12} \le \varphi_{12}(t) < l_{12}\ , \quad \varphi_{21}(t) < -l_{21} \\ W_5(\varphi(t)), & -l_{12} \le \varphi_{12}(t) < l_{12}\ , \ -l_{21} \le \varphi_{21}(t) < -l_{21} \\ W_6(\varphi(t)), & -l_{12} \le \varphi_{12}(t) < l_{12}\ , \quad l_{21} \le \varphi_{21}(t) \\ W_7(\varphi(t)), & l_{12} \le \varphi_{12}(t)\ , \quad \varphi_{21}(t) < -l_{21} \\ W_8(\varphi(t)), & l_{12} \le \varphi_{12}(t)\ , \ -l_{21} \le \varphi_{21}(t) < l_{21} \\ W_9(\varphi(t)), & l_{12} \le \varphi_{12}(t)\ , \quad l_{21} \le \varphi_{21}(t) \end{cases}$$

where:

$$W_1(\varphi(t)) = \begin{bmatrix} -a_{12}\ , & a_{12} \\ a_{21}\ , & -a_{21} \end{bmatrix}, W_2(\varphi(t)) = \begin{bmatrix} -b_{12}\ , & b_{12} \\ a_{21}\ , & -a_{21} \end{bmatrix},$$
$$W_3(\varphi(t)) = \begin{bmatrix} -a_{12}\ , & a_{12} \\ a_{21}\ , & -a_{21} \end{bmatrix}, W_4(\varphi(t)) = \begin{bmatrix} -a_{12}\ , & a_{12} \\ b_{21}\ , & -b_{21} \end{bmatrix},$$

$$W_5(\varphi(t)) = \begin{bmatrix} -a_{12} & , & a_{12} \\ b_{21} & , & -b_{21} \end{bmatrix}, W_6(\varphi(t)) = \begin{bmatrix} -a_{12} & , & a_{12} \\ b_{21} & , & -b_{21} \end{bmatrix},$$
$$W_7(\varphi(t)) = \begin{bmatrix} -a_{12} & , & a_{12} \\ a_{21} & , & -a_{21} \end{bmatrix}, W_8(\varphi(t)) = \begin{bmatrix} -b_{12} & , & b_{12} \\ a_{21} & , & -a_{21} \end{bmatrix},$$
$$W_9(\varphi(t)) = \begin{bmatrix} -a_{12} & , & a_{12} \\ a_{21} & , & -a_{21} \end{bmatrix}.$$

$$G = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \tag{20}$$

where $W(\varphi(t))$ is non-symmetric and the sum of its rows is zero uniformly in time, therefore it is negative semidefinite uniformly in time, furthermore its eigenvalues are $\lambda_{1,2}(W(\varphi(t)))$, where $\lambda_1(W(\varphi(t))) = 0, \forall\varphi \in \mathbb{R}^2$ and $\lambda_2(W(\varphi))$ is given by (21).

Equation (18a) is rewritten:

$$\dot{\mathbf{e}}(t) = \hat{F}(t, \mathbf{e}(t)) \tag{22}$$

where $\hat{F}(t, \mathbf{e}(t)) = F(\mathbf{X}(t)) - F(\mathbf{S}(t)) - W(\varphi(t)) \otimes \Gamma\mathbf{e}(t)$.

We aim to determine if $\mathbf{e}(t) \to 0$ exponentially in time at lest locally, which means that $\mathbf{S}(t)$ is a solution exponentially stable of (18a) and consequently $\varphi(t) \to \bar{\varphi}$ in (18b), where $\bar{\varphi} = [\bar{\varphi}_{12}, \bar{\varphi}_{21}]^\top$ is a constant value denominated the memory states of the memristive synapses.

