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Optimal investment in a market with borrowing, quadratic-affine interest rates, and Heston stochastic volatility

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Abstract: We consider the problem of optimal investment in an incomplete market with borrowing, random and possibly unbounded coefficients, and the power utility from terminal wealth. We use the *Heston* model for stochastic volatility, and the *quadratic-affine* model for interest rates. The resulting problem is an example of optimal stochastic control problem with a *nonlinear* system dynamics which is due to borrowing, the square-root non-linearity of Heston model, and the quadratic non-linearity of the interest rates. Explicit closed-form solution is obtained by a certain *piece-wise* completion of squares method. The resulting optimal control law is of linear state-feedback form the gain of which can be in up to three different regimes.

Keywords: Optimal control, stochastic nonlinear systems, optimal investment, borrowing, quadratic-affine interest rates, Heston volatility.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t),\, t \geq 0), \mathbb{P})$ be a complete filtered probability space on which an m-dimensional standard Brownian motion $(W(t),\, t \geq 0)$ and an $\tilde{m}$-dimensional standard Brownian motion $(\tilde{W}(t),\, t \geq 0)$ are defined. We further assume that W and $\tilde{W}$ are independent, and that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s), \tilde{W}(s)\colon 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. If E is an Euclidean space, then we denote by $L^{\infty}(0,T;E)$ the set of all E-valued uniformly bounded functions. Consider a two asset financial market consisting of a bond and of a stock, the prices of which B and S, respectively, are:

$$\begin{cases} dB(t) = B(t)r(t)dt, \quad t \in [0,T], \\ dS(t) = S(t)[\mu(t)dt + V'(t)dW(t)], \quad t \in [0,T] \\ B(0) > 0, \quad S(0) > 0, \quad \text{are given,} \end{cases} \tag{1}$$

and for some $T > 0$. The interest rate r, the appreciation rate of the stock μ, and the m-dimensional volatility of the stock V, are random processes in general and such that equations (1) have unique strong solutions. In this market, consider an investor that has an initial wealth y_0, and holds $v_B(t)$ number of shares in the bond and $v_S(t)$ number of shares in the stock at time t, i. e. the investor's wealth (or the value of investor's portfolio) at time t is $y(t) := v_B(t)B(t) + v_S(t)S(t)$. The investor is said to hold a self-financing portfolio if:

$$dy(t) = v_B(t)dB(t) + v_S(t)dS(t). \tag{2}$$

By substituting the differentials of B and S from (1) into (2), and knowing that $v_B(t)B(t) = y(t) - v_S(t)S(t)$, we obtain:

$$dy(t) = [r(t)y(t) + (\mu(t) - r(t))u(t)]dt + u(t)V'(t)dW(t), \tag{3}$$

where $u(t) := v_S(t)S(t)$. The well-known *optimal investment problem* with expected utility from terminal wealth, as one of fundamental mathematical finance problems, is the following optimal stochastic control problem:

$$\begin{cases} \max\limits_{u(\cdot)\in\mathcal{B}} \mathbb{E}\left[U(y(T))\right], \\ \text{s.t. (3),} \end{cases} \tag{4}$$

for some suitable admissible set of controls $\mathcal{B}$ and a utility function U (see, for example, Korn (1997), Kartzas and Shreve (1998), Duffie (2010), for a textbook account). If the coefficients r, μ, and V are deterministic, and $U(x) = x^{\gamma}/\gamma$ with $\gamma \in (0,1)$, then this problem is known as the *Merton problem*, and it admits an explicit closed-form solution in a linear wealth-feedback form (see, for example, Kartzas and Shreve (1998), Korn (1997), Merton (1992)). A more general market model than (1) is the one where an investor can *borrow* at a rate R that is higher than the bond rate r. The *borrowed amount* ζ in this case is:

$$\zeta(t) := [u(t) - y(t)]^{+} := \max[0, u(t) - y(t)], \quad t \in [0,T],$$

as it is not reasonable to borrow with the higher rate R and at the same time invest in the bond with a lower rate r. The equation of investor's wealth (3) now becomes:

$$\begin{aligned} dy(t) &= [r(t)y(t) + (\mu(t) - r(t))u(t)]dt - (R(t) - r(t))\zeta(t)dt \\ &+ u(t)V'(t)dW(t) = [r(t)y(t) + (\mu(t) - r(t))u(t)]dt \\ &- (R(t) - r(t))[u(t) - y(t)]^{+}dt + u(t)V'(t)dW(t). \end{aligned} \tag{5}$$

