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# On stabilization, synchronization and dynamical analysis of a fractional-order chaotic map

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**Abstract:** This paper presents a chaotic map's fractionalization, dynamical analysis, control, and synchronization in a leader-follower configuration. The fractional-order version of the chaotic map is obtained based on the Caputo-like delta difference operator. Then, the dynamical behaviors associated with the fractional-order difference system are analyzed by employing the phase portraits, bifurcations diagrams, and largest Lyapunov exponent. Afterward, the control and synchronization are achieved by proposing a controller for the fractional-order map. Finally, the synchronization error based on the proposed control scheme is proven, and numerical simulations confirm that the control technique can quickly stabilize and synchronize the fractional-order chaotic maps.

*Keywords:* Fractional-Order; Chaotic map; Control; Synchronization; Caputo-like delta operator.

## 1. INTRODUCTION

Fractional calculus presents a history of more than three centuries. Researchers have observed that the description of some phenomena is more accurate when the fractional integrals and derivatives are considered [Jahanshahi et al. (2022)]. In recent years its application in biology, physics and engineering caused to attract attention because it shows genetic and memory characteristics [Fernández-Carreón et al. (2022)]. For instance, it has been applied to model the interaction of immune system with tumor cells and the HIV infection of CD4+ T cells [Ucar et al. (2019)]. Moreover, it was used in a glucose-insulin regulatory system to determine the population densities of insulin, glucose, and $\beta$-cells as a relation to the fractional-order [Munoz-Pacheco et al. (2020)]. Similarly, fractional calculus was introduced to capture the time-dependent behavior, hereditary properties and aging of concrete beams [Beltempo et al. (2018)]. [Zambrano-Serrano et al. (2021c)] show that the voltage in a RC circuit depends directly from the fractional-order parameter, when it is modelled in terms of a fractional Caputo derivative.

Also, there has been considerable attention to chaotic systems with noninteger-order. It has been reported that many fractional-order systems, such as Liu, Rössler, unstable dissipative and PWL, [Platas-Garza et al. (2021); Zambrano-Serrano et al. (2021b); Gilardi-Velázquez et al. (2022)] can exhibit chaotic behavior. Although chaos was analyzed in continuous-time systems, its existence and features in discrete-time systems have also been a subject of interest. It is worth mentioning that because of chaotic systems' high susceptibility to the initial conditions, parameters, and inherent randomness, the application of chaotic systems is one of the challenging frontier topics. Thus, developing a chaotic map with some complex dynamical characteristics is essential. [García-Grimaldo and Campos (2021)] propose a map with hidden dynamics, and [Jiang et al. (2016)] propose a class of three-dimensional maps with hidden chaotic dynamics.

Recently, considering the Caputo-like delta difference operator, the focus has shifted towards fractional-order chaotic maps. To date, only a few fractional maps have been proposed [Zambrano-Serrano et al. (2021a); Danca (2022)], which has motivated the research presented in this paper. It is noted that these maps have outstanding attributes over their integer counterparts. [Danca (2022)] point out that the chaotic behavior generated by the fractional logistic map depends on the discrete memory and the fractional-order. He suggests that the fractional-order difference map includes a new degree of freedom being more appropriate for secure communications and encryption applications. This added degree of freedom can also capture the hidden characteristics of real-world phenomena encountered in different areas.

This investigation enhances our knowledge of fractional-order maps. To do it, we examine the fractional-order map corresponding to a three-dimensional map with hidden dynamics, study its dynamics by considering the phase portraits, bifurcations diagrams, and Lyapunov exponent, and focus on control and synchronization employing a leader-follower configuration. Sometimes, when we talk about systems with chaotic behavior, control and synchronization are the particular interest. The control of chaos is understood by stabilizing discrete or continuous chaotic systems to a steady-state, usually zero or a periodic orbit. In contrast, synchronization is concerned with forcing a follower system to mimic the same trajectory of a master system with different initial conditions. In integer-order systems a lot of papers have paid attention to this subject and have controlled different continuous and discrete chaotic systems; being the method proposed by [Ott et al. (1990)] the first method in integer-order domain. Nonetheless, compared with the extensive works on the synchronization of various systems, studies on the control and synchronization of the fractional-order discrete-time systems are rare. Therefore, we have proposed a fast control technique for the system. Furthermore, the controller's gains are tuned based on the error of synchronization, which will make the controller agile. The stability of the proposed control scheme is confirmed. Numerical simulation of both control and synchronization confirm the proposed theory.

