Memorias del 2023 Congreso Nacional de Control Automático
25-27 de Octubre, 2023. Acapulco, Guerrero, México.

Harvesting optimal policy in three species food chain model as an optimal control problem with path constraints ⋆

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Abstract: In this paper, we address the issue of harvesting prey and intermediate predators in a tritrophic food chain. Our approach is based on a model that characterizes the interactions among the three species. We assume that the intermediate predator has alternative food sources (it is a generalist), while the top predator relies solely on the intermediate predator (it is a specialist). This model has been previously explored in the literature, but representing the harvesting effort as a scalar control variable. In this study, we treat it as a vector variable, offering a more comprehensive representation, particularly relevant for terrestrial species hunting. Our primary objective is to determine optimal harvesting policies that ensure the persistence of all three species. To achieve this, we formulate the problem as an optimal control problem with a finite horizon and path constraints. We present a numerical example solved using ICLOCS2.

Keywords: Optimal control, state constraints, harvesting policies, persistence.

1. INTRODUCTION

This study is based on the model proposed in Dawed and Kebedow (2021), where a tritrophic food chain is analyzed. The chain consists of a prey, an intermediate predator and a top predator, the latter depending solely on the intermediate predator while the intermediate predator has alternative food sources and, as well as the prey, it is harvested.

There are many works in the literature exploring harvest-prey-predator models considering different type of interactions between the species as in Rojas-Palma and González-Olivares (2012) and Chen et al. (2013). In the context of tritrophic chains, some examples can be found in Upadhyay and Raw (2011); Chen et al. (2013); Panja and Mondal (2015); Blé et al. (2018); Dawed et al. (2020), to mention just a few. In these papers, the dynamics describing the system and conditions for subsistence of the three species are analyzed. In other works, optimal control techniques are used to determine optimal harvesting policies, usually considering infinite horizon and rate discount, as in Fleming and Rishel (1975), Mortoja et al. (2020) and Dawed and Kebedow (2021).

In the last reference, the authors consider the harvesting effort as a scalar control variable acting on the dynamics of both, prey and intermediate predator. In a first step, considering this control variable as a constant parameter and transforming the system into one with dimensionless variables, they determine conditions for the existence and stability of certain equilibrium points for each Holling functional response. Of particular interest are the equilibrium points where persistence occurs, i.e., where the three species can coexist. In a second stage, they analyze the maximum sustainable yield, the bionomic equilibrium and solve an optimal control problem with infinite horizon and discount rate to determine the optimal harvest policy.

The previous model is particularly useful for situations when the harvesting of a species affects the other, as usually occurs in fishery (commonly known as non-selective fishery). However, it does not consider different efforts for harvesting each species. In this work, we deal with a situation in which the interaction between the species is like in the previous model, but we consider a vector control variable since it allows us to deal with the efforts to harvest the prey and the intermediate predator separately, as is the case in some terrestrial hunting practices.

We pose the problem of finding an optimal harvesting policy as a finite horizon optimal control problem with path constraints. The choice of a finite horizon is rooted in the need for more effective planning, as it allows for adaptation to potential variations in parameters caused by unforeseen events. The inclusion of path constraints serves the purpose of ensuring that the population of each species remains above a defined threshold, safeguarding

⋆ This work was supported by UAM-PEAPDI 2023: "Políticas de cosecha óptimas en un modelo de cadena alimentaria de tres especies con respuesta funcional Holling-II".

them from the risk of extinction. To the best of our knowledge, this is a new approach.

The paper is organized as follows: in the next section, we introduce the dynamics describing the interaction between the three species, as well as a simplified version that considers dimensionless variables. In section 3, we state the optimal control problem of interest and some well-known necessary optimality conditions taken from Vinter (2000). In section 4, we solve numerically an illustrative example using ICLOCS2, the Imperial College of London optimal control solver, and use the necessary conditions of the previous section to partially validate our results. We finish with some conclusions and future work.