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The optimal investment (and consumption) problem with borrowing at the higher rate R was introduced in Fleming and Zariphopoulou (1991) for the market with constant coefficients. In Cvitanić and Karatzas (1992), this problem in a market with random coefficients is considered, and explicit solution to the optimal investment problem with logarithmic utility is obtained, whereas for the power utility it was assumed that the coefficients are deterministic. In the recent series papers Alasmi and Gashi (2023), Alasmi and Gashi (2024), Aljalal and Gashi (2022b), Aljalal and Gashi (2022a), Aljalal and Gashi (2022d), Aljalal and Gashi (2022c), Aljalal and Gashi (2023), several cases of the optimal investment problem in market with borrowing, random interest rates, and the power utility have been considered, and it was succeeded in obtaining an explicit closed-form solution in the following cases: in Aljalal and Gashi (2022b) the interest rate is assumed to be quadratic-affine with independent source of uncertainty as compared to the stock; in Aljalal and Gashi (2022c) this was generalised further to permit for a general class of such coefficients; in Aljalal and Gashi (2022d) the Hull-White model for the interest rate was used which has the same source of uncertainty as the stock; in Aljalal and Gashi (2022a) the market with a Markovian switching coefficients is considered; whereas in Alasmi and Gashi (2023), Alasmi and Gashi (2024), Aljalal and Gashi (2023), markets with certain combined random interest rate models are introduced.

Although the previously mentioned works have considered several market models with borrowing and random coefficients, they do not cover the important case of a market with volatility V having the *Heston model* (see, for example, Heston (1993), Shreve (2004), Desmettre et al. (2015), Bergomi (2015), Rouah (2015), Rouah (2013), Mikhailov and Nögel (2004))). In order to define this model, let η denote the solution to the following nonlinear stochastic differential equation of Cox-Ingersoll-Ross (CIR) type (for $t \in [0,T]$):

$$\begin{cases} d\eta(t) = [\kappa(t)(\theta(t) - \eta(t))]dt + \sqrt{\eta(t)}\tilde{\sigma}'(t)dW(t), \\ \eta(0) = \eta_0 > 0, \quad \text{is given,} \end{cases} \tag{6}$$

where $0 < \kappa(\cdot) \in L^\infty(0,T;\mathbb{R})$, $0 < \theta(\cdot) \in L^\infty(0,T;\mathbb{R})$, $\tilde{\sigma}(\cdot) \in L^\infty(0,T;\mathbb{R}^m)$, and $\tilde{\sigma}'(t)\tilde{\sigma}(t) > 0$ for all $t \in [0,T]$. The Heston volatility model is now defined as:

$$V(t) := \sigma(t)\sqrt{\eta(t)}, \quad t \in [0,T],$$

where $\sigma(\cdot) \in L^\infty(0,T;\mathbb{R}^m)$, and $\sigma'(t)\sigma(t) > 0$ for all $t \in [0,T]$. The wealth equation (5) now becomes (for $t \in [0,T]$):

$$\begin{aligned} dy(t) &= \left[r(t)y(t) + a(t)u(t) - b(t)[u(t) - y(t)]^+\right] dt \\ &\quad + u(t)\sqrt{\eta(t)}\sigma'(t)dW(t), \quad y(0) = y_0, \end{aligned} \tag{7}$$

where $a(t) := \mu(t) - r(t)$, $b(t) := R(t) - r(t)$, for $t \in [0,T]$. In this paper, we consider the problem of optimal investment in a market with borrowing, and the Heston stochastic volatility model. Moreover, our assumptions on the other market coefficients are as follows. Consider the following n-dimensional *factor process* (for $t \in [0,T]$):

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)]\, dt + C(t)d\widetilde{W}(t), \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given,} \end{cases} \tag{8}$$

where $A(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $B(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$, $C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times\tilde{m}})$. We assume that the interest rate r is defined as (for $t \in [0,T]$):

$$r(t) := x'(t)D_2(t)x(t) + x'(t)D_1(t) + D_0(t) + \beta(t)\eta(t), \tag{9}$$

where $D_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$ and symmetric, $D_1(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$, $D_0(\cdot) \in L^\infty(0,T;\mathbb{R})$, and $\beta(\cdot) \in L^\infty(0,T;\mathbb{R})$. We further assume that (for $t \in [0,T]$):

$$a(t) := \phi(t)\eta(t), \quad b(t) := \xi(t)\eta(t), \tag{10}$$

where $0 < \phi(\cdot) \in L^\infty(0,T;\mathbb{R})$, $0 < \xi(\cdot) \in L^\infty(0,T;\mathbb{R})$ (see, for example, Liu (2007) for a similar assumption on a). The model (9) with $\beta(t) = 0$, $t \in [0,T]$, is the well-known quadratic-affine interest rate model (see, for example, Liu (2007), Date and Gashi (2013), Gashi and Zhang (2023), Hua et al. (2023), Alasmi and Gashi (2023), Aljalal and Gashi (2022b), Aljalal and Gashi (2022d), Aljalal and Gashi (2023)), Algoulity and Gashi (2023), and by considering a not-necessarily zero β, i.e. by permitting for the interest rate r to be influenced by the additional factor η, the model (9) is a generalisation of the quadratic-affine interest rate.