Let $\|\cdot\|$ be the euclidean norm, with $B_r = \{\mathbf{e} \in \mathbb{R}^6 : \|\mathbf{e}\| < r\}$ the following properties of $\hat{F}(\cdot)$ are satisfied: (I) $\hat{F}(t, \mathbf{e})$ is Locally Lipschitz on $B_r$ and piecewise continuous with respect to $t$. (II) Linearizing (22) around the origin we obtain:

$$\dot{\mathbf{e}}(t) = A(\mathbf{S}(t))\mathbf{e}(t) + W(\varphi(t)) \otimes \Gamma\mathbf{e}(t) \tag{23}$$

where $A(\mathbf{S}(t))$ is black diagonal matrix:

$$A(\mathbf{S}(t)) = \begin{bmatrix} Df(s(t)) & 0 \\ 0 & Df(s(t)) \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

which is locally Lipschitz in $B_r$ uniformly in $t$ and $Df(s(t)) \in \mathbb{R}^{3\times 3}$ is the Jacobian of $f(\cdot)$.

*Theorem 1.* Assume:
(**A1**) $s(t)$ is an exponentially stable solution of single node dynamics (2), and
(**A2**) $||D(f(s(t)))|| < \alpha$, where $||\cdot||$ is a matrix induced norm and $\alpha > 0$ a positive constant.
If the memductance matrix $W(\cdot)$ is negative semidefinite uniformly in time, then the linearized error dynamics (23) are exponentially in time ( $\mathbf{e}(t) \to 0$). Furthermore, since the origin is a locally exponentially stable solution of the nonlinear system (22) identical synchronization between the neurons is achieved.

**Proof.** The system (23) is rewritten:

$$\dot{\nu}(t) = Df(s(t))\nu(t) + \Gamma\nu(t)(W\varphi(t))^\top \tag{24}$$

where $\nu(t) = [\mathbf{e}_1(t), \mathbf{e}_2(t)] \in \mathbb{R}^{3\times 2}$ for the given $W(\varphi(t))$ there exists a non singular matrix $Z(t) \in \mathbb{R}^{2\times 2}$ such that:

$$\Lambda(t) = Z^{-1}(\varphi(t))W(\varphi(t))Z(\varphi(t))$$

where $\Lambda(t) = \mathrm{diag}(\lambda_1(t), \lambda_2(t)) \in \mathbb{R}^{2\times 2}$ and

$$Z(\varphi(t)) = \begin{bmatrix} -\dfrac{w_{12}(\varphi_{12}(t))}{w_{21}(\varphi_{21}(t))} & 1 \\ 1 & 1 \end{bmatrix}$$

a change of base is considered:

$$\eta(t) = \nu(t)Z^{-1}(\varphi(t)) \tag{25}$$

taking the derivative of (25) is obtained:

$$\dot{\eta}(t) = \dot{\nu}(t)Z(\varphi(t)) + \nu(t)\dot{Z}(\varphi(t)) \tag{26}$$

substituting (23) in (26):

$$\dot{\eta}(t) = Df(s(t))\eta(t) + \Gamma\eta(t)\Lambda(t) - \eta(t)\dot{Z}^{-1}(\varphi(t))Z(\varphi(t)) \tag{27}$$

given that $Z(\varphi(t))$ is a piece-wise constant matrix $\dot{Z}(\varphi(t)) = \mathrm{diag}(0,0) \in \mathbb{R}^{2\times 2}$, then equation (27) becomes:

$$\dot{\eta}(t) = Df(s(t))\eta(t) + \Gamma\eta(t)\Lambda(t) \tag{28}$$

expanding (28) by columns and considering $\lambda_1(t) = 0\ \forall t$ is obtained:

$$\dot{\eta}_1(t) = Df(s(t))\eta_1(t) \tag{29a}$$
$$\dot{\eta}_2(t) = Df(s(t))\eta_2(t) + \lambda_2(t)\Gamma\eta_2(t) \tag{29b}$$

given that $s(t)$ is a exponentially stable solution of (2) by converse Lyapunov theorem $\eta_1(t) \to 0$ exponentially in (29a), to determine if $\eta_2(t)$ converges exponentially to the origin, we propose a Lyapunov candidate function:

$$V(\eta_2(t)) = \frac{1}{2}\eta_2^\top(t)\eta_2(t) \tag{30}$$

taking the derivative of (30) is obtained:

$$\dot{V}(\eta_2(t)) = \eta_2^\top(t)\dot{\eta}_2(t) \tag{31}$$

substituting (29b) in (31) is obtained:

$$\dot{V}(\eta_2(t)) = \eta_2^\top(t)D(f(s(t)))\eta_2(t) + \lambda_2(t)\eta_2^\top(t)\Gamma\eta_2(t) \tag{32}$$

given that $Df(s(t))$ is bounded and $\Gamma = \mathrm{diag}(1,0,0) \in \mathbb{R}^{3\times 3}$, equation (32) becomes

$$\dot{V}(\eta_2(t)) = c\|\eta_2(t)\|^2 + \lambda_2(t)\eta_{12}^2(t) \tag{33}$$

we know $\eta_{12}^2(t) < \|\eta_2(t)\|^2, \forall\eta_2(t) \neq 0$, therefore (33) becomes:

$$\begin{aligned} \dot{V}(\eta_2(t)) &< c\|\eta_2(t)\|^2 + \lambda_2(t)\|\eta_2(t)\|^2 \\ \dot{V}(\eta_2(t)) &< (c + \lambda_2(t))\|\eta_2(t)\|^2 \end{aligned} \tag{34}$$

considering $-(b_{12} + b_{21}) \leq \lambda_2(t) \leq -(a_{12} + a_{21})$, if $a_{12} + a_{21} > c$, then $\dot{V}(\eta_2(t)) < 0\ \forall\eta_2 \neq 0$. We conclude that $\eta_2(t) = 0$ is an exponential solution (29b), therefore $\mathbf{e}(t)$ converge exponentially in time to zero solution in the linearized error dynamics (23); by converse Lyapunov theorem $\mathbf{e} = 0$ is an exponentially stable equilibrium point of nonlinear dynamics (22). □

## 4. SIMULATION EXAMPLES

Consider two identical HR neurons bidirectionally coupled through memristors $M_{21}$ and $M_{12}$, the circuit of such system is presented in figure 3. As noticed this circuit is composed of two memristors $M_{21}$ and $M_{12}$ which can be implemented as in [Sanchez-Lopez (2014)], four operational amplifiers (OPAM) $U_1$-$U_4$ and two positive second generation current conveyors (CCII+) $U_4$-$U_5$ and resistors $R$. Considering the elements of this circuit are

$$\lambda_2(W(\varphi(t))) = \begin{cases} -(a_{12}+a_{21}), & \varphi_{12}(t) < -l_{12} \;, \quad \varphi_{21}(t) < -l_{21} \\ -(a_{12}+b_{21}), & \varphi_{12}(t) < -l_{12} \;, \; -l_{21} \leq \varphi_{21}(t) < \; l_{21} \\ -(a_{12}+a_{21}), & \varphi_{12}(t) < -l_{12} \;, \quad l_{21} \leq \varphi_{21}(t) \\ -(b_{12}+a_{21}), & -l_{12} \leq \varphi_{12}(t) < \; l_{12} \;, \quad \varphi_{21}(t) < -l_{21} \\ -(b_{12}+b_{21}), & -l_{12} \leq \varphi_{12}(t) < \; l_{12} \;, \; -l_{21} \leq \varphi_{21}(t) < -l_{21} \\ -(b_{12}+a_{21}), & -l_{12} \leq \varphi_{12}(t) < \; l_{12} \;, \quad l_{21} \leq \varphi_{21}(t) \\ -(a_{12}+a_{21}), & l_{12} \leq \; \varphi_{12}(t) \;, \quad \varphi_{21}(t) < -l_{21} \\ -(a_{12}+b_{21}), & l_{12} \leq \; \varphi_{12}(t) \;, \; -l_{21} \leq \varphi_{21}(t) < \; l_{21} \\ -(a_{12}+a_{21}), & l_{12} \leq \; \varphi_{12}(t) \;, \quad l_{21} \leq \varphi_{21}(t) \end{cases} \tag{21}$$

