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An LMI-based Proportional-Derivative Observer for Singular Linear Systems ⋆

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Abstract: This paper proposes a solution to design an observer for singular linear systems in terms of linear matrix inequalities whose solvability is verified in polynomial time. The observer under design shares the idea of a Luenberger one but adding the derivative of the outputs, which leads to the regularization of the error system. The direct Lyapunov method is employed for designing both the proportional and derivative gains and thus guaranteeing the stability of the error system. Two examples illustrate the effectiveness of the proposal.

Keywords: Observers, Descriptor Systems, Lyapunov Method, Linear Matrix Inequality, Singular Systems, Pantelides algorithm.

1. INTRODUCTION

The study of descriptor systems has a rich history dating back to 1867 (Lewis, 1986). In general, these systems are characterized by the presence of differential algebraic equations (DAEs), which makes difficult their analysis and control (Luenberger, 1977). The interest on them arises from their relevance in various engineering applications, such as electrical circuits (Wells, 1967; Campbell, 1982), mechanical systems (Duan, 2010), and chemical processes (Kumar and Daoutidis, 1998). Because of their structure, DAEs are also known as singular systems, they often appear when modeling systems with constraints, such as those imposed by physical laws or engineering design specifications (Arceo et al., 2018). Moreover, notice that these systems do not necessarily have a solution for any set of initial conditions (Pantelides, 1988; Gear, 1988; Rabier and Rheinboldt, 1994), some authors (Chadli et al., 2008; Chadli and Darouach, 2013; Zhang et al., 2014) do not take into account this fact, thus, they can generate inconsistent initial conditions. The of Pantelides algorithm helps to characterize the restricted subspace of initial conditions of the constrained system (Pantelides, 1988).

In practice, not all state variables are available for control tasks; therefore, it is necessary to estimate them. A state observer estimates the state variables based on measurements of known input and output variables Luenberger (1971). Different observers can be designed such as sliding mode (Spurgeon, 2008) or adaptive ones (Carroll and Lindorff, 1973). However, this paper is based on the well-known Luenberger state observer (Luenberger, 1966). There are various extensions of Luenberger observers, some examples are PI (proportional-integral) with an advantage of being robust to uncertainties (Gupta et al., 2014). (Gao, 2005) and (Wu and Duan, 2007) proposed different PD (proportional-derivative) observers based-on parametrization and generalized Sylvester matrix equations; respectively.

On the other hand, Lyapunov analysis have been widely used for analysis and design of standard systems Khalil (2014); in the case of singular ones, some works exist (Arceo et al., 2018; Gupta et al., 2014; Ishihara and Terra, 2002; Chadli and Darouach, 2012), they led to conditions in terms of linear matrix inequalities (LMIs) (Boyd et al., 1994; Scherer and Weiland, 2000). LMIs are of particular interest as they can be solved numerically through convex optimization methods readily accessible in commercial software (Gahinet et al., 1995).

Contribution: The approach provides LMI conditions guaranteeing a faster convergence (Bernal et al., 2019) for estimating the states of singular linear systems. It is suitable for regular descriptor systems. Moreover, the proposal employs a structure of Luenberger-like observer, but including the derivative of the estimation error, yielding a PD observer.

The rest of the paper is organized as follows: Section 2 establishes the problem to be solved and introduces singular systems. Section 3 presents the main results and states LMI conditions for the PD observer design. Section 4 illustrates the effectiveness of the proposal using two numerical examples. Section 5 concludes the paper.

Notation: In what follows, an asterisk $(*)$ denotes the transpose of the symmetric element in matrix expressions, this is:

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$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} A & (*) \\ B & C \end{bmatrix};$$

for in-line expressions, it stands for the transpose of terms on its left side, i.e.,

$$A + B + A^T + B^T + C = A + B + (*) + C.$$

2. PROBLEM STATEMENT

Consider the following descriptor linear system (Duan, 2010):

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^o$ is the output vector. Matrices $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known matrices with constant entries. System (1) is called normal if E is nonsingular, in this case, a standard state-space representation can be computed [1] :

$$\dot{x}(t) = E^{-1}\left(Ax(t) + Bu(t)\right), \tag{2}$$

and then traditional techniques can be used (Kailath, 1980), for standard linear systems, the system dynamical order n is the same with the usual system order n, while for a general descriptor is generally less (Duan, 2010). If $\operatorname{rank}(E) = r < n$, system (1) is called singular (Luenberger, 1977); moreover, it is also regular if

$$\det(sE - A) \neq 0 \text{ for some } s \in \mathbf{C}.$$

This paper is concerned with regular and singular linear systems of the form (1) under the following assumptions:

Assumption 1. System (1) satisfies

$$\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$$

and it is called (Verghese et al., 1981; Cobb, 1984; Dai, 1989; Yip and Sincovec, 1981; Ishihara and Terra, 2002):

(1) $\boldsymbol{C}$-observable if

$$\operatorname{rank} \begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \quad \text{for all } s \in \mathbf{C} \quad \text{holds;}$$

(2) completely detectable if

$$\operatorname{rank} \begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \quad \text{for all } s \in \mathbf{C}^+ \quad \text{holds.}$$

Definition 2. (Arceo et al., 2018) Let be $\mathcal{X}$ a the state space $\mathbb{R}^n$ where the trajectories $x(t)$ of a singular system are restricted. If $0 \in \mathcal{X}$ and $x = 0$ is an equilibrium point; then it is asymptotically stable if there exists a neighborhood $N(0,r) \subset \mathcal{X}$, where $N(0,r) = \{x \in \mathcal{X} : ||x|| \leq r, r > 0\}$, for all $x(0) \in N(0,r)$ and $\epsilon > 0$, exists $\delta > 0$ such that $||x(0)|| < \delta < r$, then $||x(t)|| < \epsilon$ for all $t \geq 0$, and $\lim_{t\to\infty} x(t) = 0$.

[1] Inverting E may not be ideal, as matrix inversion can lead to numerical issues and the computational error introduced during the initial conversion may result in inaccurate analysis or synthesis results (Lewis et al., 2003; Duan, 2010).

Thus, under Assumption 1, the task is to design a PD Luenberger observer for (1), that is:

$$\begin{aligned} E\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) + F(\dot{y} - \dot{\hat{y}}), \\ \hat{y} &= C\hat{x}, \end{aligned} \tag{3}$$

where $\hat{x} \in \mathbb{R}^n$ is the vector of estimated states, $L, F \in \mathbb{R}^{n\times o}$ are the proportional an derivative gains of the observer, respectively. Next section provides LMI condition for the design of both gains while adding constraints con the speed convergence.

3. MAIN RESULTS

Let the observer error be $e = x - \hat{x}$; thus the error system dynamics is:

$$E\dot{e} = A(x - \hat{x}) - LC(x - \hat{x}) - FC(\dot{x} - \dot{\hat{x}})$$

or

$$(E + FC)\dot{e} = (A - LC)e, \tag{4}$$

the task is design L and F such as

$$\lim_{t\to\infty} e(t) = 0.$$

To study system (4), it can be rewritten as descriptor redundancy form:

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{\bar{E}} \underbrace{\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix}}_{\dot{\bar{e}}} = \left(\underbrace{\begin{bmatrix} 0 & I \\ A & -E \end{bmatrix}}_{\bar{A}} - \underbrace{\begin{bmatrix} 0 \\ L \end{bmatrix}}_{\bar{L}} \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{\bar{C}_1} - \underbrace{\begin{bmatrix} 0 \\ F \end{bmatrix}}_{\bar{F}} \underbrace{\begin{bmatrix} 0 & C \end{bmatrix}}_{\bar{C}_2} \right) \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \tag{5}$$

Notice that in (5), the augmented state vector is constituted by e and $\dot{e}$.

3.1 LMI conditions

The direct Lyapunov method may allow obtaining LMI conditions; thus if we define

$$\bar{e} = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} e - \hat{e} \\ \dot{e} - \dot{\hat{e}} \end{bmatrix},$$

now we can consider the following Lyapunov function candidate:

$$V(\bar{e}) = \bar{e}^T \bar{E}^T \bar{P} \bar{e}, \quad \bar{P} = \begin{bmatrix} P_1 & 0 \\ P_2 & P_2 \end{bmatrix}, \tag{6}$$

where $P_1 > 0$, $P_2 > 0 \in \mathbb{R}^{n\times n}$. With this in mind the following result can be stated.

Theorem 3. The error $e = x - \hat{x}$ in (4), under Definition 2, is asymptotically stable, if there exist matrices $P_1 > 0$, $P_2 > 0 \in \mathbb{R}^{n\times n}$, $N, M \in \mathbb{R}^{n\times o}$ such that LMI:

$$\begin{bmatrix} P_2A - MC + (*) & (*) \\ P_2A - MC + P_1 - E^TP_2 - C^TN^T & -P_2E - NC + (*) \end{bmatrix} < 0 \tag{7}$$

holds; if found feasible, the observer gains are defined as $L = P_2^{-1}M$ and $F = P_2^{-1}N$.