The structure of the paper is as follows, preliminaries are discussed in Section 2, which includes the Caputo-like delta difference operator. In Section 3, the generalized version (fractionalized version) of the difference chaotic system is proposed. Moreover, the chaotic characteristics of the fractional chaotic map are analyzed via phase plane, largest Lyapunov exponent, and bifurcation diagram. Section 4 shows the outcomes of the stabilization and synchronization of fractional chaotic maps. A summary of our conclusions closes the paper in Section 5.

## 2. PRELIMINARIES

This section provides some definitions, theorems, and remarks to be used in the paper. Herein, we will use the general $n$-th order difference. It can be written as

$$\begin{aligned}\Delta^n f(t) &= \Delta^{n-1} f(t+1) - \Delta^{n-1} f(t), \\ &= \sum_{k=0}^{n} C_n^k (-1)^k f(t+n-k),\end{aligned} \tag{1}$$

where $C_n^k$ is the binomial coefficient, $C_n^k = \frac{k!}{n!(n-k)!}$. Extending the concept to fractional-order difference, the fractional sum of order $v$ is defined as follows.

*Definition 1.* [Atici and Eloe (2009)]. If $f(\cdot)$ is a real-valued function defined on $\mathbb{N}_\phi$ and $v > 0$, then, the discrete fractional-order sum of $v$ denoted as $\Delta_\phi^{-v}$, it is defined as

$$\Delta_\phi^{-v} f(t) = \frac{1}{\Gamma(v)} \sum_{s=\phi}^{t-v} (t-\sigma(s))^{v-1} f(s), \quad t \in \mathbb{N}_{\phi+v}, \tag{2}$$

where $\mathbb{N}_\phi = \{\phi, \phi+1, \phi+2, \ldots\}$, $\phi$ is the starting point, $\sigma(s) = s+1$, is the forward shift operator, $t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}$, with $t \neq -1, -2, -3, \ldots$, is the falling function, and $\Gamma(\cdot)$ is the gamma function, denoted as $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

*Definition 2.* [Abdeljawad (2011)]. Let $v > 0$ with $v \in \mathbb{N}$. The $v$-order Caputo-like delta difference of a function $f(t)$ defined on $\mathbb{N}_\phi$ is denoted by

$$\begin{aligned}{}^C\Delta_\phi^v f(t) &= \Delta_\phi^{-(m-v)} \Delta^m f(t), \\ &= \frac{1}{\Gamma(m-v)} \sum_{s=\phi}^{(t-(m-v))} (t-\sigma(s))^{(m-v-1)} \Delta^m f(s),\end{aligned} \tag{3}$$

where $t \in \mathbb{N}_{\phi+m-v}$, with $m = \lceil v \rceil + 1$, $v$ the fractional-order, and $\phi$ the lower bound.

*Remark* 1. For $m = 1$, the $v$-th Caputo-like delta difference is defined by

$${}^C\Delta_\phi^v f(t) = \frac{1}{\Gamma(1-v)} \sum_{s=\phi}^{(t-(1-v))} (t-\sigma(s))^{(-v)} \Delta f(s), \tag{4}$$

with $t \in \mathbb{N}_{\phi+1-v}$.

*Theorem 1.* [Fulai et al. (2011)]. The delta fractional difference equation

$$\begin{aligned}{}^C\Delta_\phi^v f(t) &= f(t^+, u(t^+)), \\ \Delta^k u(\phi) &= u_k,\end{aligned} \tag{5}$$

with $m = \lceil v \rceil + 1$, $k = 0, \ldots, m-1$. Where the equivalent discrete integral equation can be expressed for $t \in \mathbb{N}_{\phi+m}$ as

$$u(t) = u_0(t) + \frac{1}{\Gamma(v)} \sum_{s=\phi+m-v}^{t-v} \left( (t-\sigma(s))^{(v-1)} \right) f(s^+, u(s^+)), \tag{6}$$

the term $u_0(t) = \sum_{k=0}^{m-1} \frac{(t-\phi)^{(k)}}{k!} \Delta^k u(\phi)$ corresponds to initial iteration, $t^+ = t + v - 1$, and $s^+ = s + v - 1$.

