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Attitude stabilization controller for quadrotors based on the unit quaternion ⋆

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Abstract: This paper addresses the attitude stabilization problem of a flying robot with four rotors. We propose an alternative attitude dynamic model based on the unit quaternion. The proposed dynamic model resembles the equation of motion of a four degrees of freedom robot manipulator and shares similar properties. If the desired attitude is constant, we show that a PD controller renders the equilibrium point of the closed-loop dynamics exponentially stable. The validity of the dynamic model and effectiveness of the controller were assessed by experimental results on a small quadrotor.

Keywords: attitude stabilization, exponential stability, flying robot, modelling, quadrotor

1. INTRODUCTION

Aerial vehicles have been a topic of great relevance in recent years, mainly due to technological advances. The development and control of the quadrotor has been of great interest to the scientific community due to its maneuverability and low cost. The quadrotor is a type of unmanned aerial vehicle with four rotors with the ability to take off and land in limited spaces (Abdelhay and Zakriti, 2019). Although there are a variety of sizes, they have a simple structure since they are quick to manufacture and economical (Xuan-Mung and Hong, 2019; Bashi et al., 2017). Controlling the robot's position and attitude have been the subject of studies for years with different control schemes and applications mainly based on Euler angles (Castillo et al., 2004; Bouabdallah et al., 2005; Madani and Benallegue, 2006; Luukkonen, 2011; Mahony et al., 2012). However, Euler angles cannot globally describe any attitude of the robot and present singularities; thus, these type of controllers do not allow angles greater than ± 90[deg].

An alternative to Euler's angles is the quaternions discovered by William Hamilton, which have one real part and three imaginary parts (Kuipers, 1999). They have a large number of applications mainly for three-dimensional rotations, providing advantages over traditional methods, since it only requires 4 parameters compared to the 9 elements of rotation matrices, which means that it is less sensitive to computational errors (Salamin, 1979).

In computer graphics, unit quaternions are used to interpolate rotations, for instance inanimations (Mukundan, 2012; Shoemake, 1985). The combination of Euler parameter kinematics and Hamiltonian mechanics provides a model of rigid body dynamics suitable for use in strongly nonlinear problems involving large arbitrary rotations (Shivarama and Fahrenthold, 2004).

Attitude control of rigid bodies has important applications for airplanes, helicopters, spacecraft, satellites and even robots with the use of non minimal attitude representations (Wen and Kreutz-Delgado, 1991). Attitude control can also be applied to water vehicles such as submarines with 6 degrees of freedom (Fjellstad and Fossen, 1994a,b).

One of the first works to study the attitude problem by means of the unit quaternion in quadrotors was reported in (Tayebi and McGilvray, 2006) where the authors propose a feedback control scheme for the attitude exponential stabilization problem. The authors report two controllers, the first one is based on the compensation of Coriolis and gyroscopic terms with the use of a PD^2 structure and the second one is based on the classical PD controller where the proportional action is in terms of Quaternion and derivative action in terms of angular velocity providing asymptotic stability. In (Tayebi, 2008) a dynamic feedback based on unit quaternions is proposed for attitude tracking without velocity measurement, this approach introduces an auxiliary system based on the unit quaternion, using it in the control law together with the attitude error, the authors developed a velocity-free scheme that guarantees almost global asymptotic stability. In the case of regulation, the control law is a pure feedback of quaternions.

Other authors, such as in (Fresk and Nikolakopoulos, 2013) address the attitude problem by proposing a nonlinear proportional squared P^2 control algorithm. Stingu and Lewis (2009) proposed simple PD controllers that guarantee stable flight in hover motion. The authors de-

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scribe the implementation of the simple PD algorithms for flight in a controlled environment equipped with low-cost sensors.

Carino et al. (2015) present the design and implementation of a control scheme capable of globally stabilizing the position and attitude of a quadrotor. Using position references, they calculate a smooth trajectory that the attitude controller must follow to stabilize the vehicle's position. The control law proposed by the authors can be analyzed as a linear system.

The problems of altitude and attitude tracking in quadrotors have also been studied by means of computational tools such as neural networks (Xian et al., 2015) where the authors present a robust nonlinear output feedback tracking controller based on quaternions. The problem is subject to structural uncertainties and unknown external perturbations. Neural networks are used to estimate modeling uncertainties. They use a Lyapunov-based stability analysis to demonstrate semi-global asymptotic tracking.