2. THE MODEL

The dynamics of the tritrophic chain under study is described with the next model which is a slight modification of the one proposed in Dawed and Kebedow (2021)

$$\dot{X} = r_x X(1 - X/K_x) - \phi_x(X)Y - \nu_x E_x X \tag{1}$$

$$\dot{Y} = r_y Y(1 - Y/K_y) + c_{xy}\phi_x(X)Y - \phi_y(Y)Z - \nu_y E_y Y \tag{2}$$

$$\dot{Z} = -r_z Z + c_{yz}\phi_y(Y)Z, \tag{3}$$

The variables and parameters appearing in the model are:

- X, Y and Z represent, respectively, the density of prey, intermediate predator and top predator in the ecosystem. They are usually measured in g/m^2 or, sometimes in individuals/m^2.
- K_x and K_y are, respectively, prey and intermediate predator carrying capacity, i.e., the maximum population density that the ecosystem can sustainably support over an extended period of time.
- r_x (resp. r_y) represents the annual growth rate of the prey (resp.intermediate predator) while r_z is the annual reduction rate of the top predator due to other factors.
- $\phi_x(X)Y$ (resp. $\phi_y(Y)Z$) is the Holling response of the intermediate predator (resp. top predator) to the prey (resp. intermediate predator). In predator-prey interactions, the Holling response is a fundamental aspect of the dynamics. It refers to how the predator's consumption rate varies in response to changes in prey density or availability.

  There are various functional forms of Holling response, encompassing the Holling Type I, II, III, and IV functional responses, each describing a distinct pattern of predator-prey interaction. A detailed description of each type can be found, for example, in Dawed et al. (2020).
- c_{xy} (resp. c_{yz}) is the conversion proportion of prey (resp. intermediate predator) biomass into intermediate predator (resp. top predator) biomass, c_{xy}, $c_{yz} \in (0, 1)$.

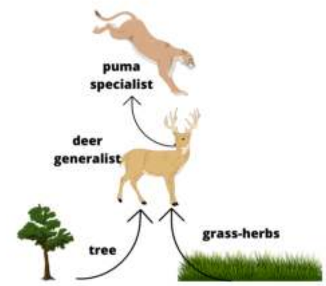

Fig. 1. Diagram created using Canva.com. The trophic chain shown in the diagram illustrates the relationship described by the model presented here. Trees (Prey) provide the leaves that deer consume. Deer (Intermediate Predators) feed on the leaves of trees as their primary food source but can adapt their diet to the resources available, also consuming grass and other herbs. Pumas (Top Predators) primarily prey on deer for their food. Additionally, both trees and deer are subject to exploitation by human activity.

- The control variables representing the harvesting efforts for the prey and the intermediate predator are denoted by E_x and E_y, respectively.
- ν_x and ν_y stand for the harvesting success coefficients (known as catchability coefficients in fishery) of the prey and intermediate predator, respectively.

2.1 Equilibrium points

As in the original formulation (with scalar control), it is easy to see that the system has the following equilibria:

- $E_0 = (0, 0, 0)$. The eigenvalues of its linealized Jacobian matrix are $\lambda_1 = r_x - \nu_x E_x$, $\lambda_2 = r_y - \nu_y E_y$ and $\lambda_3 = -r_z$, i.e., a simultaneous extinction could occur if $\nu_x E_x \geq r_x$ and $\nu_y E_y \geq r_y$, which means that the prey and the intermediate predator are harvested more than their natural growth rate.
- $E_p = (X^*, 0, 0)$. This point indicates that the prey can survive alone. It is straightforward to see that $X^* = \frac{K_x(r_x - \nu_x E_x)}{r_x}$ which has physical meaning only if $\nu_x E_x < r_x$ and is asymptotically stable if $\nu_y E_y > r_y + c_{xy}\phi_x(X^*)$. That indicates the system could be driven to this point if the prey is harvested less than its growth rate but the intermediate predator is harvested more than its growth rate, leading it and the top predator to extinction.
- $E_i = (0, Y^*, 0)$. This point indicates that the intermediate predator can survive alone. It is easy to verify that $Y* = \frac{K_y(r_y - \nu_y E_y)}{r_y}$ which has physical meaning only if $\nu_y E_y < r_y$ and is asymptotically stable if $r_x < \nu_x E_x + \phi'_x(0)Y^*$ and $c_{yz}\phi_y(Y^*) < r_z$. To clarify, this situation can occur if the prey's growth rate is not sufficient to offset the harvesting and the attacks of the intermediate predator and,

on the other hand, the top predator cannot obtain enough resources.

