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# Optimal regulator for linear stochastic systems with multiple state-delay

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**Abstract:** We consider an optimal control problem for linear stochastic systems with *multiple state delays*. By formulating a general *quadratic-linear* cost functional, we provide an explicit solution involving coupled Riccati and partial differential equations. The derived optimal control law is in an *affine* feedback form with respect to the current state, the delayed state, and the integral of past state values. Additionally, we demonstrate the application of this solution to an optimal investment problem with logarithmic utility in market with interest rate influenced by a multi-delayed factor process.

*Keywords:* Optimal regulator, stochastic linear systems, multiple state-delay.

## 1. INTRODUCTION AND PROBLEM FORMULATION

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s)\colon 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the following *linear* stochastic control system with a *single state-delay* (for $t \in [0, T]$):

$$\begin{cases} dx(t) = [A^*x(t) + A_1^*x(t-h) + B^*u(t)]dt \\ \qquad +[\underline{A}^*x(t) + \underline{A}_1^*x(t-h) + \underline{B}^*u(t)]dW(t), \\ x(s) = \eta(s), \quad s \in [-h, 0], \quad \text{is given.} \end{cases} \tag{1}$$

Here $0 < h \in \mathbb{R}$ is the system delay; the constant coefficients $A^*, A_1^*, \underline{A}^*, \underline{A}_1^* \in \mathbb{R}^{n\times n}$, and $B^*, \underline{B}^* \in \mathbb{R}^{n\times m}$ are given; the $n$-dimensional initial value $\eta$ is assumed to be a continuous function on the interval $[-h, 0]$; and the adapted $m$-dimensional control process $u$ is such that equation (1) has a unique strong solution for the $n$-dimensional system state $x$ on the interval $[0, T]$ (see Mao (2007) for some sufficient conditions that ensure this). We associate with (1) the following *quadratic* cost functional:

$$I(u(\cdot)) := \mathbb{E}\bigg\{ \int_0^T [x'(t)Q^*x(t) + u'(t)R^*u(t)]dt + x'(T)H^*x'(T) \bigg\}, \tag{2}$$

for constant symmetric coefficients $Q^*, H^* \in \mathbb{R}^{n\times n}$, and $R^* \in \mathbb{R}^{m\times m}$. In Liang et al. (2018), the optimal control problem

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}^*} I(\cdot) \\ s.t. \quad (1) \end{cases} \tag{3}$$

for some suitable admissible se of controls $\mathcal{A}^*$, was considered and explicit closed-form solution obtained. The solution turns out to be of feedback form in the system state, the delayed state, and the integral of past system state values. The coefficients of such a control law are determined by the solution to certain system of coupled Riccati and partial differential equations, the solvability of which is assumed (see Liang et al. (2018) for details). This is an interesting result as not only it gives the solution in an explicit feedback closed-form, which is *rare* in optimal control, but it does so for the important class of systems with state-delay, which, as is well-known, appear in many applications and have been studied extensively for decades. The solution to (3) in an *open-loop* form can be obtained from the result of Chen and Zhang (2023). For some other related results, see, for example, Chen and Wu (2010), Huang et al. (2012), Kong and Chen (2016), Li et al. (2018), Li et al. (2020), Wu and Shu (2017).

In this paper, we *generalise* the problem (3) and find its solution in an explicit feedback closed-form. This generalisation involves the inclusion of *additive* noise, the system and cost coefficients that can be *time-varying*, the criterion of general *quadratic-linear* form, and a *two-period* state delay is also included. More precisely, let $L^\infty(0, T; E)$ denote the set of $E$-valued uniformly bounded functions, and $L^2_{\mathcal{F}}(0, T; E)$ the set of $E$-valued square-integrable adapted processes (with $E$ being an Euclidean space). Consider the following linear stochastic control system with up to two-period state-delay (for $t \in [0, T]$):

$$\begin{cases} dx(t) = \bigg[A(t)x(t) + \sum\limits_{i=1}^{2} A_i(t)x(t-ih) + B(t)u(t) \\ +C(t)\bigg]dt + \bigg[\bar{A}(t)x(t) + \sum\limits_{i=1}^{2} \bar{A}_i(t)x(t-ih) \\ +\bar{B}(t)u(t) + \bar{C}(t)\bigg]dW(t), \\ x(s) = \zeta(s), \quad s \in [-2h, 0], \quad \text{is given.} \end{cases} \tag{4}$$

Here the $n$-dimensional initial value $\zeta$ is assumed to be a continuous function on the interval $[-2h, 0]$, whereas for the other coefficients we assume that: $A(\cdot), A_i(\cdot), \bar{A}(\cdot), \bar{A}_i(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times n})$; $B(\cdot)$, $\bar{B}(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times m})$; $C(\cdot)$, $\bar{C}(\cdot) \in L^\infty(0, T; \mathbb{R}^n)$, where $i = 1, 2$. It is clear

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that (4), as compared to (1), is more general since the coefficients can be time-varying, additive noise is included, i.e. $\bar{C}$ is not necessarily zero, a *disturbance* $C$ is permitted in the system, and there is an up to two-period state-delay. Further, we consider the following cost functional:

$$\begin{aligned} J(u(t)) := \mathbb{E}\Bigg\{ \int_0^T \Bigg[ & x'(t)Q(t)x(t) + \sum_{i=1}^{2}\sum_{j=1}^{2} x'(t-ih) \\ & \times Q_{ij}(t)x(t-jh) + u'(t)R(t)u(t) + x'(t)\sum_{i=1}^{2} L_{1i}(t) \\ & \times x(t-ih) + \sum_{i=1}^{2} x'(t-ih) L_{2i}(t)x(t) + x'(t)Z(t) \\ & + u'(t)F(t)x(t) + u'(t)\sum_{i=1}^{2} F_i(t)x(t-ih) + \sum_{i=1}^{2} b_i'(t) \\ & \times x(t-ih) + S(t)'u(t) \Bigg] dt + x'(T)Hx(T) + f'x(T) \Bigg\} \end{aligned} \tag{5}$$