Proof. The time-derivative of (6) is

$$\dot{V}(\bar{e}) = \dot{\bar{e}}^T \bar{E}^T \bar{P} \bar{e} + \bar{e}^T \bar{E}^T \bar{P} \dot{\bar{e}},$$

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where once the system (5) is substituted yields:

$$\dot{V} = \bar{e}^T \left(\bar{P}^T \left(\bar{A} - \bar{L}\bar{C}_1 - \bar{F}\bar{C}_2 \right) + (*) \right) \bar{e} \tag{8}$$

therefore, $\dot{V}(\bar{e}) < 0$ if

$$\bar{P}^T \left(\bar{A} - \bar{L}\bar{C}_1 - \bar{F}\bar{C}_2 \right) + (*) < 0$$

or

$$\begin{bmatrix} P_2A - P_2LC_1 + (*) & (*) \\ \Gamma^{(2,1)} & -P_2E - P_2FC_1 + (*) \end{bmatrix} < 0 \tag{9}$$

where

$$\Gamma^{(2,1)} = P_2A - P_2LC_1 + P_1 - E^T P_2 - C_2^T F^T P_2;$$

finally using the substitution $M = P_2L$ and $N = P_2F$ in (9) leads to LMI (7). □

The speed converge of the error signal towards the origin can be increased by means of LMIs too. The following result establishes this fact.

Corollary 4. The error $e = x - \hat{x}$ in (4), under Definition 2, is asymptotically stable holding a decay rate $\alpha > 0$, if there exist matrices $P_1 > 0$, $P_2 > 0 \in \mathbb{R}^{n\times n}$, $N, M \in \mathbb{R}^{n\times o}$ such that LMI:

$$\begin{bmatrix} P_2A - MC + (*) + 2\alpha P_1 & (*) \\ P_2A - MC + P_1 - E^T P_2 - C^T N^T & -P_2E - NC + (*) \end{bmatrix} < 0 \tag{10}$$

holds; if found feasible, the observer gains are defined as $L = P_2^{-1}M$ and $F = P_2^{-1}N$.

Proof. It follows similar lines as Theorem 3 but with $\dot{V}(\bar{e}) \leq -2\alpha V(\bar{e})$. □

Remark 5. If system (1) is a DAE one, it can be converted into a minimal set of ordinary differential equations with constraints on their initial conditions (Pantelides, 1988). The method begins by differentiating the algebraic constraints in system (1) as many times as necessary to write the system as an ODE. The minimum number of derivatives necessary is called index (Gear, 1988; Kumar and Daoutidis, 1995). Then, a set of initial conditions is proposed such that they satisfy the algebraic constraints obtained in the previous step.

For consistent initialization of simulations purposes, the Pantelides algorithm is employed as shown in Figure 1.

4. EXAMPLES

The proposed methodology is illustrated via examples taken from the literature.

Example 6. Consider the descriptor system borrowed from (Gupta et al., 2014), in the form (1), with matrices

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Notice that the system is completely detectable. Through the methodology in Section 3 and with LMIs in Theorem 3, the following gains are obtained:

$$L = \begin{bmatrix} 1.5 & 0 \\ 0 & -1.5 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \\ 0 & 0 \end{bmatrix}.$$

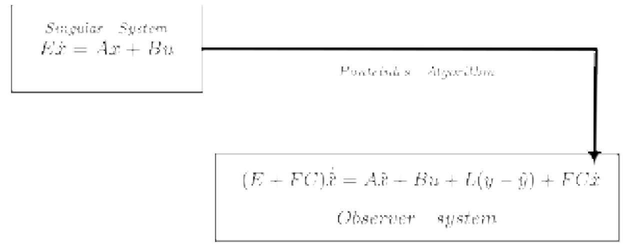

Fig. 1. The use of Pantelides algorithm.