Conditions for the existence and stability of equilibria of the form $E_{pi} = (\hat{X}, \hat{Y}, 0)$, $E_{it} = (0, \bar{Y}, \bar{Z})$ and $E_{pit} = (X^*, Y^*, Z^*)$ obtained in Dawed and Kebedow (2021) for each Holling functional type can be easily translated into this formulation. However, their interpretation is not as clear as for the previous three, although, we can say that a necessary condition for the existence of a persistence equilibrium is for r_z/c_{yz} to be less than the attack rate of the top predator, otherwise this would be driven to extinction regardless of how much the efforts are moderated.

2.2 Profit and costs

We consider the cost of harvesting every species to be proportional to the harvesting effort, which means the total cost can be expressed as:

$$C_T = C_x E_x + C_y E_y, \tag{4}$$

where C_x, C_y denote the harvesting cost per unit effort for population X and Y respectively.

We also assume that the income from the harvesting of each population is proportional to the harvesting yield, then, the total income can be written as

$$I_T = P_x \nu_x E_x X + P_y \nu_y E_y Y, \tag{5}$$

where P_x and P_y represent the unit biomass prices for populations X and Y, respectively, and $\nu_x E_x$, $\nu_y E_y$ represent the harvesting yield functions. The profit function is then given by

$$P = I_T - C_T = (P_x \nu_x X - C_x)E_x + (P_y \nu_y Y - C_y)E_y. \tag{6}$$

2.3 Normalized model

For the sake of simplicity and comparison, we continue our analysis with a simplified version of the previous model. This simplification is achieved by considering the dimensionless variables

$$\tau = r_x t,\ x = X/K_X,\ y = Y/K_Y \text{ and } z = r_Y Z/(r_X K_Y),$$

(see Dawed et al. (2020) and Dawed and Kebedow (2021)) here, τ represents a rescaling of time, considering the prey's growth rate, x and y represent, respectively, the proportion of the prey's carrying capacity and the proportion of the intermediate predator's carrying capacity that are occupied. Providing an interpretation for z is somewhat more complex; however, it is evident that there is a direct proportionality between z and Z and, actually, $z = 0$ would imply the extinction of the top predator.

Now, set the parameters

$$\kappa = \frac{K_y}{c_{xy} K_x},\ \epsilon = \frac{r_y}{r_x},\ \beta = c_{yz}\epsilon,\ \gamma = \frac{r_z}{r_x},$$

the "normalized" Holling functions

$$\varphi_x(x) = c_{xy}\phi_x(X)/r_x \text{ and } \varphi_y(y) = \phi_y(Y)/Y$$

and the control variables

$$u_x = \nu_x E_x/r_x,\ u_y = \nu_y E_y/r_x.$$

Then, the scaled version of system (1)-(3) is:

$$\frac{dx}{d\tau} = x(1-x) - \kappa\varphi_x(x)y - u_x x \tag{7}$$

$$\frac{dy}{d\tau} = \epsilon y(1-y) + \varphi_x(x)y - \varphi_y(y)z - u_y y \tag{8}$$

$$\frac{dz}{d\tau} = -\gamma z + \beta\varphi_y(y)z \tag{9}$$

With this reformulation, we can work with inherently continuous variables and also reduce the number of parameters simplifying the analysis. However, it is important to emphasize that interpreting both the variables and parameters in the new model is not as straightforward as in the original. It is also worth noting that, under this framework, one unit of time τ corresponds to a period of $1/r_x$ years, for example, if $r_x = 0.4$ we have $\tau = 2.5$ years whereas for smaller growth rates, like $r_x = 0.05$, $\tau = 20$ years. The ideal values for x and y are 1 since it would mean there are as much prey and intermediate predator density as the ecosystem can support.

In the next section we pose an optimal control problem in terms of this dimensionless model to determine an optimal harvesting policy that allows the persistence of all species.

3. OPTIMAL CONTROL

3.1 Statement of the problem

Most papers that use optimal control techniques to determine optimal harvesting policies consider a cost functional of the form:

$$\int_0^\infty e^{\delta t}\left[(p_x x - \theta_x)u_x + (p_y y - \theta_y)u_y\right]dt,$$

where δ is the discount rate of the net revenue, $p_x = P_x r_x K_x$, $p_y = P_y r_y K_x$ and $\theta_x = \frac{r_x C_x}{\nu_x}$ $\theta_y = \frac{r_x C_y}{\nu_y}$ (see, e.g. Rojas-Palma and González-Olivares (2012); Tchepmo-Djomegni et al. (2019); Dawed and Kebedow (2021).