Unit quaternion has been also used to describe the attitude of hybrid aerial robots, that is, flying robots that integrate manipulator arms fixed in the body (Alvarez-Munoz et al., 2018). Mo et al. (2019) use a quadrotor with an arm of 2 degrees of freedom to follow a time-varying reference. For the model, they take into account external disturbances such as the torque generated by the gravitational force of the manipulator.

On the other hand, Espíndola and Tang (2023) present a Lagrangian approach to model the attitude of rigid bodies. The Lagrangian dynamics of 4 degrees of freedom evolve on the unit sphere, and exploit the property of conservation of energy.

This paper focuses on the attitude stabilization problem of quadrotors. We employ the unit quaternion to describe the attitude of the flying robots. Exploiting the properties of the quaternion kinematics, we develop an alternative attitude dynamic model with four degrees of freedom. Using the proposed dynamic model, we design and analyze the stability properties of a PD control law. The paper is organized as follows: the quaternion kinematics and the Newton-Euler equations of the quadrotors are given in Section 2. The dynamic model based on the unit quaternion is presented in Section 3. Section 4 presents the control law and stability analysis. The experimental results are discussed in Section 5. The paper ends in Section 6, where we give some conclusions and future directions.

2. PRELIMINARES

The flying robot considered in this paper is shown in Figure 1. The attitude of the flying robot is described by the rotation matrix $R \in SO(3)$ that relates the body frame $\Sigma_{\mathcal{B}}$ with the inertial frame $\Sigma_{\mathcal{I}}$. A useful parametrization of the rotation matrix R is the unit quaternion

$$\boldsymbol{q} = \left[\, q_{\rm s} \ \boldsymbol{q}_{\rm v}^\top \,\right]^\top \in \mathbb{S}^3$$

where $q_{\rm s} \in \mathbb{R}$ is the scalar part and $\boldsymbol{q}_{\rm v} \in \mathbb{R}^3$ is the vector part and $\mathbb{S}^3 = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \boldsymbol{x}^\top \boldsymbol{x} = 1\}$ denotes the three-sphere. The kinematics of the unit quaternion is given

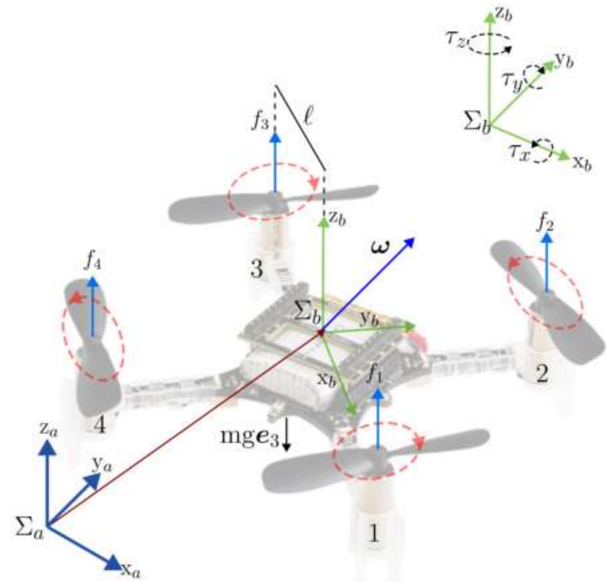

Fig. 1. Flying robot with four rotors

$$\dot{\boldsymbol{q}} = \frac{1}{2} J(\boldsymbol{q})\boldsymbol{\omega}, \quad J(\boldsymbol{q}) \triangleq \begin{bmatrix} -\boldsymbol{q}_{\rm v}^\top \\ q_{\rm s} I_3 + S(\boldsymbol{q}_{\rm v}) \end{bmatrix} \in \mathbb{R}^{4\times 3} \tag{1}$$

where $\boldsymbol{\omega} \in \mathbb{R}^3$ is the quadrotor angular velocity expressed in the body frame I_3 is the 3×3 identity matrix and $S(\cdot) \in \mathbb{R}^{3\times 3}$ is a skew-symmetric matrix such that $S(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x}\times\boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. The Jacobian matrix $J(\boldsymbol{q})$ in (1) satisfies

$$J^\top(\boldsymbol{x})\boldsymbol{x} = \mathbf{0} \tag{2a}$$

$$J^\top(\boldsymbol{q})J(\boldsymbol{q}) = I_3 \tag{2b}$$

for all $\boldsymbol{x} \in \mathbb{R}^4$, $\boldsymbol{q} \in \mathbb{S}^3$.