The cost functional that we consider is the following power utility from terminal wealth:

$$J(u(\cdot)) := -\frac{1}{\gamma}\mathbb{E}\left[y^\gamma(T)\right], \quad \gamma \in (0,1). \tag{11}$$

which, due to its minus sign, is to be minimized. The optimal investment problem to be solved is the following optimal stochastic control problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \text{s.t. (7),} \end{cases} \tag{12}$$

where $\mathcal{A}$ is a suitable set of admissible controls to be defined precisely in the next section. Despite the considerable progress on the optimal investment problem in a market with borrowing that we previously mentioned, none of those works fully covers problem (12). For example, the existence result of Cvitanić and Karatzas (1992) only applies to complete markets (i.e. to the case of $m = 1$) and no explicit solution is given. We use a certain piecewise completion of squares method to find the solution in an explicit closed form. This approach has been successfully used in Alasmi and Gashi (2023), Alasmi and Gashi (2024), Aljalal and Gashi (2022b), Aljalal and Gashi (2022a), Aljalal and Gashi (2022d), Aljalal and Gashi (2022c), Aljalal and Gashi (2023), to find the solution to optimal investment problem with borrowing in market models that do not cover the one considered in the present paper. An added difficulty here is the poof of admissibly for optimal control, which is more involved due to the Heston stochastic volatility model. The solution turns out to be of a linear state-feedback form despite the fact that dynamics of y, r, and η are nonlinear. In §2 we give the precise formulation of admissible set $\mathcal{A}$ and derive the solution to problem (12).

2. SOLUTION TO THE OPTIMAL INVESTMENT PROBLEM WITH POWER UTILITY

In order to complete the formulation of problem (12), we now give the definition of admissible set of controls

$\mathcal{A}$. Consider the following nonlinear backward ordinary differential equation:

$$\begin{cases} \dot{g} + 0.5g^2\tilde{\sigma}'\tilde{\sigma} - g\kappa + \gamma\beta - \gamma N = 0, & t \in [0,T], \\ g(T) = 0. \end{cases} \quad (13)$$

Here N is a certain function of g defined as ($t \in [0,T]$):

$$N(t) := \begin{cases} -\dfrac{\Psi^2(t)}{2\rho(t)} & \text{if } \frac{\Psi(t)}{\rho(t)} \leq 1, \\ \dfrac{1}{2}\rho(t) - \Psi(t) & \text{if } 1 < \frac{\Psi(t)}{\rho(t)} \leq 1 + \frac{\xi(t)}{\rho(t)}, \\ -\dfrac{(\Psi(t)-\xi(t))^2}{2\rho(t)} - \xi(t) & \text{if } \frac{\Psi(t)}{\rho(t)} > 1 + \frac{\xi(t)}{\rho(t)}, \end{cases}$$

where $\rho(t) := (1-\gamma)\sigma'(t)\sigma(t)$ and $\Psi(t) := \phi(t) + \sigma'(t)\tilde{\sigma}(t)g(t)$. As Ψ is a linear function of g, equation (13) has an at most quadratic growth in g.

Assumption 1. *The nonlinear backward ordinary differential equation (13) has a unique global solution.*

In the special case of $\sigma'(t)\tilde{\sigma}(t) = 0$ for $t \in [0,T]$, equation (13) reduces to a Riccati backward ordinary differential equation (see, e.g., Rami and Zhou (2000) and Rami et al. (2001), for sufficient conditions under which Assumption 1 holds). Further consider the following *Riccati* backward ordinary differential equation (for $t \in [0,T]$):

$$\begin{cases} \dot{H}_2 + H_2 A + A' H_2 + 2H_2 CC' H_2 + \gamma D_2 = 0, \\ H_2(T) = 0, \end{cases} \quad (14)$$

Assumption 2. *The Riccati backward ordinary differential equation (14) has a unique global solution.*

Sufficient conditions for this assumption to hold can be derived from the results in Rami and Zhou (2000) and Rami et al. (2001). As an example, if $D_2(t) = 0$ for $t \in [0,T]$, then by inspection we conclude that $H_2(t) = 0$, $t \in [0,T]$, is the unique solution to (14). Furthermore, we introduce the following backward ordinary differential equations (for $t \in [0,T]$):

$$\begin{cases} \dot{H}_1 + 2H_2 B + A' H_1 + 2H_2 CC' H_1 + \gamma D_1 = 0, \\ H_1(T) = 0, \end{cases} \quad (15)$$

$$\begin{cases} \dot{P} + \gamma P D_0 + P H_1' B + P\, tr\,(C' H_2 C) \\ + P g\kappa\theta + P H_1' CC' H_1/2 = 0, \\ P(T) = -\gamma^{-1}. \end{cases} \quad (16)$$

As (15) and (16) are linear in H_1 and P, respectively, our Assumption 1 and Assumption 2, ensure the existence of unique global solutions for these equations.