Here, we propose a finite horizon problem since we are interested in finding short-term policies that allow periodic replanning that considers changes in the parameters. We express the persistence condition as path constraints. Then, the optimal control problem at hand is:

$$(P) = \begin{cases} \text{Max} & \int_0^T \left[(p_x x - \theta_x)u_x + (p_y y - \theta_y)u_y\right]dt \\ \text{s.t.} & (7)-(9) \\ & (u_x(t), u_y(t)) \in U\ \ [0,T] - a.e. \\ & (x(0), y(0), z(0)) = (x_0, y_0, z_0), \\ & (x(t), y(t), z(t)) \geq (x_m, y_m, z_m),\ \forall t, \end{cases}$$

where $U = [0, u_{xM}] \times [0, u_{yM}]$ and the last inequality is componentwise. Note that u_{xM} represents the maximum harvesting effort E_x that can be made, multiplied by ν_x/r_x. Similarly, u_{yM} corresponds to the maximum possible value of E_y multiplied by ν_y/r_x.

We solve an illustrative example using the Imperial College of London Optimal Control Software (ICLOCS2) to interface the interior point optimizer (IPOPT). We consider parameters similar to those in Dawed and Kebedow (2021) and Holling functional responses of type II. To partially validate our results, we use necessary optimality conditions derived in Vinter (2000).

To do so, first, observe that the problem can be reformulated in Mayer form as:

$$(P_M) = \begin{cases} \text{Min} & -w(T) \\ \text{s.t.} & (7)-(9) \\ & \dot{w} = P(s,u), \\ & u(t) \in U, \quad [0,T]-a.e. \\ & s(0) = s_0, \\ & g_i(s) \le 0,\ i=1,2,3,\ \forall t \in [0,T]. \end{cases}$$

where $s = (x, y, z, w)$, $u = (u_x, u_y)$ denote the state and the control, respectively,

$$P(s,u) = (p_x x - \theta_x)u_x + (p_y y - \theta_y)u_y, \qquad (10)$$

$U = [0, u_{xM}] \times [0, u_{yM}]$ is the control set, the initial conditions are $s(0) = (x_0, y_0, z_0, 0)$ and the path constraints are given by

$$g_1(s) = x_m - x(t) \le 0, \quad g_2(s) = y_m - y(t) \le 0, \quad g_3(s) = z_m - z(t) \le 0$$

For a state function $s : [0,T] \to \mathbb{R}^4$, we say that the constraint g_i is active at $t_0 \in [0,T]$ if $g_i(s(t_0)) = 0$.

Remark 3.1.1. *If the third constraint is active in a nontrivial interval $[t_0, t_1] \subset [0,T]$, called boundary interval, this would imply not only $\dot{z} = 0$ but also $\dot{y} = 0$, i.e., $y \equiv y_*$, as a consequence, if u_y is also constant in that interval, then $x \equiv x_*$, which means that (x_*, y_*, z_m) are the coordinates of a persistence equilibrium point e_{pit} (corresponding to E_{pit}) in which the system will remain throughout the interval.*

Another implication of the previous observation is that if $y_m < y_*$, then the second and third constraints cannot be simultaneously active within a nontrivial interval, as having $y \equiv y_m$ would result in $\dot{z} < 0$.

3.2 Necessary Conditions

To state the theorem mentioned in this section, we employ the following standard notation: $C(I, \mathbb{R}^n)$ and $W^{1,1}(I, \mathbb{R}^n)$ denote sets of continuous and absolutely continuous functions, respectively, defined on I and taking values in $\mathbb{R}^n$. Additionally, $C^{\oplus}(I, \mathbb{R}^n)$ represents the elements of the dual space $C^*(I, \mathbb{R}^n)$ that assign nonnegative values to nonnegative-valued functions, while $supp\{\mu\}$ signifies the support of a measure μ.