*Remark* 2. If we consider that the starting point is $\phi = 0$, and set $0 < v \leq 1$, then (9) changes to

$$u(t) = u_0(t) + \frac{1}{\Gamma(v)} \sum_{s=1-v}^{t-v} \left( (t-\sigma(s))^{(v-1)} \right) f(s^+, u(s^+)), \tag{7}$$

where $(s+v) \in \mathbb{N}$, let $s + v = j$ and employing the expansion $(t-\sigma(s))^{(v-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s-v+1)}$ as a result, the numerical formula with global memory effect can be presented explicitly as

 

$$u(t) = u_0(t) + \frac{1}{\Gamma(v)} \sum_{j=1}^{t} \frac{\Gamma(t-j+v)}{\Gamma(t-j+1)} f(j-1, u(j-1)). \quad (8)$$

*Lemma 1.* [Cermák et al. (2015)] Let the following fractional-order map:

$$\begin{aligned} {}^C\Delta_\phi^v f(t) &= A f(t^+), \\ f(0) &= f_0. \end{aligned} \quad (9)$$

The zero equilibrium of the system is asymptotically stable, if

$$\lambda \in \left\{ z \in \mathbb{Z} : |z| < \left( 2\cos\frac{|\arg(z)| - \pi}{2-v} \right)^v \text{ and } \quad |\arg(z)| > \frac{v\pi}{2} \right\}, \quad (10)$$

where $\lambda$ indicate the eigenvalues of the matrix $A$, $t^+ = t + v - 1$, $0 < v \leq 1$, $f(t) = (f_1(t), \ldots, x_n(t))$ and $t \in \mathbb{N}_{\phi+1-v}$. For more details on the fractional difference operators reference please refer to [Abdeljawad (2011)] and the references therein may consult.

## 3. FRACTIONAL-ORDER CHAOTIC MAP

In this section, by employing the Caputo-like delta difference operator given in (4), the generalized version (fractionalized version) of the difference chaotic system is proposed. Afterward, the chaotic behavior from the fractional map is analyzed because of the state portraits, bifurcation diagrams, and Lyapunov exponent.

### *3.1 Fractional three-dimensional map*

The three-dimensional map with hidden dynamics was proposed by [Jiang et al. (2016)] and defined as

$$\begin{aligned} x_{n+1} &= y_n, \\ y_{n+1} &= z_n, \\ z_{n+1} &= x_n - c z_n^2 + b x_n y_n + 1, \end{aligned} \quad (11)$$

being $c$ and $b$ positive parameters.

Employing the integer-order difference equation (1) in the map (11) and the Caputo difference operator of Definition 2, we obtain the fractional-order map as follows

$$\begin{aligned} {}^C\Delta_\phi^v x(t) &= y(t^+) - x(t^+), \\ {}^C\Delta_\phi^v y(t) &= z(t^+) - y(t^+), \\ {}^C\Delta_\phi^v z(t) &= x(t^+) - z(t^+)(cz(t^+) + 1) + bx(t^+)y(t^+) + 1, \end{aligned} \quad (12)$$

where $0 < v \leq 1$, is the fractional-order, $t \in \mathbb{N}_{\phi+1-v}$, with $\phi$ defining the starting point, the initial conditions are $\Delta^k x(\phi) = x_k$, $\Delta^k y(\phi) = y_k$, $\Delta^k z(\phi) = z_k$. By considering the starting point $\phi = 0$, and the Theorem 1, the numerical iterative process for (12) can be expressed as follows

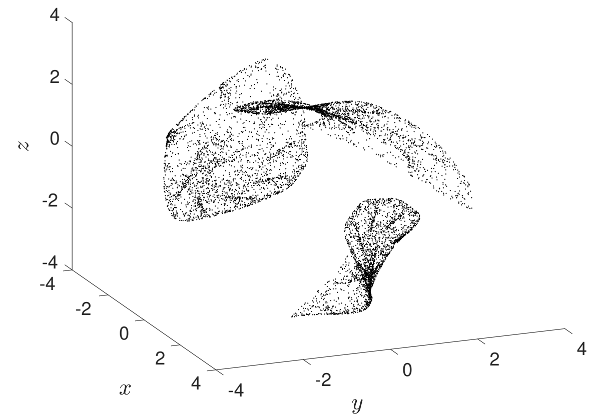


Fig. 1. Phase portrait of system (12), with $x(0) = 0.0199$, $y(0) = 0.0001$, $z(0) = -0.3298$, $v = 0.987$, $c = 0.2$, and $b = 0.33$, respectively.