Here $Q(\cdot)$, $Q_{ij}(\cdot) \in L^\infty(0,T;\mathbb{S}^n)$, $L_{1i}(\cdot)$, $L_{2i}'(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $R(\cdot) \in L^\infty(0,T;\mathbb{S}^m)$, $F_1(\cdot)$, $F_i(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times n})$, $Z(\cdot)$, $b_i(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$, $S(\cdot) \in L^\infty(0,T;\mathbb{R}^m)$, $H \in \mathbb{S}^n$, $f \in \mathbb{R}^n$, $L_{1i}(t) = L_{2i}'(t)$ for $t \in [0,T]$, where $\mathbb{S}^n$ denotes the set of real-valued symmetric $n \times n$ matrices. The cost functional (5) is considerably more general than (1) since: the coefficients can be time-varying; it contains quadratic terms in the delayed state $x(t-ih)$, $i = 1,\ 2$; various cross products between the state, delayed state, and control appear; and linear terms in the state and control are included. Thus, the cost functional (5) represents a general quadratic-linear criterion. The *optimal regulator problem* to be considered is:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \text{s.t. } (4), \end{cases} \tag{6}$$

where $\mathcal{A} := L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ (which in particular ensures the existence of a unique strong solution of (4), see. for example, Mao (2007)). The motivation for considering this problem is two fold. Firstly, it is an important optimal control problem that is more general than the existing ones, and for which the explicit closed-form solution is derived. This solution turns out to be of an *affine* feedback form with respect to the system state, the delayed state, and the integral of past system state values. Our approach to finding the solution is a certain generalisation of the approach of Liang et al. (2018), and is given in §2 below. Secondly, examples of problem (6) appear in optimal investment. Indeed, the inclusion of additive noise permits for consideration of certain well-known interest rate models, such as the *Hull-White* model (see, for example, Shreve (2004), Musiela and Rutkowski (2006)), and the linear cost appears in an application given in §3. In what follows, we omit the argument $t$ whenever convenient for notation simplicity.

## 2. SOLUTION TO THE OPTIMAL REGULATOR PROBLEM

In order to state the solution to optimal regulator problem (6), we introduce two sets of linear and Riccati ordinary differential equations that are coupled with a system of partial differential equations. Firstly, on the intervals $t \in [T-2h, T]$, $\theta \in [t,T]$, $s \in [t,T]$, $i = 1,\ 2$, $j = 1,\ 2$, consider the equations:

$$\begin{cases} \dot{P} + PA + A'P + Q - G_1'\bar{R}^{-1}G_1 + \bar{A}'P\bar{A} = 0, \\ \bar{R} := R + \bar{B}'P\bar{B} > 0, \\ P(T) = H, \end{cases} \tag{7}$$

$$\begin{cases} \dot{q}' + q'A + 2C'P + 2\bar{C}'P\bar{A} - 2G_0'\bar{R}^{-1}G_1 + Z = 0, \\ q'(T) = f', \end{cases} \tag{8}$$

$$\begin{cases} \dot{k} + q'C + \bar{C}'P\bar{C} - G_0'\bar{R}^{-1}G_0 = 0, \\ k(T) = 0, \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{\partial N_{1i}'(t,\theta)}{\partial t} + 2C'N_{2i}(t,\theta) - 2G_0'\bar{R}^{-1}G_{3i}(t,\theta) = 0, \\ b_i + A_i'q + 2\bar{A}_i'P\bar{C} - 2G_{2i}'\bar{R}^{-1}G_0 - N_{1i}(t,t) = 0, \end{cases} \tag{10}$$

$$\begin{cases} \dfrac{2\partial N_{2i}(t,\theta)}{\partial t} + 2A'N_{2i}(t,\theta) - 2G_1'\bar{R}^{-1}G_{3i}(t,\theta) = 0, \\ 2A_i'P + 2\bar{A}_i'P\bar{A} - 2N_{2i}'(t,t) - 2G_{2i}'\bar{R}^{-1}G_1 + 2L_{1i}' = 0, \end{cases} \tag{11}$$

$$\begin{cases} \dfrac{\partial N_{3ij}(t,\theta)}{\partial t} = 0, \\ Q_{ij} + N_{3ij}(t,t) - \bar{G}_{2i}'\bar{R}^{-1}\bar{G}_{2j} + \bar{A}_i'P\bar{A}_j = 0, \end{cases} \tag{12}$$

$$\begin{cases} \dfrac{\partial N_{4ij}(t,s,\theta)}{\partial t} - G_{3i}'(t,s)\bar{R}^{-1}G_{3j}(t,\theta) = 0, \\ A_i'N_{2j}(t,\theta) - N_{4ij}'(t,t,\theta) - G_{2i}'\bar{R}^{-1}G_{3j}(t,\theta) = 0, \\ A_jN_{2i}'(t,\theta) - N_{4ij}'(t,\theta,t) - G_{3j}'(t,\theta)\bar{R}^{-1}G_{2i} = 0, \end{cases} \tag{13}$$

where

$$G_0 := 0.5B'q + \bar{B}'P\bar{C} + 0.5S, \quad G_1 := B'P + \bar{B}'P\bar{A} + 0.5F,$$
$$G_{2i} := \bar{B}'P\bar{A}_i + 0.5F_i, \quad G_{3i}(t,\theta) := B'N_{2i}(t,\theta).$$