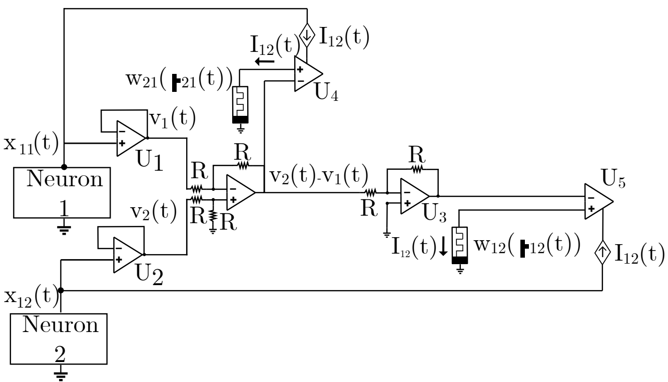


Fig. 3. Circuit implementation

in the ideal region, because the voltages of both neurons are below saturating voltage of OPAM. Its mathematical described by:

$$\dot{\mathbf{x}}_1(t) = f(\mathbf{x}_1(t)) + w_{21}(\varphi_{21}(t))\Gamma(\mathbf{x}_2(t) - \mathbf{x}_1(t)) + \zeta(t) \tag{35a}$$

$$\dot{\mathbf{x}}_2(t) = f(\mathbf{x}_2(t)) + w_{12}(\varphi_{12}(t))\Gamma(\mathbf{x}_1(t) - \mathbf{x}_2(t)) \tag{35b}$$

$$\dot{\varphi}_{21}(t) = v_2(t) - v_1(t) \tag{35c}$$

$$\dot{\varphi}_{12}(t) = v_1(t) - v_2(t) \tag{35d}$$

where $\mathbf{x}_1(t) = [x_{11}(t), x_{21}(t), x_{31}(t)]^\top$ is the state of neuron one, and $\mathbf{x}_2(t) = [x_{12}(t), x_{22}(t), x_{32}(t)]^\top$ is that of neuron two, $\Gamma = \text{diag}(1,0,0) \in \mathbb{R}^{3\times 3}$ is the internal coupling matrix, and the voltage of neurons are $v_1(t) = \gamma\mathbf{x}_1(t), v_2(t) = \gamma\mathbf{x}_2(t)$, where $\gamma = [1,0,0]$.

The memristor $M_{12}$, connects neuron 1 with neuron 2, its magnetic flux is $\varphi_{12}(t)$ described by (8a). While $M_{21}$ is the memristor that connects neuron 2 with neuron 1, its magnetic flux is $\varphi_{21}(t)$ given by (8b). Here is considered a perturbation signal $\zeta(t) = 0.3e^{-0.005t}$ in the first neuron.

The characteristic function of memristor $M_{12}$ is:

$$g_{12}(\varphi_{12}) = 0.9\varphi_{12} - 0.4(|\varphi_{12}+140| - |\varphi_{12}-140|) \tag{36}$$

where its parameters are $a_{12} = 0.9$, $b_{12} = 0.1$ and $l_{12} = 140$, the function that describes its memristance is:

$$w_{12}(\varphi_{12}) = \frac{dg_{12}(\varphi_{12})}{d\varphi_{12}} = \begin{cases} 0.9 \;, & \varphi_{12} < -140 \\ 0.1 \;, & -140 \leq \varphi_{12} \leq \;140 \\ 0.9 \;, & 140 < \varphi_{12} \end{cases} \tag{37}$$

While the characteristic function of memristor $M_{21}$ is:

$$g_{21}(\varphi_{21}) = \varphi_{21} - 0.425(|\varphi_{21}+120| - |\varphi_{21}-120|) \tag{38}$$