The admissible set of controls $\mathcal{A}$ is defined as the set of all $\mathbb{R}$-valued adapted processes u under which the wealth equation (7) has a unique and strictly positive strong solution, i.e. $y(t) > 0$ *a.s.* for all $t \in [0,T]$, and the following integrability conditions hold:

$$\mathbb{E}\left[\int_0^T P e^Z y^\gamma \sqrt{\eta}\left(\gamma\frac{u}{y}\sigma' + g\tilde{\sigma}'\right) dW\right] = 0, \quad (17)$$

$$\mathbb{E}\left[\int_0^T P e^Z y^\gamma \left(2x' H_2 + H_1'\right) C d\widetilde{W}\right] = 0. \quad (18)$$

Here the process Z is defined as the following quadratic affine form in x (for $t \in [0,T]$):

$$Z(t) := x'(t)H_2(t)x(t) + x'(t)H_1(t) + g(t)\eta(t).$$

The strict positivity requirement on the wealth y avoids investor's bankruptcy, whereas the above integrability conditions ensures that certain stochastic integrals appearing in the proof of Theorem 2.1 have a zero expectation. This completes the formulation of problem (12) to be solved below.

The expected value and the variance-covariance matrix of the factor process (8) are denoted as (for $t \in [0,T]$) :

$$\mu_x(t) := \mathbb{E}[x(t)], \quad \Sigma_x(t) = \mathbb{E}[(x(t)-\mu_x(t))(x(t)-\mu_x(t))'].$$

These are solutions to the following forward differential equations (see, for example, Kloeden and Platen (1992)):

$$\dot{\mu}_x = A\mu_x + B, \quad \mu_x(0) = x_0,$$

$$\dot{\Sigma}_x = A\Sigma_x + \Sigma_x A' + B\mu_x' + \mu_x B + CC', \quad \Sigma_x(0) = 0,$$

In order to state our assumptions under which we solve problem (12), as well as its solution, we introduce the function M as (for $t \in [0,T]$):

$$M(t) := \begin{cases} \dfrac{\Psi(t)}{\rho(t)} & \text{if } \frac{\Psi(t)}{\rho(t)} \leq 1, \\ 1 & \text{if } 1 < \frac{\Psi(t)}{\rho(t)} \leq 1 + \frac{\xi(t)}{\rho(t)}, \\ \dfrac{\Psi(t)-\xi(t)}{\rho(t)} & \text{if } \frac{\Psi(t)}{\rho(t)} > 1 + \frac{\xi(t)}{\rho(t)}. \end{cases}$$

Assumption 3. *Let $q_1, q_2, p_1, p_2 > 1$ be such that $q_1^{-1} + q_2^{-1} = 1$ and $p_1^{-1} + p_2^{-1} = 1$. The following hold:*
(i) $\Sigma_x(t) > 0$, $\Sigma_x^{-1}(t) - 32\, H_2(t) > 0$ for $t \in [0,T]$,
(ii) $4\kappa(t)\theta(t) \leq \tilde{\sigma}'(t)\tilde{\sigma}(t)$ for $t \in [0,T]$,
(iii) $O(t)t\displaystyle\int_0^t e^{\int_0^s \kappa(\tau)d\tau}\tilde{\sigma}'(s)\tilde{\sigma}(s)ds < 2$ for $t \in [0,T]$,
where $O(t) := \sup\limits_{s\in[0,t]} [m_2(-16g\kappa + 128g^2 m_1\tilde{\sigma}'\tilde{\sigma})]$,
(iv) $G(t)t\displaystyle\int_0^t e^{\int_0^s \kappa(\tau)d\tau}\tilde{\sigma}'(s)\tilde{\sigma}(s)ds < 2$ for $t \in [0,T]$,
where $G(t) := \sup\limits_{s\in[0,t]} [8\gamma p_2 q_2\alpha + 32\gamma^2 q_1 q_2 p_2^2 M\sigma'\sigma]$
with $\alpha(t) := \beta + \phi M - \xi[M-1]^+ - \frac{1}{2}M^2\sigma'\sigma$,
(v) $\mathbb{E}\Big[\exp\Big(8\gamma p_1\displaystyle\int_0^t F ds\Big)\Big] < \infty$ for $t \in [0,T]$,
where $F(t) := x'(t)D_2(t)x(t) + x'(t)D_1(t) + D_0(t)$.