On the other hand, the attitude dynamics of the quadrotor in the body frame is given by

$$M\dot{\boldsymbol{\omega}} = \boldsymbol{\tau} + S(M\boldsymbol{\omega})\boldsymbol{\omega} \tag{3}$$

where $M = M^\top \in \mathbb{R}^{3\times 3}$ is the constant inertia matrix and $\boldsymbol{\tau} \in \mathbb{R}^3$ is the input torque.

3. QUATERNION-BASED DYNAMIC MODEL

In this section, we develop an alternative dynamic model for the quadrotor. Using the properties of the Jacobian matrix $J(\boldsymbol{q})$ in (2), the angular velocity can be expressed as follows

$$\boldsymbol{\omega} = 2J^\top(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{4}$$

Now, we define the quaternion $\bar{\boldsymbol{\omega}} = \left[\, 0 \ \boldsymbol{\omega}^\top \,\right]^\top \in \mathbb{R}^4$ and the matrix (Espíndola and Tang, 2023)

$$P(\boldsymbol{q}) = \left[\, \boldsymbol{q} \ J(\boldsymbol{q}) \,\right] = \begin{bmatrix} q_{\rm s} & -q_{\rm x} & -q_{\rm y} & -q_{\rm z} \\ q_{\rm x} & q_{\rm s} & -q_{\rm z} & q_{\rm y} \\ q_{\rm y} & q_{\rm z} & q_{\rm s} & -q_{\rm x} \\ q_{\rm z} & -q_{\rm y} & q_{\rm x} & q_{\rm s} \end{bmatrix}. \tag{5}$$

where $\boldsymbol{q} = \left[\, q_{\rm s} \ q_{\rm x} \ q_{\rm y} \ q_{\rm z} \,\right]^\top \in \mathbb{S}^3$. By direct computation we can prove that $P(\boldsymbol{q})P^\top(\boldsymbol{q}) = P^\top(\boldsymbol{q})P(\boldsymbol{q}) = I_4$ and $\det(P(\boldsymbol{q})) = 1$. Therefore, the matrix $P(\boldsymbol{q})$ is orthogonal and in fact is a rotation matrix, that is, $P(\boldsymbol{q}) \in SO(4)$. Some interesting properties of the structure of the matrix $P(\cdot)$ are listed below

$$P(\boldsymbol{x}) + P(\boldsymbol{y}) = P(\boldsymbol{x}+\boldsymbol{y}) \tag{6a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}P(\boldsymbol{x}) = P(\dot{\boldsymbol{x}}) \tag{6b}$$

for all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^4$.

Using the definitions of $\bar{\boldsymbol{\omega}}$ and $P(\boldsymbol{q})$, the kinematics of the unit quaternion can be written as

$$\dot{\boldsymbol{q}} = \frac{1}{2}P(\boldsymbol{q})\bar{\boldsymbol{\omega}} \quad \Leftrightarrow \quad \frac{1}{2}\bar{\boldsymbol{\omega}} = P^\top(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{7}$$

Differentiating with respect to time the previous expression yields

$$\frac{1}{2}\dot{\bar{\boldsymbol{\omega}}} = P(\boldsymbol{q})\ddot{\boldsymbol{q}} + P(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}. \tag{8}$$

Note that the first element of the previous equation is given by

$$\boldsymbol{q}^\top\ddot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^\top\dot{\boldsymbol{q}} = 0 \tag{9}$$

which correspond to the second time derivative of the unit quaternion constraint $\boldsymbol{q}^\top\boldsymbol{q} = 1$.