$$\begin{aligned} x(n) =& x(0) + \\ & \frac{1}{\Gamma(v)} \sum_{j=1}^{n} \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} \left( y(j-1) - x(j-1) \right), \\ y(n) =& y(0) + \\ & \frac{1}{\Gamma(v)} \sum_{j=1}^{n} \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} \left( z(j-1) - y(j-1) \right), \\ z(n) =& z(0) + \\ & \frac{1}{\Gamma(v)} \sum_{j=1}^{n} \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} (x(j-1) - \\ & z(j-1)(az(j-1)+1) + \\ & bx(j-1)y(j-1)+1). \end{aligned} \quad (13)$$

When the order of (13) is $v = 1$, it is reduces to the classical integer-order difference system given in (11) respectively.

### *3.2 Phase portrait and bifurcation diagram*

Figure 1, shows the phase portrait of fractionalized system (12), considering the numerical formula expressed in (13). By choosing the initial conditions as $x(0) = 0.0199$, $y(0) = 0.0001$, $z(0) = -0.3298$, a fractional-order $v = 0.987$, the parameters $c = 0.2$, and $b = 0.33$ respectively. The phase plane was performed considering $n = 8000$ iterations and discarding the first 100 values. The largest Lyapunov exponent of (12), taking into account the previous considerations is $\lambda = 0.25$. The Lyapunov exponent describes an average rate of convergence or divergence of adjacent orbits on an attractor; it characterizes the stability of the system. As is well known, there are some algorithms to compute the Lyapunov exponent in integer-order systems, however, there are few related results reported fractional-order discrete maps. To compute the

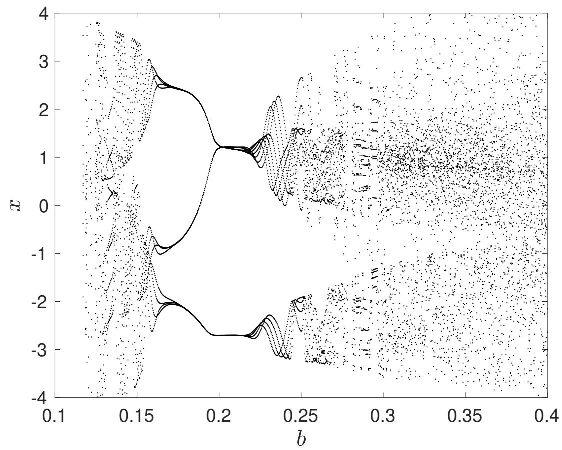

Fig. 2. Bifurcation diagram of parameter $b$, for a fractional-order $v = 0.987$.

Lyapunov exponent we consider the algorithm given in [Wu and Baleanu (2015)]. It is called the Jacobian matrix algorithm which employs a tangent map with memory effect, for more detailed information see [Wu and Baleanu (2015); Zambrano-Serrano et al. (2021a)].

The bifurcation diagram of the system (12) is shown in Fig. 2, where $b$ performs as a critical parameter by selecting a step-size $\Delta b = 0.0003$. The bifurcation diagram was obtained considering $v = 0.987$, and changing the parameter in an interval $b \in [0.1, 0.4]$. From Fig. 2, is observed that there are two principal regions where chaotic behavior occurs, given as $b \in [0.1171, 0.15] \cup [0.3, 0.4]$.

## 4. STABILIZATION AND SYNCHRONIZATION

The stabilization (control) and synchronization are fascinating topics in dynamical systems, including integer and fractional-order, whether in discrete or continuous systems.