All of the above assumption are in terms of known functions and processes, and can thus be verified for each concrete market example. These are more involved than in the previous works that use quadratic-affine interest rates (see Alasmi and Gashi (2023) and Aljalal and Gashi (2022b)) due to the Heston volatility model.

Theorem 2.1. *There exists a unique solution to the optimal investment problem (12). This solution is given as:*

$$u^*(t) = M(t)y(t), \qquad t \in [0,T].$$

The optimal cost functional is $J(u^*(\cdot)) = y_0^\gamma P(0)e^{Z(0)}$. □

Prior to giving the proof of this theorem, which is our main result, we derive sufficient conditions for the finiteness of $\mathbb{E}[e^{\lambda\eta(t)}]$, where $\lambda \in \mathbb{R}$. Let $m_1, m_2 > 1$ be such that ${m_1}^{-1} + {m_2}^{-1} = 1$. Also let $\ell(t) := m_2(-\lambda\kappa + \lambda^2 m_1\tilde{\sigma}'\tilde{\sigma}/2)$, and define the function Q as:

$$Q(t) := \sup_{s\in[0,t]} \ell(s), \quad t \in [0,T]$$

Lemma 2.2. *If* $4\kappa(t)\theta(t) \le \tilde{\sigma}'(t)\tilde{\sigma}(t)$ *and*

$$Q(t)t \int_0^t e^{\int_0^s \kappa(\tau)d\tau} \tilde{\sigma}'(s)\tilde{\sigma}(s)ds < 2,$$

then $\mathbb{E}\left[e^{\lambda\eta(t)}\right] < \infty$.

Proof. As the implicit solution of (6) is

$$\eta(t) = \eta_0 + \int_0^t \kappa(\theta-\eta)ds + \int_0^t \tilde{\sigma}'\sqrt{\eta}d,$$

we can write $\mathbb{E}\left[e^{\lambda\eta(t)}\right]$ as:

$$\mathbb{E}\left[e^{\lambda\eta(t)}\right] = e^{\left(\lambda\eta_0+\lambda\int_0^t \kappa\theta ds\right)} \mathbb{E}\Bigg[\exp\Bigg(-\lambda\int_0^t \kappa\eta ds$$
$$+\lambda\int_0^t \tilde{\sigma}'\sqrt{\eta}dW\Bigg)\Bigg]$$
$$= e^{\left(\lambda\eta_0+\lambda\int_0^t \kappa\theta ds\right)} \mathbb{E}\Bigg[\exp\Bigg(-\lambda\int_0^t \kappa\eta ds - \frac{1}{m_1}\int_0^t -\lambda m_1$$
$$\times\tilde{\sigma}'\sqrt{\eta}dW - \frac{1}{2m_1}\int_0^t (\lambda m_1\sqrt{\eta})^2\tilde{\sigma}'\tilde{\sigma}ds + \frac{1}{2m_1}$$
$$\times\int_0^t (\lambda m_1\sqrt{\eta})^2\tilde{\sigma}'\tilde{\sigma}ds\Bigg)\Bigg]$$
$$\le e^{\left(\lambda\eta_0+\lambda\int_0^t \kappa\theta ds\right)} \left\{\mathbb{E}\left[\exp\left(\int_0^t \ell\eta ds\right)\right]\right\}^{\frac{1}{m_2}}$$
$$\times\left\{\mathbb{E}\left[\exp\left\{\left(\frac{-1}{2}\int_0^t (\lambda m_1\sqrt{\eta})^2\tilde{\sigma}'\tilde{\sigma}ds\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\int_0^t -\lambda m_1\sqrt{\eta}\tilde{\sigma}'dW\right)\right\}\right]\right\}^{\frac{1}{m_1}}$$
$$\le e^{\lambda\eta_0+\lambda\int_0^t \kappa\theta ds}\left\{\mathbb{E}\left[\exp\left(Q\int_0^t \eta ds\right)\right]\right\}^{\frac{1}{m_2}}$$
$$< \infty,$$

where the first inequality above is due to the Hölder inequality, the second due to the supermartingale property, and the third due to Theorem 4.1 of Yong (2004). □

Proof of Theorem 2.1 By Itô's formula, the differential of y^γ is:

$$dy^\gamma = y^\gamma[\gamma r + \gamma a\nu - \gamma b[\nu-1]^+ + \gamma(\gamma-1)\sigma'\sigma\eta\nu^2/2]dt$$
$$+y^\gamma\nu\gamma\sqrt{\eta}\sigma'dW,$$

where $\nu(t) := u(t)/y(t)$, $t \in [0,T]$. Further, as H_2 is symmetric, the differential of Z by Itô's formula is:

$$dZ = \Big\{x'\left[\dot{H}_2 + H_2A + A'H_2\right]x + x'\left[\dot{H}_1 + 2H_2B + A'H_1\right]$$
$$+\left[H_1'B + tr\left(C'H_2C\right) + g\kappa\theta\right] + [\dot{g} - g\kappa]\eta\Big\}dt$$
$$+\left[2x'H_2 + H_1'\right]Cd\widetilde{W} + g\tilde{\sigma}'\sqrt{\eta}dW.$$