Multiplying (8) by the following symmetric positive definite matrix

$$\bar{M} = \begin{bmatrix} \mu & \boldsymbol{0}^\top \\ \boldsymbol{0} & M \end{bmatrix} \in \mathbb{R}^{4\times 4}, \quad \mu > 0, \tag{10}$$

yields the following expression

$$\frac{1}{2}\bar{M}\dot{\bar{\boldsymbol{\omega}}} = \frac{1}{2}\begin{bmatrix} 0 \\ M\dot{\boldsymbol{\omega}} \end{bmatrix} = \bar{M}P^\top(\boldsymbol{q})\ddot{\boldsymbol{q}} + \bar{M}P^\top(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}. \tag{11}$$

Finally, multiplying (11) by $P(\boldsymbol{q})$ and by taking into account the quadrotor dynamics (3), we obtain the quaternion-based dynamic model given by

$$H(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \bar{\boldsymbol{\tau}} \tag{12}$$

where:

- $H(\boldsymbol{q}) = P(\boldsymbol{q})\bar{M}P^\top(\boldsymbol{q}) \in \mathbb{R}^{4\times 4}$ is a symmetric positive definite matrix.
- $C(\boldsymbol{q},\dot{\boldsymbol{q}}) = P(\boldsymbol{q})\bar{M}P^\top(\dot{\boldsymbol{q}}) - P(\boldsymbol{q})\bar{S}(M\boldsymbol{\omega})P^\top(\boldsymbol{q}) \in \mathbb{R}^{4\times 4}$ is the Coriolis matrix and
$$\bar{S}(M\boldsymbol{\omega}) = \begin{bmatrix} 0 & \boldsymbol{0}^\top \\ \boldsymbol{0} & S(M\boldsymbol{\omega}) \end{bmatrix} \in \mathbb{R}^{4\times 4}$$
is a skew-symmetric matrix.
- $\bar{\boldsymbol{\tau}} = \frac{1}{2}J(\boldsymbol{q})\boldsymbol{\tau} \in \mathbb{R}^4$ is torque input in the unit quaternion coordinates.

From the definition of the inertia matrix $H(\boldsymbol{q})$ it follows $P^\top(\boldsymbol{q})H(\boldsymbol{q}) = \bar{M}P^\top(\boldsymbol{q})$, this expression implies that $H(\boldsymbol{q})$ and $\bar{M}$ are similar matrices and hence they have the same eigenvalues. Therefore, if the parameter μ is selected as $\mu = \lambda_{\min}\{M\}$, the inertia matrix $H(\boldsymbol{q})$ satisfies

$$\lambda_{\min}\{M\}\|\boldsymbol{x}\|^2 \le \boldsymbol{x}^\top H(\boldsymbol{q})\boldsymbol{x} \le \lambda_{\max}\{M\}\|\boldsymbol{x}\|^2$$

for all $\boldsymbol{x} \in \mathbb{R}^4$.

Since the matrix $\bar{S}(\cdot)$ is skew-symmetric, we can show by direct computation that the matrix $\dot{H}(\boldsymbol{q}) - 2C(\boldsymbol{q},\dot{\boldsymbol{q}})$ is also a skew-symmetric matrix, that is,

$$\boldsymbol{x}^\top(\dot{H}(\boldsymbol{q}) - 2C(\boldsymbol{q},\dot{\boldsymbol{q}}))\boldsymbol{x} = 0,$$

for all $\boldsymbol{x} \in \mathbb{R}^4$. In addition, we have the following result

$$\dot{H}(\boldsymbol{q}) = C(\boldsymbol{q},\dot{\boldsymbol{q}}) + C^\top(\boldsymbol{q},\dot{\boldsymbol{q}}).$$

4. ATTITUDE CONTROL LAW

This section presents a control law that stabilize the quadrotor's attitude $\boldsymbol{q}(t) \in \mathbb{S}^3$ to some desired constant attitude $\boldsymbol{q}_\mathrm{d} \in \mathbb{S}^3$. The proposed control law is given by

$$\bar{\boldsymbol{\tau}} = K_1\tilde{\boldsymbol{q}} - K_2\dot{\boldsymbol{q}} \tag{13}$$

where $\tilde{\boldsymbol{q}} = \boldsymbol{q}_\mathrm{d} - \boldsymbol{q} \in \mathbb{R}^4$ is the attitude error and $K_1 \in \mathbb{R}^{4\times 4}$, $K_2 \in \mathbb{R}^{4\times 4}$ are diagonal positive definite matrices. In the original coordinates, the control law is given by