The Hamiltonian function for this problem is given by:

$$\begin{aligned} H(s,q,u) = &\, q_x(x(1-x) - \kappa\varphi_x(x)y - u_x x) \\ &+ q_y(\epsilon y(1-y) + \varphi_x(x)y - \varphi_y(y)z - u_y y) \\ &+ q_z(-\gamma z + \beta\varphi_y(y)z) \\ &+ q_w(p_x x - \theta_x)u_x + (p_y y - \theta_y)u_y \end{aligned}$$

By Vinter (2000)[Theorem 9.5.1], if $(\bar{s}, \bar{u})$ is an optimal solution for (P_M) then there exists $p \in W^{1,1}\left([0,T], \mathbb{R}^4\right)$, $\lambda \ge 0$, $\mu_i \in C^{\oplus}(0,T)$, $(i = 1,2,3)$ satisfying:

i) Nontriviality condition: $(p, \{\mu_i\}_{i=1}^3, \lambda) \neq 0$
ii) Costate equation: $-\dot{p} = H_s(\bar{s}(t), q(t), \bar{u}(t))$
iii) Transversality condition: $q(1) = (0,0,0,\lambda)$
iv) Maximum condition:
$$H(\bar{s}(t), q(t), \bar{u}(t)) = \max_{u \in U} H(\bar{s}(t), q(t), u)$$
v) For $i = 1,2,3$, we have $supp\{\mu_i\} \subset I_i(\bar{s})$
vi) There exists a constant r such that
$$H(\bar{s}(t), q(t), \bar{u}(t)) = r - \text{a.e.}$$

where

$$q(t) := \begin{cases} p(t) - \displaystyle\int_{[0,t)} \sum_{i=1}^{3} e_i \mu_i(d\zeta) & \text{if } t \in [0,T) \\ p(1) - \displaystyle\int_{[0,T]} \sum_{i=1}^{3} e_i \mu_i(d\zeta) & \text{if } t = T \end{cases}$$

e_i is the i-th element of the canonical base of $\mathbb{R}^4$ and

$$I_i(\bar{s}) = \{t \in [0,T] \mid g_i(\bar{s}(t)) = 0\}$$

Since the controls appear linearly in the Hamiltonian, by condition iv) we know that the optimal control will be either, bang-bang or singular. Then, by conditions ii) and iii), we have that $q_w \equiv \lambda$, therefore, the switching functions are given by

$$Q_1(\bar{s}(t), q(t), \bar{u}(t)) = x(\lambda p_x - q_x) - \lambda\theta_x$$
$$Q_2(\bar{s}(t), q(t), \bar{u}(t)) = y(\lambda p_y - q_y) - \lambda\theta_y$$

Regarding normality, (i.e. $\lambda = 1$), an essential condition to avoid loosing information from the cost functional, it becomes apparent that if none of the state constraints are active, the previous conditions hold in normal form.

4. NUMERICAL SOLUTION

To illustrate this approach we are going to consider a Holling functional response of type II which is, as far as we know, the most studied type in the literature. This Holling response is characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited by its capacity to process food.

We use the following parameters: $\kappa = 0.15$, $\epsilon = 0.5$, $\beta = 0.3$, $\gamma = 0.04$. This choice was made because in Dawed and Kebedow (2021), these values were also selected for the case where both Holling responses were type II. It is worth recalling that that paper discussed non-selective harvesting.

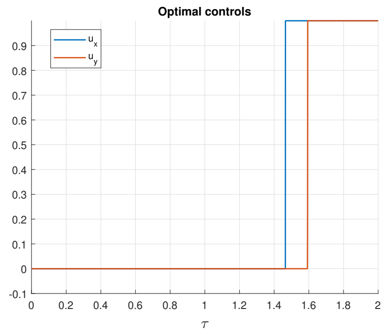

Fig. 2. Optimal controls are bang-bang. They are different in a nontrivial interval. Recall, here the τ units of time are r_x^{-1} then, for this initial conditions, the calculations suggest that harvesting of the prey should commence at approximately 1.45τ and continue thereafter with maximum effort. Meanwhile, harvesting of the intermediate predator should commence at 1.6τ and also continue with maximum effort until the end of the time horizon.