Again by Itô's formula, the differential of Pe^Z is:

$$d(Pe^Z) = e^Z\Big\{\dot{P} + P\Big[x'(\dot{H}_2 + H_2A + A'H_2)x + x'(\dot{H}_1$$
$$+2H_2B + A'H_1) + H_1'B + tr(C'H_2C) + g\kappa\theta + [\dot{g} - g\kappa]\eta$$
$$+\frac{1}{2}(2x'H_2 + H_1')CC'(2x'H_2 + H_1')' + \frac{1}{2}g^2\tilde{\sigma}'\tilde{\sigma}\eta\Big]\Big\}dt$$
$$+Pe^Z(2x'H_2 + H_1')Cd\widetilde{W} + Pe^Z g\tilde{\sigma}\sqrt{\eta}dW.$$

By Ito's product rule, the differential of $y^\gamma Pe^Z$ is:

$$d\left(y^\gamma Pe^Z\right) = (dy^\gamma)Pe^Z + y^\gamma d\left(Pe^Z\right) + (dy^\gamma)d\left(Pe^Z\right)$$
$$= y^\gamma e^Z\Big\{P\gamma\eta\Big[\phi\nu - \xi[\nu-1]^+ + \frac{1}{2}(\gamma-1)\nu^2\sigma'\sigma$$
$$+\nu\sigma'\tilde{\sigma}g\Big] + x'\Big(P\dot{H}_2 + PH_2A + PA'H_2 + 2PH_2CC'H_2$$
$$+\gamma PD_2\Big)x + x'\Big(P\dot{H}_1 + 2PH_2B + PA'H_1 + 2PH_2CC'H_1$$
$$+\gamma PD_1\Big) + \dot{P} + PH_1'B + P\ tr\left(C'H_2C\right) + \gamma PD_0$$
$$+Pg\kappa\theta + \frac{P}{2}H_1'CC'H_1 + P\eta\Big[\dot{g} - \kappa g + \gamma\beta + \frac{1}{2}g^2\tilde{\sigma}'\tilde{\sigma}\Big]\Big\}dt$$
$$+y^\gamma Pe^Z\left(2x'H_2 + H_1'\right)Cd\widetilde{W} + y^\gamma Pe^Z\sqrt{\eta}(\gamma v\sigma' + g\tilde{\sigma}')dW.$$

By integrating both sides of the above equation from 0 to T, and then taking the expectation, we obtain the following for any admissible control (note that due to the integrability requirements (17) and (18) the expectation of stochastic integrals are zero):

$$\frac{-1}{\gamma}\mathbb{E}[y^\gamma(T)] = y_0^\gamma P_0 e^{Z_0} + \mathbb{E}\Big[\int_0^T y^\gamma e^Z\Big\{P\gamma\eta\Big[\phi\nu$$
$$-\xi[\nu-1]^+ + \frac{1}{2}(\gamma-1)\nu^2\sigma'\sigma + \nu\sigma'\tilde{\sigma}g\Big] + x'\Big(P\dot{H}_2$$
$$+PH_2A + PA'H_2 + 2PH_2CC'H_2 + \gamma PD_2\Big)x + x'\Big(P\dot{H}_1$$
$$+2PH_2B + PA'H_1 + 2PH_2CC'H_1 + \gamma PD_1\Big) + PH_1'B$$
$$+\dot{P} + P\ tr\left(C'H_2C\right) + P\gamma D_0 + Pg\kappa\theta + \frac{P}{2}H_1'CC'H_1$$
$$+P\eta\Big[\dot{g} - \kappa g + \gamma\beta + \frac{1}{2}g^2\tilde{\sigma}'\tilde{\sigma}\Big]\Big\}dt.$$

An integral representation of J for all admissible controls is thus:

$$J(u(\cdot)) = y_0^\gamma P_0 e^{Z_0} + \mathbb{E}\Big[\int_0^T y^\gamma e^Z\Big\{P\gamma\eta\Big[\phi\nu - \xi[\nu-1]^+$$
$$+\frac{1}{2}(\gamma-1)\nu^2\sigma'\sigma + \nu\sigma'\tilde{\sigma}g\Big] + P\eta\Big[\dot{g} - \kappa g + \gamma\beta$$
$$+\frac{1}{2}g^2\tilde{\sigma}'\tilde{\sigma}\Big]\Big\}dt. \tag{19}$$