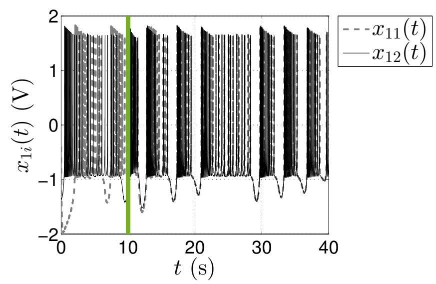


Fig. 4. (a) Neurons voltages $x_{11}(t)$ and $x_{21}(t)$

where its parameters are $a_{21} = 1$ and $b_{21} = 0.15$,$l_{21} = 120$, the function that describes its memristance:

$$w_{21}(\varphi_{21}) = \frac{dg_{21}(\varphi_{21})}{d\varphi_{21}} = \begin{cases} 1 \;, & \varphi_{21} < -120 \\ 0.15 \;, & -120 \leq \varphi_{21} \leq \;120 \\ 1 \;, & 120 < \varphi_{21} \end{cases} \tag{39}$$

the time dependent coupling matrix $W(\varphi(t))$ is piecewise constant, non symmetric and negative semidefinite uniformly in time, therefore condition of Theorem 1 is met, in this case $b_{12}$,$b_{21}$ are chosen big enough so that the conditions of Theorem 1 are satisfied.

The results of simulating numerically the model (7a)-(7d), with initial conditions $\mathbf{x}_1(0) = [-0.3945, -0.5858, 4.709]^\top$, $\varphi_{21}(0) = 10$,$\mathbf{x}_2(0) = [-1.361, -8.26, 3.11]^\top$, $\varphi_{12}(0) = 50$ and internal connection matrix $\Gamma = \text{diag}(1,0,0) \in \mathbb{R}^{3\times 3}$, are shown in Figures 4-5.

Initially the pair of neurons are uncoupled, at $t = 10$ the neurons are coupled, then after $t = 20$ the voltages $x_{11}(t)$, $x_{12}(t)$ converge towards each other in spite of perturbation signal, as shown in Figure 4 ; while the error in neurons states (see Figure 5) are basically zero. In Figure 6 is shown that magnetic flux of memristors $\varphi_{12}(t)$ and $\varphi_{21}(t)$, converge to constant values, notice that their convergence value is different, this is because initial conditions are not equal $\varphi_{12}(0) = 10$,$\varphi_{21}(0) = 50$ .

As observed in figure 7, the memristor $M_{12}$ reaches high conductance region instantly when $t = 30$, while $M_{21}$ is in low conductance region, having a big enough coupling strength, for neurons to synchronize as observed in Figure 4 and 5.

*Remark 1.* The emergence of identical synchronization is dependent on the properties of the memristor synapses as long as they have positive memductance, *i.e.* the

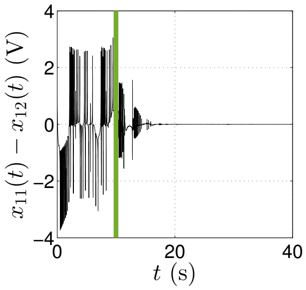


Fig. 5. Error in neurons voltages

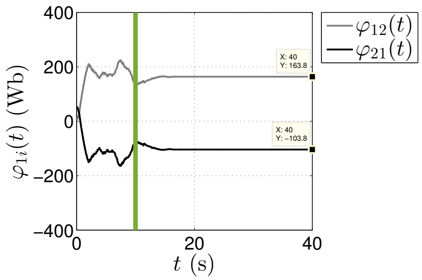


Fig. 6. Magnetic flux of memristor $M_{12}$ (gray) and memristor $M_{21}$ (black)

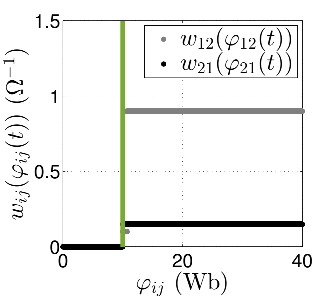


Fig. 7. Memductance

time dependent connection matrix is negative semidefinite uniformly in time, synchronization is achieved although they are not identical.