Secondly, on the intervals $t \in [0, T-2h]$, $\theta \in [t, t+h]$, $s \in [t, t+h]$, $i = 1,\ 2$, $j = 1,\ 2$, consider the equations:

$$\begin{cases} \dot{\bar{P}} + Q - \bar{G}_1'\underline{R}^{-1}\bar{G}_1 + 2\bar{N}_{21}(t,t+h) \\ \quad + \bar{A}'\bar{P}\bar{A} + A'\bar{P} + \bar{P}A + \bar{N}_{311}(t,t+h) = 0, \\ \underline{R} := R + \bar{B}'\bar{P}\bar{B} > 0, \\ \bar{P}(T-2h) = P(T-2h), \end{cases} \tag{14}$$

$$\begin{cases} \dot{\bar{k}} + \bar{q}'C + \bar{C}'\bar{P}\bar{C} - \bar{G}_0'\underline{R}^{-1}\bar{G}_0 = 0 \\ \bar{k}(T-2h) = k(T-2h), \end{cases} \tag{15}$$

$$\begin{cases} \dot{\bar{q}}' + \bar{q}'A + 2C'\bar{P} - 2\bar{G}_0'\underline{R}^{-1}\bar{G}_1 + \bar{N}_{11}'(t,t+h) + Z = 0, \\ \bar{q}(T-2h) = q(T-2h), \end{cases} \tag{16}$$

$$\begin{cases} \dfrac{\partial \bar{N}_{1i}'(t,\theta)}{\partial t} - 2C'\bar{N}_{2i}'(t,\theta) - 2\bar{G}_0'\underline{R}^{-1}\bar{G}_{3i}(t,\theta) = 0, \\ A_i'\bar{q} + b_i' - \bar{N}_{1i}'(t,t) - 2\bar{G}_{2i}'\underline{R}^{-1}\bar{G}_0 = 0, \\ \bar{N}_{1i}(T-2h,z) = N_1 i(T-2h,z) \quad \text{for} \quad z \in [T-2h,T], \end{cases} \tag{17}$$

$$\begin{cases} 2\dfrac{\partial \bar{N}_{2i}(t,\theta)}{\partial t} + 2A'\bar{N}_{2i}(t,\theta) - 2\bar{G}_1'\underline{R}^{-1}\bar{G}_{3i}(t,\theta) = 0, \\ 2A_i'\bar{P}\bar{A} + 2A_i'\bar{P} - 2\bar{N}_{2i}'(t,t) - 2\bar{G}_{2i}'\underline{R}^{-1}\bar{G}_1 + 2L_{2i} = 0, \\ \bar{N}_{2i}(T-2h,z) = N_{2i}(T-2h,z) \text{ for } \quad z \in [T-2h,T], \end{cases} \tag{18}$$

$$\begin{cases} \dfrac{\partial \bar{N}_{3ij}(t,\theta)}{\partial t} = 0, \\ Q_{ij} - \bar{N}_{3ij}(t,t) - \bar{G}_{2i}'\underline{R}^{-1}\bar{G}_{2j} + \bar{A}_i'\bar{P}\bar{A}_j = 0, \\ \bar{N}_{22}(t,t+h) + \bar{N}_{321}(t,t+h) = 0, \\ \bar{N}_{312}'(t,t+h) + \bar{N}_{22}'(t,t+h) = 0, \\ \bar{N}_{322}(t,t+h) = 0, \\ \bar{N}_{3ij}(T-2h,z) = N_{3ij}(T-2h,z) \text{ for } \quad z \in [T-2h,T], \end{cases} \tag{19}$$

 

$$\begin{cases} \dfrac{\partial \bar{N}_{4ij}(t,s,\theta)}{\partial t} - \bar{G}'_{3j}(t,\theta)\underline{R}^{-1}\bar{G}_{3i}(t,s) = 0, \\ A'_i\bar{N}_{2j}(t,\theta) - \bar{N}'_{4ij}(t,t,\theta) - \bar{G}'_{2i}\underline{R}^{-1}\bar{G}_{3j}(t,\theta) = 0, \\ \bar{N}'_{2i}(t,\theta)A_j - \bar{N}'_{4ji}(t,\theta,t) - \bar{G}'_{3i}(t,\theta)\underline{R}^{-1}\bar{G}_{2j} = 0, \\ \bar{N}_{411}(t,t+h,\theta) + \bar{N}'_{411}(t,\theta,t+h) = 0, \\ \bar{N}_{422}(t,t+h,\theta) + \bar{N}'_{422}(t,\theta,t+h) = 0, \\ \bar{N}_{421}(t,t+h,\theta) + \bar{N}'_{412}(t,\theta,t+h) = 0, \\ \bar{N}_{4ij}(T-2h,\ell,z) = N_{4ij}(T-2h,\ell,z) \text{ for } \ell \in [T-2h,T] \\ \text{and } z \in [T-2h,T], \end{cases} \tag{20}$$

where

$$\bar{G}_0 := 0.5B'\bar{q} + \bar{B}'\bar{P}\bar{C} + 0.5S, \quad \bar{G}_1 := B'\bar{P} + \bar{B}'\bar{P}\bar{A} + 0.5F,$$
$$\bar{G}_{2i} := \bar{B}'\bar{P}\bar{A}_i + 0.5F_i, \quad \bar{G}_{3i}(t,\theta) := B'\bar{N}_{2i}(t,\theta).$$

The system of coupled equations (7)-(20) is more general than the ones that appear in Liang et al. (2018). We raise their solvability as our standing assumption below, and give an example in §3 to illustrate the reasonableness of this assumption.