### *4.1 Stabilization of fractional-order chaotic map*

When the state of a system achieves asymptotic stability in the presence of a suitable controller means stabilization. The following control law is considered to achieve the stable state in the fractional-order map (12) defined in subsection 3.1.

$$\begin{aligned} {}^C\Delta_\phi^v x(t) &= y(t^+) - x(t^+) + f_1(t^+), \\ {}^C\Delta_\phi^v y(t) &= z(t^+) - y(t^+) + f_2(t^+), \\ {}^C\Delta_\phi^v z(t) &= x(t^+) - z(t^+)(cz(t^+) + 1) + \\ &\quad bx(t^+)y(t^+) + 1 + f_3(t^+), \end{aligned} \tag{14}$$

where $0 < v \leq 1$ is the fractional-order, $t^+ = t+v-1$, with $f_1(t^+) = a_1x(t^+) + a_2y(t^+)$, $f_2(t^+) = a_3y(t^+) + a_4z(t^+)$, and $f_3(t^+) = a_5x(t^+)+cz^2(t^+)-bx(t^+)y(t^+)-1+a_6z(t^+)$. By considering the Lemma 1, the coefficients of matrix $A \in \mathbb{R}^{3\times 3}$ of (9) are as follows

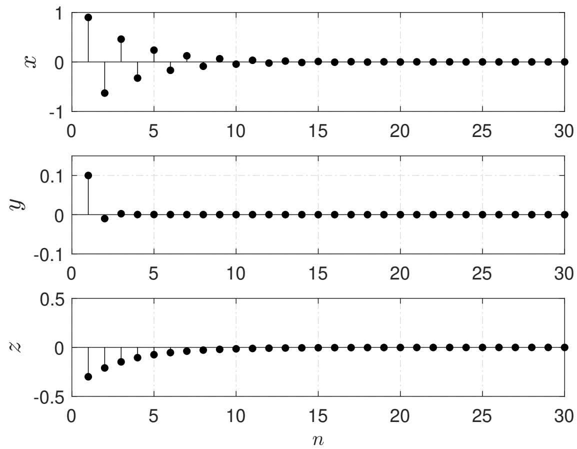

Fig. 3. Stabilization of the variables $x, y, z$ of (14) with a fractional-order $v = 0.987$.

$$A = \begin{bmatrix} (a_1 - 1) & (a_2 + 1) & 0 \\ 0 & (a_3 - 1) & (a_4 + 1) \\ (a_5 + 1) & 0 & (a_6 - 1) \end{bmatrix}, \tag{15}$$

the initial conditions and parameters are $x(0) = 0.9$, $y(0) = 0.1$, $z(0) = -0.3$, $v = 0.987$, $c = 0.2$, $b = 0.33$, respectively, and setting $a_1 = -0.7$, $a_3 = -0.1$, $a_6 = 0.7$ and $a_2 = a_4 = a_5 = -1$ for the controller. Then, the state of the system (14) its a asymptotically stable which is observed in Fig. 3. Moreover the eigenvalues of (15) are $\lambda_1 = -1.7$, $\lambda_2 = -1.1$, and $\lambda_3 = -0.3$ which fulfills the conditions of Lemma 1. Then the system is asymptotically stable.

### *4.2 Synchronization of fractional-order chaotic maps*

In this section, two fractional-order chaotic maps proposed as in subsection 3.1 are considered as drive and response systems, respectively.

The fractional-order difference chaotic system could be written as follows

$${}^C\Delta_\phi^v x(t) = Ax(t^+) + Bx(t^+), \quad x(0) = x_0; \tag{16}$$

where $A$ is a linear matrix, $Bx(t^+)$ are the nonlinear terms, and the response system is expressed as

$${}^C\Delta_\phi^v x_1(t) = Ax_1(t^+) + Bx_1(t^+) + f(t^+), \quad x_1(0) = x_{10}; \tag{17}$$

with $v \in (0, 1]$. Considering $e_1(t) = x_1(t) - x(t)$, the error system is obtained as

$${}^C\Delta_\phi^v e(t) = Ae(t^+)+Bx_1(t^+)-Bx_1(t^+)+f(t^+), \; e(0) = e_0; \tag{18}$$