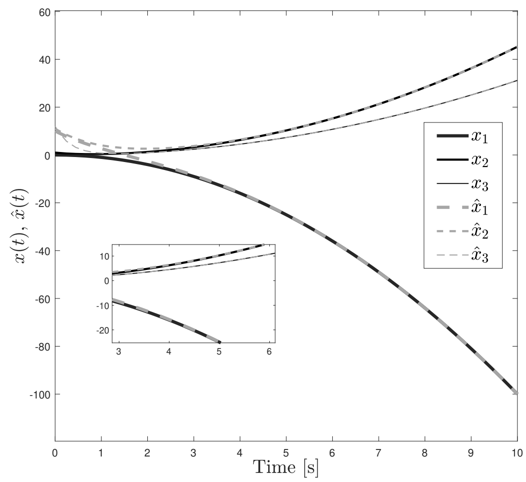

Fig. 2. States and their estimations in Example 6.

Simulation results are shown in Figure 2. Consistent initial conditions have been obtained by means of Remark 5; thus $x_0 = [0\ 1\ 0]^T$ and $\hat{x}_0 = [10\ 11\ 12]^T$, together with $u = t^2$ have been implemented. Figure 3 plots the corresponding error signals, this plot illustrate that the error is asymptotically stable.

Example 7. Let us consider the two-mass damper system shown in Figure 4, taken from (Bernal et al., 2022), where states are the velocities of the mass and an external force, i.e., $x^T = [v_1\ v_2\ F]$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system is C-observable.

LMIs in Corollary 4 with $\alpha = 0.9$ have been run, it yielded feasible result with

$$F = \begin{bmatrix} -0.7075 & -0.4980 \\ -2.1721 & -6.5818 \\ 1.7625 & 4.2115 \end{bmatrix} \text{ and } L = \begin{bmatrix} 0.8208 & -1.4980 \\ -2.1721 & -12.0191 \\ 2.7625 & 9.8056 \end{bmatrix}.$$

Figure 5 shows the behavior of the plant together with the observer for initial conditions $x_0 = [1\ 1\ -0.5]^T$ and $\hat{x}_0 = [0\ 0\ 0]^T$, once again Remark 5 has been used. As expected, $\hat{x}$ goes towards x, thus state estimation is achieved.

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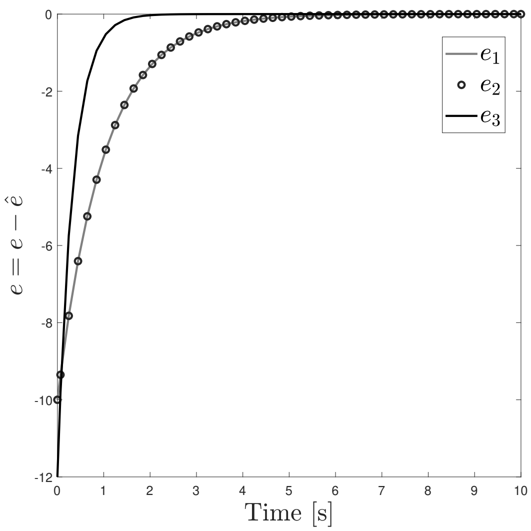

Fig. 3. Error signals in Example 6.

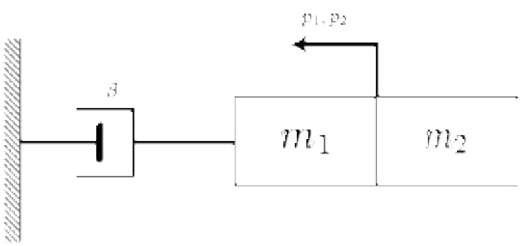

Fig. 4. Two-Mass-damper.

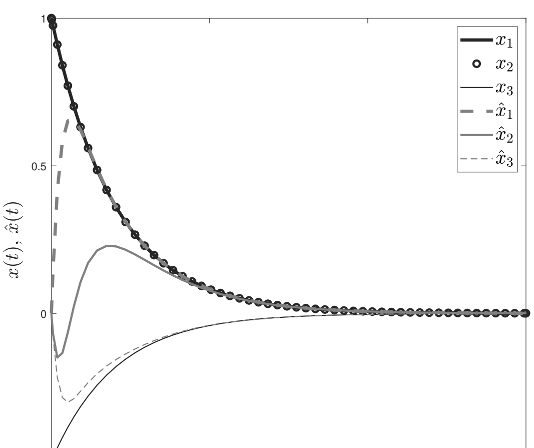

Fig. 5. States and their estimations in Example 7.

5. CONCLUSION

It has been presented a PD observer for singular linear systems. The main objective of this observer is to accurately estimate the state variables of the system, even under singular conditions. The observer gains have been computed by means of LMIs. Through numerical examples, the performance and behavior of the proposed observer have been thoroughly evaluated. The design of unknown input observers for diagnosis of singular systems is left as future work.

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