*Remark 2.* When a combination of memductances $w_{12}(\cdot)$, $w_{21}(\cdot)$ is not greater than the required coupling strength $c$ to achieve synchronization, the pair of neurons will remain unsynchronized. All code scripts are available upon reasonable request.

*Remark 3.* All code scripts are available upon reasonable request.

## 5. CONCLUSION

In this paper, synchronization in a MNN of two HR neurons bidirectionally coupled by nonidentical ideal memristors is investigated, we find sufficient conditions in memristor properties for the identical synchronization, our results show that for memristance sufficiently large and positive definite at all times, the neurons will synchronize with the magnetic flux of the memristors converge to constant values. The analysis of synchronization is based on a linearized error dynamics which restrict our results to a local neighborhood of synchronous state.

## REFERENCES

A. L. Hodgkin, A. F. Huxley, B. Katz "Ionic currents underlying activity in the giant axon of the squid," Arch. Sci. Physiologiques, 3, 129–150, 1949.

J.L. Hindmarsh, R.M. Rose. A model of neuronal bursting using 3 coupled 1st order differential-equations, Proc. R. Soc. Lond. B, 221(1222),87-102,1984.

E.R. Kandel, T.M. Jessell, J.H. Schwartz, S.A. Siegelbaum, A.J. Hudspeth Principles of Neural Science,, McGraw-Hill, 2013.

D. Serrat, B. Graham, A. Gillies, D. Willshaw Principles of Computational Modeling in Neuroscience Cambridge University Press, 2011.

A. Amirsoleimani, M. Ahmadi, A. Ahmadi, M. Boukadoum. Pattern classification with memristive neural network using the Hodgkin-Huxley neuron IEEE Int. Conf. on Electron. Circuits and Syst. ICECS, 81-84,2016.

Y. Pershin, M. Di Ventra. Experimental demonstration of associative memory with memristive neural networks Nat Prec, 2009.

Y. Nishitani, Y. Kaneko, M. Ueda. Supervised Learning Using Spike-Timing-Dependent Plasticity of Memristive Synapses IEEE Trans Neural Netw Learn Syst,26(12),2999-3008, 2015.

Y. Li, B. Luo, D. Liu, Y. Yang and Z. Yang. Robust Exponential Synchronization for Memristor Neural Networks With Nonidentical Characteristics by Pinning Control IEEE Trans. Syst. Man Cybern.: Syst.,51(3),1966-1980, 2021.

C. Sánchez-López, V.H. Carbajal-Gómez, M.A. Carrasco-Aguilar PID controller design based on memductor AEU - Int. J. Electron, 101,9-11, 2019.

I. Carro-Perez ,C. Sánchez-López, H.G. Gonzalez-Hernandez, PID controller design based on memductor AEU - Int. J. Electron, 101,9-11, 2019.

G. Innocenti, A. Morelli, R. Genesio, A. Torcini Dynamical phases of the Hindmarsh-Rose neuronal model: Studies of the transition from bursting to spiking chaos, Chaos,17(4), 2007.

L. O. Chua Memristor-The missing circuit element, IEEE Trans. Circuits Theo., 18(5), 507-519, 1971.

M. Itoh , L. O. Chua Memristor Oscillators, Int. J. Bifurc. Chaos, 18(5), 18(11),3183-3206.

D. Purves, G. Augustine, D. Fitzpatrick, W. C. Hall, A. LaMantia, R. Mooney ,L. E White. Neuroscience, Sinauer, EE. UU.,2018

H.K. Khalil. Nonlinear Systems, Prentice Hall, 2002.

C. Sanchez-Lopez, J. Mendoza-Lopez, M.A. Carrasco-Aguilar, C. Muñiz-Montero A Floating Analog Memristor Emulator Circuit, IEEE Trans. Circuits Sys. II: Express Br., 61(5), 309-313, 2014.