**Assumption 1.** *The system of coupled equations (7)-(20) has a unique solution.*

**Theorem 1.** *There exists a unique solution $u^*$ to the optimal regulator problem (6). If $T-2h \le 0$, then the solution is given by:*

$$u^*(t) = -\bar{R}^{-1}\left[G_0 + G_1x(t) + \sum_{i=1}^{2} G_{2i}x(t-ih)\right]$$
$$-\bar{R}^{-1}\int_t^T \sum_{i=1}^{2} G_{3i}(t,\theta)x(\theta-ih)d\theta, \quad t \in [0,T].$$

*If $T-2h > 0$, then the solution is given by:*

$$u^*(t) = -\underline{R}^{-1}\left[\bar{G}_0 + \bar{G}_1x(t) + \sum_{i=1}^{2} \bar{G}_{2i}x(t-ih)\right]$$
$$-\underline{R}^{-1}\sum_{i=1}^{2}\int_t^{t+h} \bar{G}_{3i}(t,\theta)x(\theta-ih)d\theta, \; t \in [0,T-2h],$$
$$u^*(t) = -\bar{R}^{-1}\left[G_0 + G_1x(t) + \sum_{i=1}^{2} G_{2i}x(t-ih)\right]$$
$$-\bar{R}^{-1}\int_t^T \sum_{i=1}^{2} G_{3i}(t,\theta)x(\theta-ih)d\theta, \quad t \in [T-2h,T].$$

**Proof.** We only consider the case of $T-2h > 0$, as the case of $T-2h \le 0$ is very similar and simpler. Thus, let $T-2h > 0$. We split (5) in two parts as $J(u(\cdot)) = \mathbb{E}[J_1(u(\cdot))] + \mathbb{E}[J_2(u(\cdot))]$ where:

$$J_1(u(\cdot)) := \mathbb{E}\Bigg\{\int_0^{T-2h}\Bigg[x'(t)Q(t)x(t) + \sum_{i=1}^{2}\sum_{j=1}^{2} x'(t-ih)$$
$$\times Q_{ij}(t)x(t-jh) + u'(t)R(t)u(t) + x'(t)\sum_{i=1}^{2} L_{1i}(t)$$
$$\times x(t-ih) + \sum_{i=1}^{2} x'(t-ih)L_{2i}(t)x(t) + x'(t)Z(t)$$
$$+ u'(t)F(t)x(t) + u'(t)\sum_{i=1}^{2} F_i(t)x(t-ih)$$
$$+ \sum_{i=1}^{2} b'_i(t)x(t-ih) + S'(t)u(t)\Bigg]dt\Bigg\},$$

$$J_2(u(\cdot)) := \mathbb{E}\Bigg\{\int_{T-2h}^{T}\Bigg[x'(t)Q(t)x(t) + \sum_{i=1}^{2}\sum_{j=1}^{2} x'(t-ih)$$
$$\times Q_{ij}(t)x(t-jh) + u'(t)R(t)u(t) + x'(t)\sum_{i=1}^{2} L_{1i}(t)$$
$$\times x(t-ih) + \sum_{i=1}^{2} x'(t-ih)L_{2i}(t)x(t) + x'(t)Z(t)$$
$$+ u'(t)F(t)x(t) + u'(t)\sum_{i=1}^{2} F_i(t)x(t-ih) + \sum_{i=1}^{2} b'_i(t)$$
$$\times x(t-ih) + S'(t)u(t)\Bigg]dt + x'(T)Hx(T)$$
$$+ f'x(T)\Big|\mathcal{F}(T-2h)\Bigg\}.$$

For $t \in [T-2h,T]$, we define the process $v_2(t,x(t))$ as:

$$v_2(t,x(t)) := k + q'x(t) + x'(t)Px(t) + \int_t^T \sum_{i=1}^{2} N'_{1i}(t,\theta)$$
$$\times x(\theta-ih)d\theta + x'(t)\int_t^T \sum_{i=1}^{2} N_{2i}(t,\theta)x(\theta-ih)d\theta$$
$$+ \int_t^T \sum_{i=1}^{2} x'(\theta-ih)N'_{2i}(t,\theta)x(t)d\theta + \int_t^T \sum_{i=1}^{2}\sum_{j=1}^{2} x'(\theta-ih)$$
$$\times N_{3ij}(t,\theta)x(\theta-jh)d\theta + \int_t^T\int_t^T \sum_{i=1}^{2}\sum_{j=1}^{2} x'(\theta-jh)$$
$$\times N_{4ij}(t,s,\theta)x(s-ih)d\theta ds.$$

By Itô's formula, the differential of $v_2(t,x(t))$ is:

$$dv_2(t,x(t)) = \dot{k}dt + \dot{q}'x(t)dt + q'\Bigg[Ax(t) + \sum_{i=1}^{2} A_i$$
$$\times x(t-ih) + Bu(t) + C\Bigg]dt + q'\Bigg[\bar{A}x(t) + \sum_{i=1}^{2} \bar{A}_i$$
$$\times x(t-ih) + \bar{B}u(t) + \bar{C}\Bigg]dW + x'(t)\dot{P}x(t)dt + \Bigg[Ax(t)$$
$$+ \sum_{i=1}^{2} A_ix(t-ih) + Bu(t) + C\Bigg]'Px(t)dt + \Bigg[\bar{A}x(t)$$
$$+ \sum_{i=1}^{2} \bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Bigg]'Px(t)dW + x'(t)P$$
$$\times\Bigg[Ax(t) + \sum_{i=1}^{2} A_ix(t-ih) + Bu(t) + C\Bigg]dt + x'(t)P$$
$$\times\Bigg[\bar{A}x(t) + \sum_{i=1}^{2} \bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Bigg]dW + \Bigg[\bar{A}x(t)$$
$$+ \sum_{i=1}^{2} \bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Bigg]'P\Bigg[\bar{A}x(t) + \sum_{i=1}^{2} \bar{A}_i$$
$$\times x(t-ih) + \bar{B}u(t) + \bar{C}\Big]dt + \int_t^T \sum_{i=1}^{2} \frac{\partial N'_{1i}(t,\theta)}{\partial t}$$
$$\times x(\theta-ih)d\theta dt - \sum_{i=1}^{2} N'_{1i}(t,t)x(t-ih)dt + x'(t)$$
$$\times \int_t^T \sum_{i=1}^{2} \frac{\partial N_{2i}(t,\theta)}{\partial t}x(\theta-ih)d\theta dt - x'(t)\sum_{i=1}^{2} N_{2i}(t,t)$$
$$\times x(t-ih)dt + \Bigg[Ax(t) + \sum_{i=1}^{2} A_ix(t-ih) + Bu(t)$$
$$+ C\Bigg]'\int_t^T \sum_{i=1}^{2} N_{2i}(t,\theta)x(\theta-ih)d\theta dt + \Bigg[\bar{A}x(t) + \sum_{i=1}^{2} \bar{A}_i$$