$$\boldsymbol{\tau} = 2J^\top(\boldsymbol{q})\left(K_1\tilde{\boldsymbol{q}} - \frac{1}{2}K_2J(\boldsymbol{q})\boldsymbol{\omega}\right). \tag{14}$$

where we used (4). Furthermore, if we select the control gains as $K_1 = \frac{1}{2}k_1I_4$ and $K_2 = k_2I_4$ for some positive constant k_1, $k_2 \in \mathbb{R}$, the controller (14) further simplifies to

$$\begin{aligned} \boldsymbol{\tau} &= k_1J^\top(\boldsymbol{q})\boldsymbol{q}_\mathrm{d} - k_2\boldsymbol{\omega} \\ &= k_1\boldsymbol{e}_\mathrm{v} - k_2\boldsymbol{\omega} \end{aligned} \tag{15}$$

where we have used $J^\top(\boldsymbol{q})\boldsymbol{q} = \boldsymbol{0}$ and $\boldsymbol{e}_\mathrm{v} \in \mathbb{R}^3$ is the vector part of the attitude error $\boldsymbol{e} \in \mathbb{S}^3$ defined as (Pliego-Jiménez, 2021)

$$\boldsymbol{e} = \begin{bmatrix} e_\mathrm{s} \\ \boldsymbol{e}_\mathrm{v} \end{bmatrix} = \boldsymbol{q}^{-1} \otimes \boldsymbol{q}_\mathrm{d} \tag{16}$$

where $\otimes$ denotes de Hamilton product. The controller (15) is the classical PD controller reported in the literature (Wen and Kreutz-Delgado, 1991).

Theorem 1. For any positive definite matrices K_1 and K_2, the control law (13) in closed-loop with (12) drives the quadrotor's attitude $\boldsymbol{q}(t)$ to the desired constant attitude $\boldsymbol{q}_\mathrm{d}$ with zero angular velocity, that is,

$$\lim_{t\to\infty}\boldsymbol{q}(t) = \boldsymbol{q}_\mathrm{d}, \quad \lim_{t\to\infty}\boldsymbol{\omega}(t) = \boldsymbol{0} \tag{17}$$

with an exponential converge rate.

Proof. Substituting the control law (13) into (12) and by taking into account that $\dot{\tilde{\boldsymbol{q}}} = -\dot{\boldsymbol{q}}$, we obtain the closed-loop dynamics

$$\dot{\tilde{\boldsymbol{q}}} = -\dot{\boldsymbol{q}} \tag{18a}$$
$$H(\boldsymbol{q})\ddot{\boldsymbol{q}} = K_1\tilde{\boldsymbol{q}} - K_2\dot{\boldsymbol{q}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \tag{18b}$$

which has an equilibrium point at the origin, that is, $(\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) = (\boldsymbol{0},\boldsymbol{0})$. To prove Theorem 1, we need to show that the origin of (18) is exponentially stable.

Consider the candidate Lyapunov function

$$\mathcal{V} = \frac{1}{2}\dot{\boldsymbol{q}}^\top H(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^\top K_1\tilde{\boldsymbol{q}} - \epsilon\tilde{\boldsymbol{q}}^\top H(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{19}$$

with $\epsilon > 0$. The Lyapunov function can be lower and upper bounded as follows

$$\mathcal{V} \ge \frac{1}{2}\begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}^\top \begin{bmatrix} \underline{k}_1 & -\epsilon\bar{m} \\ -\epsilon\bar{m} & \mu \end{bmatrix}\begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix} \tag{20}$$

$$\mathcal{V} \le \frac{1}{2}\begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}^\top \underbrace{\begin{bmatrix} \bar{k}_1 & \epsilon\bar{m} \\ \epsilon\bar{m} & \bar{m} \end{bmatrix}}_{A}\begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix} \tag{21}$$

where $\underline{k}_1 = \lambda_{\min}\{K_1\}$ and $\bar{k}_1 = \lambda_{\max}\{K_1\}$ are, respectively, the minimum and largest eigenvalue of K_1. If the parameter ϵ is selected to satisfy

$$\frac{\mu\underline{k}_1}{\bar{m}^2} > \epsilon^2,$$

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then, the candidate Lyapunov function is positive definite and radially unbounded.