For the cost functional, we choose the values $p_x = p_y = 5$ and $\theta_x = 1.5$, $\theta_y = 2$ to make sure the profit is nonnegative, otherwise, the harvest would result in a loss. We also assume $U = [0,1]^2$, as for the minimum values defining the path constraints, we take $x_m = 0.15$ and, taking into account that for these parameters any $y < 0.22$ would drive $z \to 0$ (a scenario we aim to prevent), we consider $y_m = 0.2$ and $z_m = 0.1$. For the necessary conditions to provide useful information, (see e.g. Vinter (2000)) we need to prevent path constraints from being active at $t = 0$, then, we choose $s(0) = (0.2, 0.4, 0.3, 0)$ and a time horizon of $T = 2$. The problem is implemented by (ICLOCS2), using the large-scale nonlinear optimization solver Ipopt and an error tolerance of 10^{-9}. With these parameters, we obtain a normal solution in 7.3335 CPU time and a maximum profit of 0.6451.

As shown in Figure 2, the computed optimal controls are bang-bang and they are different in a nontrivial interval, a situation which is not possible with the original formulation in Dawed and Kebedow (2021). Based on the computed solution, the initiation of prey harvesting should occur earlier. It is worth noting that the duration of the interval where the control policies differ increases in proportion to the disparity in effort costs.

The computed optimal states are illustrated in Figure 3. It is evident that the difference between the initial and final values is minimal. In fact, there is a slight increment in the stock of the harvested species. However, the value of z has decreased. We must keep in mind that in Dawed and Kebedow (2021), a scalar control and a different optimal control problem were considered. However, in relation to a possible comparison with that work concerning the impact of harvesting on the species population, we can observe in Figure 3 of the said paper that, when utilizing the one-dimensional control $h = 0.3$ with the same

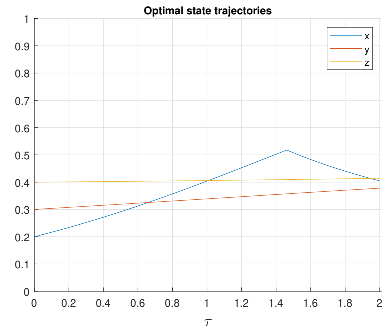

Fig. 3. Optimal states do not approach the boundary constraint. For the prey, we observe the following population dynamics: it initially constituted 20% of K_x, increased to 52% before the onset of harvesting, and ultimately settled at 40% by the end of the analyzed period. As intermediate predator's population started at 40% of K_y, rose to 60% before harvesting began, and concluded at 45%. Top predator's population remained quite stable, with only a slight decline being observed.

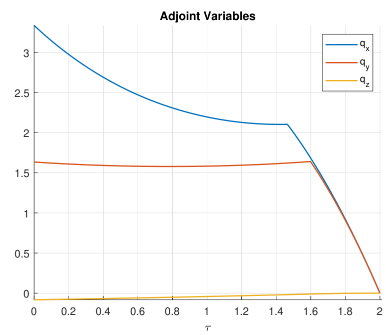

Fig. 4. Costates are continuous as a result of the constraints not being activated. We also observe that all of them converge to zero, as expected due to the transversality condition and q_x and q_y, both exhibit a peak when the harvesting begins.

parameters and initial conditions, the values of x stabilize around 0.7, while those of y and z exhibit cyclic behavior. It is also noteworthy that in that case, the time horizon is much larger than the one we are considering here. Figure 4 displays the computed costates. Given that the path constraints remain inactive, the measure μ_i is zero for $i = 1, 2, 3$. Consequently, the costates exhibit continuity and they end at zero, as expected.

In Figures 5, 6 and 7 we display the results obtained for the same data but larger values of x_0 and y_0. The most notorius difference is that, now, the optimal control corresponding to harvesting of intermediate predator is singular in a subinterval, another situation that is not possible to achieve with a scalar control. We also observed that the CPU time increased to 17.

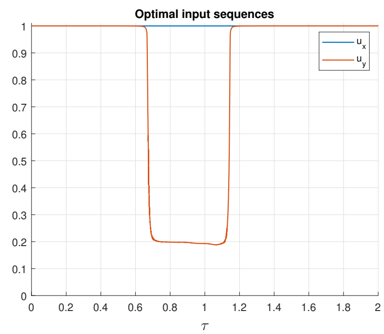

Fig. 5. The optimal control corresponding to the harvesting of prey u_x always takes its maximum value, whereas u_y is singular in a subinterval.