The terms of the integrand in the above representation of J that depend on ν are defined as:

$$f(\nu) := \frac{\rho}{2}\nu^2 - \psi\nu + \xi\left[\nu-1\right]^+.$$

By a piece-wise completion of squares, we write $f(\nu)$ as a piece-wise quadratic function:

$$f(\nu)=\left[\frac{1}{2}\rho\left(\nu-\frac{\psi}{\rho}\right)^2-\frac{\psi^2}{2\rho}\right]I_{(\nu(t)\leq 1)}$$
$$+\left[\frac{\rho}{2}\left(\nu-\frac{\psi-\xi}{\rho}\right)^2-\frac{(\psi-\xi)^2}{2\rho}-\xi\right]I_{(\nu(t)>1)},$$

where $I_{(\cdot)}$ is the indicator function. This piece-wise quadratic representation of $f(\nu)$ permits us to derive the following:

$$\min_{\nu} f(\nu)=N,$$

and the corresponding minimizer is $\nu^*=M$. We can now obtain the following lower bound for the cost functional (19) for any admissible control:

$$J(u(\cdot))=y_0^\gamma P_0 e^{Z_0}+\mathbb{E}\bigg\{\int_0^T e^Z y^\gamma\eta\Big[-P\gamma(f(\nu)-N)$$
$$+P(\dot{g}-\kappa g+\beta+\frac{1}{2}g^2\tilde{\sigma}'\tilde{\sigma}'-\gamma N)\Big]dt\bigg\}\geq y_0^\gamma P(0)e^{Z(0)}.$$

This lower bound is achieved if and only if $\nu^*(t)=\nu(t)=M(t)$ *a.e.* $t\in[0,T]$ *a.s.*, or equivalently, if and only $u^*(t)=M(t)y(t)$ *a.e.* $t\in[0,T]$ *a.s.*. The corresponding optimal cost functional is thus $J(u^*(\cdot))=y_0^\gamma P(0)e^{Z(0)}$. To show that $u^*(\cdot)\in\mathcal{A}$, we first introduce the following processes (for $t\in[0,T]$):

$$\begin{aligned}
\Pi(t) &:= r(t)+a(t)M(t)-b(t)[M(t)-1]^+,\\
&= x'(t)D_2(t)x(t)+x'(t)D_1(t)+D_0(t)+\beta(t)\eta(t)\\
&\quad+\phi(t)\eta(t)M(t)-\xi(t)\eta(t)[M(t)-1]^+,\\
\Sigma(t) &:= M(t)\sigma(t)\sqrt{\eta(t)},\\
Y(t) &:= y_0\left\{\exp\left[\int_0^t\left(\Pi(t)-\Sigma'(t)\Sigma(t)/2\right)ds\right.\right.\\
&\quad\left.\left.+\int_0^t\Sigma'(t)dW(t)\right]\right\}.
\end{aligned}$$

As u^* is of linear state-feedback form and $M(\cdot)\in L^\infty(0,T;\mathbb{R})$, it follows (in the same way as in the proof of Lemma 3.1 of Alasmi and Gashi (2023)) that under u^* the wealth equation (7) has a unique strong solution given explicitly as $y(t)=Y(t)$, which in particular means that $y(t)>0$ *a.s.* for all $t\in[0,T]$. Moreover, the requirements (17) and (18) hold if the integrands appearing there are square-integrable processes. It is thus sufficient to show that:

$$\mathbb{E}\left[e^{2Z}y^{2\gamma}x'x\right]<\infty,\quad t\in[0,T], \tag{20}$$
$$\mathbb{E}\left[e^{2Z}y^{2\gamma}\eta\right]<\infty,\quad t\in[0,T]. \tag{21}$$

An upper bound on the left-hand side of (20) is:

$$\begin{aligned}
\mathbb{E}\left[e^{2Z}y^{2\gamma}x'x\right] &\leq \mathbb{E}\left[\frac{1}{2}e^{4Z}y^{4\gamma}+\frac{1}{2}\left(x'x\right)^2\right]\\
&\leq \frac{1}{8}\mathbb{E}\left[e^{16\left(x'H_2x+x'H_1\right)}\right]+\frac{1}{8}\mathbb{E}\left[e^{16g\eta}\right]\\
&\quad+\frac{1}{4}\mathbb{E}\left[y^{8\gamma}\right]+\frac{1}{2}\mathbb{E}\left[(x'x)^2\right].
\end{aligned} \tag{22}$$