The time derivative of $\mathcal{V}$ along (18) is given by

$$\begin{aligned}\dot{\mathcal{V}} =& \dot{\boldsymbol{q}}^\top \left(K_1\tilde{\boldsymbol{q}} - K_2\dot{\boldsymbol{q}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right) - \dot{\boldsymbol{q}}^\top K_1\tilde{\boldsymbol{q}} + \epsilon\dot{\boldsymbol{q}}^\top H(\boldsymbol{q})\dot{\boldsymbol{q}} \\ &- \epsilon\tilde{\boldsymbol{q}}^\top \left(K_1\tilde{\boldsymbol{q}} - K_2\dot{\boldsymbol{q}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right) - \epsilon\tilde{\boldsymbol{q}}^\top \dot{H}(\boldsymbol{q})\dot{\boldsymbol{q}}. \quad (22)\end{aligned}$$

Using the skew-symmetric property of the matrix $\dot{H}(\boldsymbol{q}) - 2C(\boldsymbol{q},\dot{\boldsymbol{q}})$, the identity $\dot{H}(\boldsymbol{q}) = C(\boldsymbol{q},\dot{\boldsymbol{q}}) + C^\top(\boldsymbol{q},\dot{\boldsymbol{q}})$ and the inequality $\|\tilde{\boldsymbol{q}}\| \leq \|\boldsymbol{q}\| + \|\boldsymbol{q}_\mathrm{d}\| \leq 2$, an upper bound of $\dot{\mathcal{V}}$ is given by

$$\dot{\mathcal{V}} \leq -\begin{bmatrix}\|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\|\end{bmatrix}^\top Q \begin{bmatrix}\|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\|\end{bmatrix} \quad (23)$$

where we used $\|C(\boldsymbol{q},\dot{\boldsymbol{q}})\| \leq k_\mathrm{c}\|\dot{\boldsymbol{q}}\|^2$ for some $k_\mathrm{c} > 0$, see (Kelly et al., 2005). The matrix $Q \in \mathbb{R}^{2\times 2}$ is given by

$$Q = \begin{bmatrix} \epsilon\underline{k}_1 & -\frac{1}{2}\epsilon\bar{k}_2 \\ -\frac{1}{2}\epsilon\bar{k}_2 & \underline{k}_2 - \epsilon(\bar{m} + 2k_\mathrm{c}) \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad (24)$$

If the parameter ϵ also satisfies the following inequality

$$\frac{\underline{k}_1\underline{k}_2}{\bar{k}^2 + \underline{k}_1(\bar{m} + 2k_\mathrm{c})} > \epsilon > 0. \quad (25)$$

Then, the matrix Q is positive definite and hence

$$\begin{aligned}\dot{\mathcal{V}} &\leq -\lambda_{\min}\{Q\}\left(\|\tilde{\boldsymbol{q}}\|^2 + \|\dot{\boldsymbol{q}}\|^2\right) \\ &\leq -\frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{A\}}\mathcal{V}. \quad (26)\end{aligned}$$

where $P \in \mathbb{R}^{2\times 2}$ is defined in (21). The inequality (26) implies that the origin of the closed-loop dynamics (18) is exponentially stable: thus, $\boldsymbol{q}(t) \to \boldsymbol{q}_\mathrm{d}$ and $\dot{\boldsymbol{q}}(t) \to \mathbf{0}$ with an exponential convergence rate. The exponential converge of $\dot{\boldsymbol{q}}$ to zero, also implies $\boldsymbol{\omega}(t) \to \mathbf{0}$ as $t \to \infty$, see (4). This completes the proof. □

5. EXPERIMENTAL RESULTS

In this section, we present experimental results to validate the quaternion-based dynamic model and the efficacy of the control law (13). The experimental platform is composed of the low-cost quadrotor *Crazyflie* from the Swedish company *Bitcraze* [1], PC computer, and a test bench, see Figure 2. The minidrone is equipped with a microcontroller STM32F40, a nRF51822 radio and an Inertial Measurement Unit (IMU) BMI088. The attitude controller was programed onboard using C language. A Python program running on the PC sends the desired attitude and thrust to the quadrotor via the Crazyradio. The test bench is a 3-dof mechanism designed to carry out experiments safely and was manufactured by 3D printing (Sidon-Ayala, 2020).