The nonlinear term $f(t)$ is given as

$$f(t) = Bx_1(t) - Bx_1(t) - Ke(t), \tag{19}$$

then,

$${}^C\Delta_\phi^v e(t) = (A - KI)e(t^+), \quad e(0) = e_0; \tag{20}$$

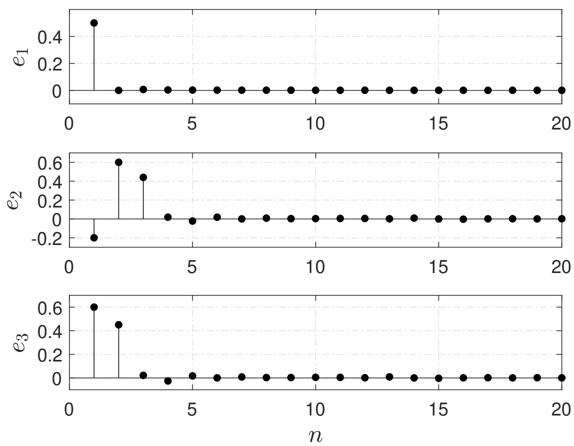


Fig. 4. Error of synchronization of (12) and (22) systems both with a fractional-order $v = 0.987$.

*Definition 3.* [Liu and Ma (2020)]. Let $x(t)$ and $x_1(t)$ the solutions of drive system (16) and response system (17). For all $\epsilon > 0$, there exists $t_0 \in \mathbb{N}_\phi$ such that

$$||x(t) - x_1(t)|| < \epsilon, \tag{21}$$

for $t > t_0$. Then the systems (16) and (17) are synchronized.

Considering the drive system such as (12) and a response system as follows

$$\begin{aligned} {}^C\Delta^v_\phi x_1(t) &= y_1(t^+) - x_1(t^+) + f_1(t^+), \\ {}^C\Delta^v_\phi y_1(t) &= z_1(t^+) - y_1(t^+) + f_2(t^+), \\ {}^C\Delta^v_\phi z_1(t) &= x_1(t^+) - z_1(t^+)(cz_1(t^+) + 1) + \\ &\quad bx_1(t^+)y_1(t^+) + 1 + f_3(t^+), \end{aligned} \tag{22}$$

with $v \in (0, 1]$. Considering $e_1(t) = x_1(t) - x(t)$, $e_2(t) = y_1(t) - y(t)$, $e_3(t) = z_1(t) - z(t)$, $f_1(t^+) = -e_2(t^+)$, $f_2(t^+) = 0$ and $f_3(t^+) = cz_1^2(t^+) - cz^2(t^+) - bx_1(t^+)y_1(t^+) + bx(t^+)y(t^+)$. Then the error system is obtained as

$$\begin{aligned} {}^C\Delta^v_\phi e_1(t) &= -e_1(t^+), \\ {}^C\Delta^v_\phi e_2(t) &= -e_2(t^+) + e_3(t^+), \\ {}^C\Delta^v_\phi e_3(t) &= e_1(t^+) - e_3(t^+), \end{aligned} \tag{23}$$

By considering the Lemma 1, the coefficients of matrix $E \in \mathbb{R}^{3\times3}$ of the error system are as follows

$$E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \tag{24}$$

The initial conditions parameters are $x(0) = 0.5$, $y(0) = 0.1$, $z(0) = -0.3$, $v = 0.987$, $c = 0.2$, $b = 0.33$, $x_1(0) = 1$, $y_1(0) = -0.1$, $z_1(0) = 0.3$, for drive (12) and response (22) systems, respectively. Then, the chaotic synchronization is achieved and the corresponding errors are displayed in Fig. 4. Moreover the eigenvalues of (24) are $\lambda_1 = -1$, $\lambda_2 = -1$, and $\lambda_3 = -1$ which fulfills the conditions of Lemma 1. Then the systems are synchronized.

## 5. CONCLUSIONS

In this paper, some basic concepts related to fractional difference operators were presented. The generalization of a three-dimensional map with hidden dynamics considering the Caputo type fractional difference operator was proposed. Chaotic behavior associated with the fractional-order map was analyzed by considering the phase planes, bifurcation diagram, and Lyapunov exponent performed by the Jacobian matrix algorithm. Besides, with the stability theory for linear fractional-order difference equations, the stabilization and chaos synchronization between a drive-response configuration for fractional-order difference chaotic system are derived by appropriate controllers. As a work in this field, the performance of the proposed controller can be enhanced by applying adaptive control techniques. Also, the control of multi-stability in the proposed map could be fruitful for future studies.

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