 

$$
\begin{aligned}
&\times x(t-ih)+\bar{B}u(t)+\bar{C}\Big]'\int_t^T\sum_{i=1}^{2}N_{2i}(t,\theta)x\left(\theta-ih\right)d\theta dW\\
&+\int_t^T\sum_{i=1}^{2}x'\left(\theta-ih\right)\frac{\partial N_{2i}'(t,\theta)}{\partial t}x(t)d\theta dt-\sum_{i=1}^{2}x'\left(t-ih\right)\\
&\times N_{2i}'(t,t)x(t)dt+\int_t^T\sum_{i=1}^{2}x'\left(\theta-ih\right)N_{2i}'(t,\theta)d\theta\Big[Ax(t)\\
&+\sum_{i=1}^{2}A_ix\left(t-ih\right)+Bu(t)+C\Big]dt+\int_t^T\sum_{i=1}^{2}x'\left(\theta-ih\right)\\
&\times N_{2i}'(t,\theta)d\theta\Big[\bar{A}x(t)+\sum_{i=1}^{2}\bar{A}_ix\left(t-ih\right)+\bar{B}u(t)+\bar{C}\Big]dW\\
&+\int_t^T\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(\theta-ih\right)\frac{\partial N_{3ij}(t,\theta)}{\partial t}x(\theta-jh)d\theta dt\\
&-\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(t-ih\right)N_{3ij}(t,t)x\left(t-jh\right)dt\\
&+\int_t^T\int_t^T\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(\theta-jh\right)\frac{\partial N_{4ij}(t,s,\theta)}{\partial t}\left(s-ih\right)d\theta\\
&\times dsdt-\int_t^T\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(\theta-jh\right)N_{4ij}(t,t,\theta)x\left(t-ih\right)d\theta\\
&\times dt-\int_t^T\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(t-jh\right)N_{4ij}(t,\theta,t)x(\theta-ih)d\theta dt
\end{aligned}
$$

The cost functional $J_2(u(\cdot))$ can be written as:

$$
\begin{aligned}
&J_2(u(\cdot)):=\mathbb{E}\Bigg\{\int_{T-2h}^{T}\Bigg[x'(t)Q(t)x(t)+\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(t-ih\right)\\
&\times Q_{ij}(t)x\left(t-jh\right)+u'(t)R(t)u(t)+x'(t)\sum_{i=1}^{2}L_{1i}(t)\\
&\times x\left(t-ih\right)+\sum_{i=1}^{2}x'\left(t-ih\right)L_{2i}(t)x(t)+x'(t)Z(t)+u'(t)\\
&\times F(t)x(t)+u'(t)\sum_{i=1}^{2}F_i(t)x\left(t-ih\right)+\sum_{i=1}^{2}b_i'(t)x\left(t-ih\right)\\
&+S'(t)u(t)\Big]dt+x'(T)Hx(T)+f'x(T)\Big|\mathcal{F}(T-2h)\Bigg\}. \quad (21)
\end{aligned}
$$

The terms of (21) that depend explicitly on the control $u$ are:

$$
\begin{aligned}
&\Big[u(t)+\bar{R}^{-1}\Big(G_0+G_1x(t)+\sum_{i=1}^{2}G_{2i}x(t-ih)\\
&+\int_t^T\sum_{i=1}^{2}G_{3i}(t,\theta)x(\theta-ih)d\theta\Big)\Big]'\bar{R}\Big[u(t)+\bar{R}^{-1}\Big(G_0\\
&+G_1x(t)+\sum_{i=1}^{2}G_{2i}x(t-ih)+\int_t^T\sum_{i=1}^{2}G_{3i}(t,\theta)\\
&\times x(\theta-ih)d\theta\Big)\Big]-\Big[G_0+G_1x(t)+\sum_{i=1}^{2}G_{2i}x(t-ih)\\
&+\int_t^T\sum_{i=1}^{2}G_{3i}(t,\theta)x(\theta-ih)d\theta\Big]'\bar{R}^{-1}\Big[G_0+G_1x(t)\\
&+\sum_{i=1}^{2}G_{2i}x(t-ih)+\int_t^T\sum_{i=1}^{2}G_{3i}(t,\theta)x(\theta-ih)d\theta\Big],
\end{aligned}
$$

where the last equality is due to the completion of squares. The cost functional $J_2(u(\cdot))$ can now be written as:

$$
\begin{aligned}
&J_2(u(\cdot))=v_2(T-2h,x(T-2h))+\mathbb{E}\Big\{\int_{T-2h}^{T}\Big\{\Big[u(t)\\
&+\bar{R}^{-1}\Big(G_0+G_1x(t)+\sum_{i=1}^{2}G_{2i}x(t-ih)\\
&+\int_{T-2h}^{T}\sum_{i=1}^{2}G_{3i}(t,\theta)x(\theta-ih)d\theta\Big)\Big]'\bar{R}\Big[u(t)+\bar{R}^{-1}\Big(G_0\\
&+G_1x(t)+\sum_{i=1}^{2}G_{2i}x(t-ih)+\int_t^T\sum_{i=1}^{2}G_{3i}(t,\theta)\\
&\times x(\theta-ih)d\theta\Big)\Big]+\Big[\dot{k}+q'C+\bar{C}'P\bar{C}-G_0'\bar{R}^{-1}G_0\Big]\\
&+\left[\dot{q}'+q'A+2C'P+2\bar{C}'P\bar{A}-2G_0'\bar{R}^{-1}G_1+Z\right]x(t)\\
&+x'(t)[\dot{P}+A'P+PA+\bar{A}'P\bar{A}+Q-G_1'\bar{R}^{-1}G_1]x(t)\\
&+\sum_{i=1}^{2}\sum_{j=1}^{2}x'(t-ih)[Q_{ij}-N_{3ij}(t,t)+\bar{A}_i'P\bar{A}_j-G_{2i}'\\
&\times\bar{R}^{-1}G_{2j}]x(t-jh)+\sum_{i=1}^{2}x'\left(t-ih\right)[b_i+A_i'q+2\bar{A}_i'\\
&\times P\bar{C}-2G_{2i}'\bar{R}^{-1}G_0-N_{1i}(t,t)]+\sum_{i=1}^{2}x'\left(t-ih\right)[2A_i'P\\
&+2\bar{A}_i'P\bar{A}-2N_{2i}'(t,t)+2L_{1i}'-2G_{2i}'\bar{R}^{-1}G_1]\,x(t)\\
&+x'(t)\int_t^T\sum_{i=1}^{2}\Big[2A'N_{2i}(t,\theta)+2\frac{\partial N_{2i}(t,\theta)}{\partial t}-2G_1'\bar{R}^{-1}\\
&\times G_{3i}(t,\theta)]\,x\left(\theta-ih\right)d\theta+\int_t^T\sum_{i=1}^{2}[2C'N_{2i}(t,\theta)\\
&+\frac{\partial N_{1i}'(t,\theta)}{\partial t}-2G_0'\bar{R}^{-1}G_{3i}(t,\theta)\Big]\,x\left(\theta-ih\right)d\theta\\
&+\sum_{i=1}^{2}x'\left(t-ih\right)\int_t^T\sum_{j=1}^{2}\left[A_i'N_{2j}(t,\theta)-G_{2i}'\bar{R}^{-1}G_{3j}(t,\theta)\right.\\
&\left.-N_{4ij}'(t,t,\theta)\right]x\left(\theta-jh\right)d\theta+\int_t^T\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(\theta-ih\right)\\
&\times\left[N_{2i}'(t,\theta)A_j-G_{2j}'\bar{R}^{-1}G_{3i}(t,\theta)-N_{4ij}'(t,\theta,t)\right]\\
&\times x\left(t-jh\right)d\theta+\int_t^T\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(\theta-ih\right)\left[\frac{\partial N_{3ij}(t,\theta)}{\partial t}\right]\\
&\times x\left(\theta-jh\right)d\theta+\int_t^T\int_t^T\sum_{i=1}^{2}\sum_{j=1}^{2}x'\left(\theta-jh\right)\Big[\frac{\partial N_{4ij}(t,s,\theta)}{\partial t}\\
&-G_{3j}'(t,\theta)\bar{R}^{-1}G_{3i}(t,s)\Big]x\left(s-ih\right)d\theta ds\Big\}dt\\
&=v_2(T-2h,x(T-2h))+\mathbb{E}\Big\{\int_{T-2h}^{T}\Big[u(t)+\bar{R}^{-1}\\
&\times\Big(G_0+G_1x(t)+\sum_{i=1}^{2}G_{2i}x(t-ih)+\int_t^T\sum_{i=1}^{2}G_{3i}(t,\theta)\\
&\times x(\theta-ih)d\theta\Big)\Big]'\bar{R}\Big[u(t)+\bar{R}^{-1}\Big(G_0+G_1x(t)\\
&+\sum_{i=1}^{2}G_{2i}x(t-ih)+\int_{T-h_2}^{T}\sum_{i=1}^{2}G_{3i}(t,\theta)\\
&\times x(\theta-ih)d\theta\Big)\Big]dt\Big|\mathcal{F}(T-2h)\Big\}. \quad (22)
\end{aligned}
$$

For $t \in [0, T-2h]$, we define the process $v_1(t, x(t))$ as:

$$v_1(t,x(t)) = \bar{k} + \bar{q}'x(t) + x'(t)\bar{P}x(t) + \sum_{i=1}^{2}\int_t^{t+h}\bar{N}_{1i}'(t,\theta)x(\theta-ih)d\theta + x'(t) \times \sum_{i=1}^{2}\int_t^{t+h}\bar{N}_{2i}(t,\theta)x(\theta-ih)d\theta + \sum_{i=1}^{2}\int_t^{t+h}x'(\theta-ih) \times \bar{N}_{2i}'(t,\theta)x(t)d\theta + \sum_{i=1}^{2}\sum_{j=1}^{2}\int_t^{t+h}x'(\theta-ih)\bar{N}_{3ij}(t,\theta) \times x(\theta-jh)d\theta + \sum_{i=1}^{2}\sum_{j=1}^{2}\int_t^{t+h}\int_t^{t+h}x'(\theta-jh) \times \bar{N}_{4ij}(t,s,\theta)x(s-ih)d\theta ds.$$