To show the stabilization capabilities of the proposed controller, we carried out two experiments with different desired attitudes. In both experiments, we selected the control gains as $K_1 = \mathrm{diag}\{0.025\ 0.0228\ 0.039\ 0.0165\}$ and $K_2 = \mathrm{diag}\{0.0005\ 0.0065\ 0.01\ 0.004\}$ and the sample time was set as 0.002 [s]. The desired attitude in the first experiment was $\boldsymbol{q}_\mathrm{d} = [\,1\ 0\ 0\ 0\,]^\top$ and the initial attitude of the robot was

$$\boldsymbol{q}(0) = [\,0.491\ 0.781\ 0.321\ 0.212\,]^\top, \quad \boldsymbol{\omega}(0) \approx \mathbf{0}\ \text{[rad/s]}.$$

Fig. 2. Test bench and Crazyflie drone

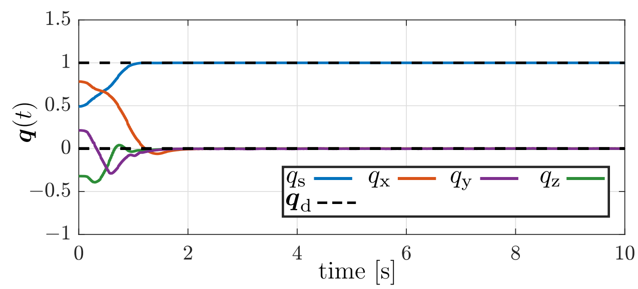

Fig. 3. Desired quaternion versus the measured quaternion in the first experiment

Figure 3 shows the time evolution of the unit quaternion in the first experiment; after a few seconds, the Crazyflie reaches the desired attitude since the attitude measurements provided by the IMU are very similar to the desired one. As can be seen in Figure 4, the angular velocity converges to zero; therefore, the control law has a good performance and achieves the control objective. Figure 5 shows the control signals generated by the control law; the control input vanishes as the robot's attitude reaches the desired one.

In the second experiment, the desired attitude is specified as a function of the *roll* (ϕ), *pitch* (θ) and *yaw* (ψ) Euler angles

$$\boldsymbol{q}_\mathrm{d} = \begin{bmatrix} \cos\left(\frac{\psi_\mathrm{d}}{2}\right) \\ 0 \\ 0 \\ \sin\left(\frac{\psi_\mathrm{d}}{2}\right) \end{bmatrix} \otimes \begin{bmatrix} \cos\left(\frac{\theta_\mathrm{d}}{2}\right) \\ 0 \\ \sin\left(\frac{\psi_\mathrm{d}}{2}\right) \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \cos\left(\frac{\phi_\mathrm{d}}{2}\right) \\ \sin\left(\frac{\psi_\mathrm{d}}{2}\right) \\ 0 \\ 0 \end{bmatrix}$$

[1] https://www.bitcraze.io

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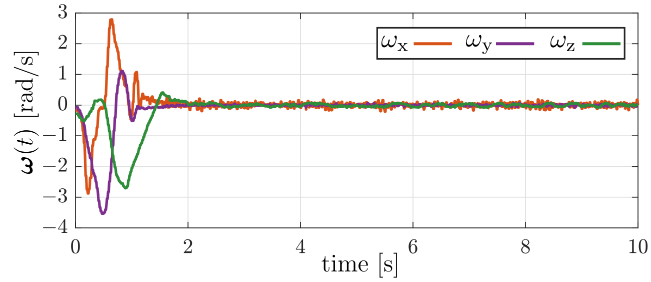