By Lemma 2.2, and Assumption 3 (ii) and (iii), we conclude that the second term $\mathbb{E}\left[e^{16g\eta}\right]$ of (22) is finite. The fourth term $\mathbb{E}\left[(x'x)^2\right]$ of (22) is finite since all moments of x are finite (see, for example, Theorem 1.6.16 of Yong and Zhou (1999)). The first term $\mathbb{E}\left[e^{16\left(x'H_2x+x'H_1\right)}\right]$ of (22) is also finite. Indeed, due to Assumption 3 (i), we have $x(t)\sim N(\mu_x(t),\Sigma_x(t))$, and since $\Sigma_x^{-1}(t)-32H_2(t)>0$, it follows that the first term of (22) finite. To show that the third term of (22) is finite, we compute $\mathbb{E}\left[y^{8\gamma}\right]$ as:

$$\begin{aligned}
&\mathbb{E}\left[y^{8\gamma}\right]=y_0^{8\gamma}\mathbb{E}\bigg\{\exp\bigg[8\gamma\int_0^t\left(\Pi-\Sigma'\Sigma/2\right)ds\\
&+8\gamma\int_0^t\Sigma'dW\bigg]\bigg\}\\
&\leq y_0^{8\gamma}\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_1\int_0^t(x'D_2x+x'D_1+D_0)ds\bigg)\bigg]\bigg\}^{\frac{1}{p_1}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_2\int_0^t\eta(\beta+\phi M-\xi[M-1]^+\\
&-M^2\sigma'\sigma/2)ds+8\gamma P_2\int_0^t M\sigma'\sqrt{\eta}dW\bigg)\bigg]\bigg\}^{\frac{1}{p_2}}\\
&\leq y_0^{8\gamma}\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_1\int_0^t(x'D_2x+x'D_1+D_0)ds\bigg)\bigg]\bigg\}^{\frac{1}{p_1}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_2\int_0^t\eta\alpha ds-\frac{1}{q_1}\int_0^t(-8q_1p_2\gamma M\sigma'\sqrt{\eta})dW\\
&-\frac{1}{2q_1}\int_0^t q_1^2(8\gamma P_2M\sqrt{\eta})^2\sigma'\sigma ds+\frac{1}{2q_1}\int_0^t q_1^2(8\gamma P_2M\sqrt{\eta})^2\\
&\times\sigma'\sigma ds\bigg)\bigg]\bigg\}^{\frac{1}{p_2}}\\
&\leq y_0^{8\gamma}\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_1\int_0^t(x'D_2x+x'D_1+D_0)ds\bigg)\bigg]\bigg\}^{\frac{1}{p_1}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_2q_2\int_0^t\eta\alpha ds+32q_1q_2p_2^2\gamma^2\\
&\int_0^t M\eta\sigma'\sigma ds\bigg)\bigg]^{\frac{1}{q_2}}\times\bigg[\mathbb{E}\bigg(\exp\bigg\{-\frac{1}{2}\int_0^t q_1^2(-8p_2\gamma\\
&\times M\sqrt{\eta})^2\sigma'\sigma ds\\
&-q_1\int_0^t(-8p_1\gamma M\sigma'\sqrt{\eta})dW\bigg\}\bigg)\bigg]^{\frac{1}{q_1}}\bigg)\bigg]\bigg\}^{\frac{1}{p_2}}\\
&\leq y_0^{8\gamma}\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_1\int_0^t F ds\bigg)\bigg]\bigg\}^{\frac{1}{p_1}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(\int_0^t\eta h ds\bigg)\bigg]\bigg\}^{\frac{1}{p_2q_1}}\\
&\leq y_0^{8\gamma}\bigg\{\mathbb{E}\bigg[\exp\bigg(8\gamma p_1\int_0^t F ds\bigg)\bigg]\bigg\}^{\frac{1}{p_1}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(G\int_0^t\eta ds\bigg)\bigg]\bigg\}^{\frac{1}{p_2q_1}}.
\end{aligned}$$

The first term in last inequality above is finite due to Assumption 3 (v), whereas the second term is finite due to Assumption 3 (ii), Assumption 3 (iv), and Lemma 2.2. We conclude that (23) is finite, and thus complete the proof that condition (20) holds. Note that the only difference between conditions (20) and (21) is that (20) contains the term $x'x$ whereas (21) the term η. Since all moments of

η are finite, we can show that condition (21) holds in the same way as we did in our proof that (20) holds. □

3. CONCLUSIONS

We have solved the optimal investment problem with power utility from terminal wealth in a market with different interest rates for borrowing and lending, the volatility follows the Heston model, and the interest rate is a certain generalised version of the quadratic-affine model. This is an optimal stochastic control problem with nonlinear system dynamics and unbounded coefficients. An explicit closed-form solution is obtained as a linear wealth-feedback control law the gain of which can be in one of possibly three regimes (see the definition of M and u^*). The consideration of this problem in the more general setting of multi-asset markets and investors that can consume, would be more challenging and is currently open.

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