By Itô's formula, the differential of $v_1(t, x(t))$ is:

$$dv_1(t,x(t)) = \dot{\bar{k}}dt + \dot{\bar{q}}'x(t)dt + \bar{q}'\Big[Ax(t) + \sum_{i=1}^{2}A_i \times x(t-ih) + Bu(t) + C\Big]dt + \bar{q}'\Big[\bar{A}x(t) + \sum_{i=1}^{2}\bar{A}_i \times x(t-ih) + \bar{B}u(t) + \bar{C}\Big]dW + x'(t)\dot{\bar{P}}x(t)dt + \Big[Ax(t) + \sum_{i=1}^{2}A_ix(t-ih) + Bu(t) + C\Big]'\bar{P}x(t)dt + \Big[\bar{A}x(t) + \sum_{i=1}^{2}\bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Big]'\bar{P}x(t)dW + x'(t)\bar{P} \times \Big[Ax(t) + \sum_{i=1}^{2}A_ix(t-ih) + Bu(t) + C\Big]dt + x'(t) \times \bar{P}\Big[\bar{A}x(t) + \sum_{i=1}^{2}\bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Big]dW + \Big[\bar{A}x(t) + \sum_{i=1}^{2}\bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Big]'\bar{P} \times \Big[\bar{A}x(t) + \sum_{i=1}^{2}\bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Big]dt + \sum_{i=1}^{2}\Big[\int_t^{t+h}\frac{\partial\bar{N}_{1i}'(t,\theta)}{\partial t}x(\theta-ih)d\theta + \bar{N}_{1i}'(t,t+h) \times x(t-(i-1)h) - \bar{N}_{1i}'(t,t)x(t-ih)\Big]dt + \Big[Ax(t) + \sum_{i=1}^{2}A_ix(t-ih) + Bu(t) + C\Big]'\sum_{i=1}^{2}\int_t^{t+h}\bar{N}_{2i}(t,\theta) \times x(\theta-ih)d\theta dt + \Big[\bar{A}x(t) + \sum_{i=1}^{2}\bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Big]'\sum_{i=1}^{2}\int_t^{t+h}\bar{N}_{2i}(t,\theta)x(\theta-ih)d\theta dW + x'(t) \times \sum_{i=1}^{2}\int_t^{t+h}\frac{\partial\bar{N}_{2i}(t,\theta)}{\partial t}x(\theta-h)d\theta dt + x'(t) \times \sum_{i=1}^{2}\bar{N}_{2i}(t,t+h)x(t-(i-1)h)dt - x'(t)\sum_{i=1}^{2}\bar{N}_{2i}(t,t) \times x(t-ih)dt + \sum_{i=1}^{2}\int_t^{t+h}x'(\theta-ih)\frac{\partial\bar{N}_{2i}'(t,\theta)}{\partial t}x(t)d\theta dt + \sum_{i=1}^{2}\Big[x'(t-(i-1)h)\bar{N}_{2i}'(t,t+h)x(t) - x'(t-ih) \times \bar{N}_{2i}'(t,t)x(t)\Big]dt + \sum_{i=1}^{2}\int_t^{t+h}x'(\theta-ih)\bar{N}_{2i}'(t,\theta)\Big[Ax(t) + \sum_{i=1}^{2}A_ix(t-ih) + Bu(t) + C\Big]d\theta dt + \sum_{i=1}^{2}\int_t^{t+h}x'(\theta-ih)\bar{N}_{2i}'(t,\theta)\Big[\bar{A}x(t) + \sum_{i=1}^{2}\bar{A}_ix(t-ih) + \bar{B}u(t) + \bar{C}\Big]d\theta dW + \sum_{i=1}^{2}\sum_{j=1}^{2}\Big[\int_t^{t+h}x'(\theta-ih) \times \frac{\partial\bar{N}_{3ij}(t,\theta)}{\partial t}x(\theta-jh)d\theta + x'(t-(i-1)h) \times \bar{N}_{3ij}(t,t+h)x(t-(j-1)h) - x'(t-ih)\bar{N}_{3ij}(t,t) \times x(t-jh)\Big]dt + \sum_{i=1}^{2}\sum_{j=1}^{2}\Big[\int_t^{t+h}\int_t^{t+h}x'(\theta-jh) \times \frac{\partial\bar{N}_{4ij}(t,s,\theta)}{\partial t}x(s-ih)d\theta ds + \int_t^{t+h}x'(\theta-jh) \times \bar{N}_{4ij}(t,t+h,\theta)x(t-(i-1)h)d\theta - \int_t^{t+h}x'(\theta-jh) \times \bar{N}_{4ij}(t,t,\theta)x(t-ih)d\theta + x'(t-(j-1)h) \int_t^{t+h}\bar{N}_{4ij}(t,\theta,t+h)x(\theta-ih)d\theta - x'(t-jh)\int_t^{t+h}\bar{N}_{4ij}(t,\theta,t)x(\theta-ih)d\theta\Big]dt.$$

Note that $v_1(T-2h, x(T-2h)) = v_2(T-2h, x(T-2h))$. The terms of $\mathbb{E}[J_1(u(\cdot))+v_1(T-2h, x(T-2h))]$ that depend explicitly on control $u$ can now be written as:

$$\Big\{u(t) + \underline{R}^{-1}\Big[\bar{G}_0 + \bar{G}_1x(t) + \sum_{i=1}^{2}\bar{G}_{2i}x(t-ih) + \sum_{i=1}^{2}\int_t^{t+h}\bar{G}_{3i}(t,\theta)x(\theta-ih)d\theta\Big]\Big\}'\underline{R}\Big\{u(t) + \underline{R}^{-1}\Big[\bar{G}_0 + \bar{G}_1x(t) + \sum_{i=1}^{2}\bar{G}_{2i}x(t-ih) + \sum_{i=1}^{2}\int_t^{t+h}\bar{G}_{3i}(t,\theta)x(\theta-ih)d\theta\Big]\Big\} - \Big[\bar{G}_0 + \bar{G}_1x(t) + \sum_{i=1}^{2}\bar{G}_{2i}x(t-ih) + \sum_{i=1}^{2}\int_t^{t+h}\bar{G}_{3i}(t,\theta)x(\theta-ih)d\theta\Big]' \times \underline{R}^{-1}\Big[\bar{G}_0 + \bar{G}_1x(t) + \sum_{i=1}^{2}\bar{G}_{2i}x(t-ih) + \sum_{i=1}^{2}\int_t^{t+h}\bar{G}_{3i}(t,\theta)x(\theta-ih)d\theta\Big].$$