Fig. 4. Angular velocity in the first experiment

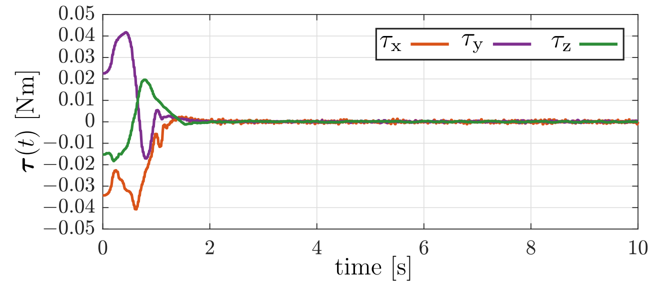

Fig. 5. Control input $\boldsymbol{\tau}$ in the first experiment

where $\phi_d = 10$ [deg], $\theta_d = 15$ [deg] and $\psi_d = 0$ [deg]. The initial conditions in the second experiment was

$$\boldsymbol{q}(0) = \left[\, 0.305 \ -0.933 \ 0.105 \ -0.157 \,\right]^\top, \quad \boldsymbol{\omega}(0) \approx \mathbf{0}[\text{rad/s}].$$

The experimental results of the second experiment are shown in Figure 6, 7 and 8. The control objective is also satisfied in the second experiment, the elements of the unit quaternion converges to the desired values and the angular velocity converges to zero, see Figure 6 and 7. A good performance is achieved even though the friction of the test bench joints and motors' dynamics were not considered in the design of the control law.

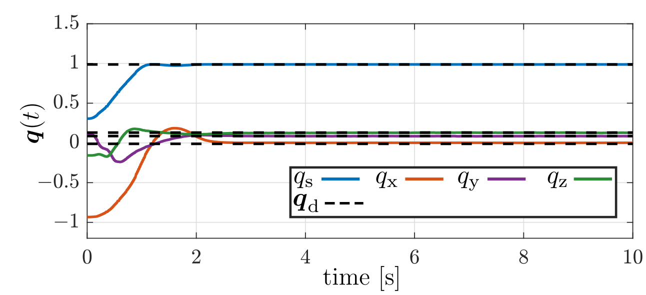

Fig. 6. Desired quaternion versus the measured quaternion in the second experiment

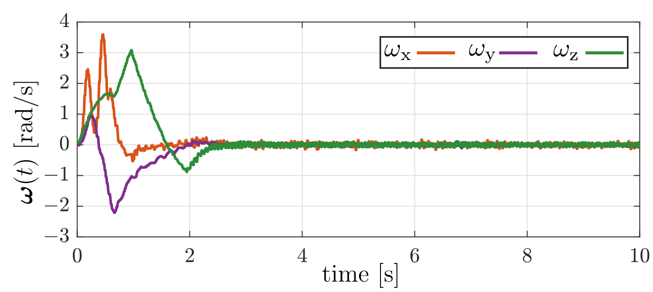

Fig. 7. Angular velocity in the second experiment

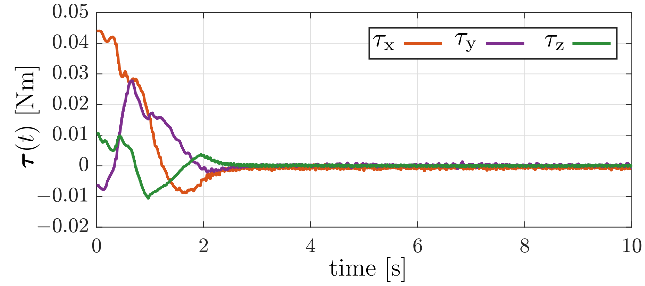

Fig. 8. Control input $\boldsymbol{\tau}$ in the second experiment

Figure 8 shows the time evolution of the control input. In a similar fashion as the first experiment (see Figure 5) the input torques tend to zero after the transient response, therefore, we conclude that in the second experiment the control law works out properly, since it supplies the torque necessary to attain the regulation goal.

6. CONCLUSION

This work proposes an attitude stabilization control law for flying robots based on the unit quaternion. Exploiting the properties of the unit quaternion kinematics, we develop a 4-dof attitude dynamic model that is very similar to the Euler-Lagrange equations of motion of robot manipulators without gravity term and presents similar properties regarding the Inertia and Coriolis matrices. Using the alternative dynamic model, we show by means of a strict Lyapunov function that the proposed stabilization controller drives the attitude and angular velocity errors to zero with an exponential convergence rate. The experimental results are consistent with the proposed theory. Since $\boldsymbol{q}$ and $-\boldsymbol{q}$ represent the same attitude, the proposed controller may induce the unwinding phenomena; thus, in future work, we will consider the problem of designing anti-unwinding or hybrid controllers to overcome this issue.

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