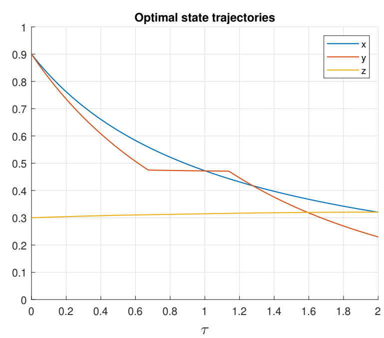

Fig. 6. Optimal states do not reach the boundary constraints. For the prey, it initially occupied 90% of K_x and gradually decreased to 32%. The intermediate predator's population began at 90% of K_y, remained stable at 47%, and then declined to 22%. The top predator' population is slightly increased.

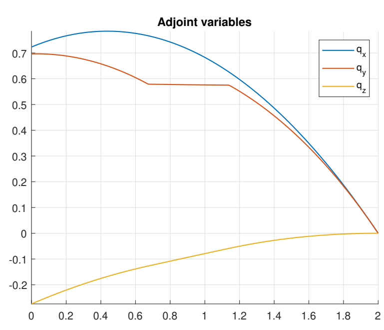

Fig. 7. Once more, the costates are continuous and all of them converge to zero. Now only q_y, has corners at the endpoints of the interval where u_y is singular.

5. CONCLUSIONS AND FUTURE WORK

In this paper we proposed a new approach to use optimal control techniques in the search for optimal harvesting policies that guarantee the persistence of the species in a tritrophic food chain where prey and intermediate predator are harvested. We consider the harvesting to be selective. Therefore, the harvesting effort is represented by a vector control variable. Numerical simulations using ICLOCS2 show that this approach is efficient for small time horizons and functional costs that combine the profits from both harvests. It would be worthwhile to examine how the results are affected by considering profits separately, as conflicting interests may arise. Naturally, a more comprehensive analysis of the system's qualitative behavior is necessary. This analysis can provide insights into the selection of appropriate path constraints for each Holling-type response, establishing conditions to ensure applicability to real-world data, and conducting a rigorous assessment of the model's robustness to determine how sensitive it is to uncertainties in the parameters. We defer these aspects to future research, including the pursuit of finding the optimal harvesting policy through the solution of an optimal control problem with boundary constraints, rather than path constraints.

REFERENCES

Blé, G., Castellanos, V., and Dela-Rosa, M. (2018). Coexistence of species in a tritrophic food chain model with holling functional response type IV. *Mathematical Methods in the Applied Sciences*, 41(16), 6683—6701.

Chen, J., Huang, J., Ruan, S., and Wang, J. (2013). Bifurcations of invariant tori in predator-prey models with seasonal prey harvesting. *SIAM Journal on Applied Mathematics*, 73(5), 1876—1905.

Dawed, M. and Kebedow, K. (2021). Coexistence and harvesting optimal policy in three species food chain model with general holling type functional response. *Natural Resource Modeling*, 34, e12316.

Dawed, M., TchepmoDjomegni, P., and Krogstad, H. (2020). Complex dynamics in a tritrophic food chain model with general functional response. *Natural Resource Modeling*, 33(2), e12260.

Fleming, W.H. and Rishel, R. (1975). *Deterministic and stochastic optimal control.* Springer Verlag.

Mortoja, S., Panja, P., Paul, A., Bhattacharya, S., and Mondal, S. (2020). Is the intermediate predator a key regulator of a tri-trophic food chain model?: An illustration through a new functional response. *Chaos, Solitons & Fractals*, 132(8), 109613.

Panja, P. and Mondal, S. (2015). Stability analysis of coexistence of three species prey-predator model. *Nonlinear Dynamics*, 81(1-2), 373–382.

Rojas-Palma, A. and González-Olivares, E. (2012). Optimal harvesting in a predator-prey model with allee effect and sigmoid functional response. *Applied Mathematical Modelling*, 36(5), 1864—1874.

Tchepmo-Djomegni, P., DoungmoGoufo, E., Sahu, S., and Mbehou, M. (2019). Coexistence and harvesting control policy in a food chain model with mutual defense of prey. *Natural Resource Modeling*, 32(4), e12230.

Upadhyay, R. and Raw, S.N. (2011). Complex dynamics of a three species food-chain model with holling type IV functional response. *Nonlinear Analysis: Modelling and Control*, 16(3), 353—374.

Vinter, R. (2000). *Optimal Control.* Birkäuser, Boston.