We thus have:

$$\mathbb{E}[J_1(u(\cdot)) + v_1(T-2h, x(T-2h))] = v_1(0, x(0))$$

 

$$+\mathbb{E}\Big\{ \int_0^{T-2h} \Big\{ \Big[ u(t) + \underline{R}^{-1}\Big( \bar{G}_0 + \bar{G}_1 x(t) + \sum_{i=1}^{2} \bar{G}_{2i} \times x(t-ih) + \sum_{i=1}^{2} \int_t^{t+h} \bar{G}_{3i}(t,\theta) x(\theta - ih) d\theta \Big) \Big]' \times \underline{R} \Big[ u(t) + \underline{R}^{-1}\Big( \bar{G}_0 + \bar{G}_1 x(t) + \sum_{i=1}^{2} \bar{G}_{2i} x(t-ih) + \sum_{i=1}^{2} \int_t^{t+h} \bar{G}_{3i}(t,\theta) x(\theta - ih) d\theta \Big) \Big] + x'(t) \Big[ \dot{\bar{P}} + Q + \bar{A}'\bar{P}\bar{A} - \bar{G}_1' \underline{R}^{-1} \bar{G}_1 + PA + A'\bar{P} + 2\bar{N}_{21}(t,t+h) + \bar{N}_{311}(t,t+h) \Big] x(t) + \sum_{i=1}^{2}\sum_{j=1}^{2} x'(t-ih) [Q_{ij} - \bar{N}_{3ij}(t,t) + \bar{A}_i'\bar{P}\bar{A}_j' - \bar{G}_{2i}'\underline{R}^{-1}\bar{G}_{2j}] x(t-jh) + \sum_{i=1}^{2} x'(t-ih) \Big[ 2A_i'\bar{P} - 2\bar{G}_{2i}'\underline{R}^{-1}\bar{G}_1 - 2\bar{N}_{2i}'(t,t) + 2L_{2i}' \Big] x(t) + x'(t) \Big[ \dot{\bar{q}} + A'\bar{q} + 2\bar{P}C + 2\bar{A}'\bar{P}\bar{C} - 2\bar{G}_1'\underline{R}^{-1}\bar{G}_0 + Z + \bar{N}_{11}'(t,t+h) \Big] + \sum_{i=1}^{2} x'(t-ih) \times [b_i' + 2\bar{A}_i'\bar{P}\bar{C} + A_i'\bar{q} - \bar{N}_{1i}'(t,t) - 2\bar{G}_{2i}'\underline{R}^{-1}\bar{G}_0] + x'(t) \sum_{i=1}^{2} \int_t^{t+h} \Big[ 2A'\bar{N}_{2i}(t,\theta) + 2\frac{\partial \bar{N}_{2i}(t,\theta)}{\partial t} - 2\bar{G}_1'\underline{R}^{-1}\bar{G}_{3i}(t,\theta) \Big] x(\theta - ih) d\theta + \sum_{i=1}^{2} \int_t^{t+h} \Big[ 2C' \times \bar{N}_{2i}(t,\theta) + \frac{\partial \bar{N}_{1i}'(t,\theta)}{\partial t} - 2\bar{G}_0'\underline{R}^{-1}\bar{G}_{3i}(t,\theta) \Big] \times x(\theta - ih) d\theta + \sum_{i=1}^{2}\sum_{j=1}^{2} x'(t-ih) \int_t^{t+h} \Big[ A_i'\bar{N}_{2j}(t,\theta) - \bar{G}_{2i}'\underline{R}^{-1}\bar{G}_{3j}(t,\theta) - \bar{N}_{4ij}'(t,t,\theta) \Big] x(\theta - jh) d\theta + \sum_{i=1}^{2} \int_t^{t+h} x'(\theta - ih) \Big[ \sum_{j=1}^{2} \bar{N}_{4ij}'(t,\theta,t) - \bar{G}_{3i}'(t,\theta) \times \underline{R}^{-1}\bar{G}_{2j} + \bar{N}_{2i}'(t,\theta) A_j \Big] x(t-jh) d\theta + \sum_{i=1}^{2}\sum_{j=1}^{2} \int_t^{t+h} x(\theta - ih) \frac{\partial \bar{N}_{3ij}(t,\theta)}{\partial t} x(\theta - jh) d\theta + \sum_{i=1}^{2}\sum_{j=1}^{2} \int_t^{t+h}\int_t^{t+h} x'(\theta - jh) \Big[ \frac{\partial \bar{N}_{4ij}(t,s,\theta)}{\partial t} - \bar{G}_{3j}'(t,\theta)\underline{R}^{-1}\bar{G}_{3i}(t,s) \Big] x(s-ih) d\theta ds + \dot{\bar{k}} + 2\bar{q}'C + \bar{C}'\bar{P}\bar{C} - \bar{G}_0'\underline{R}^{-1}\bar{G}_0 + x'(t)[\bar{N}_{22}(t,t+h) + \bar{N}_{22}'(t,t+h) + \bar{N}_{321}'(t,t+h) + \bar{N}_{312}(t,t+h)] \times x(t-h) + \bar{N}_{12}'(t,t+h) x(t-h) + x'(t-h) \times \bar{N}_{322}(t,t+h) x(t-h) + \int_t^{t+h} x'(\theta - h) [\bar{N}_{411}(t,t+h,\theta) + \bar{N}_{411}'(t,\theta,t+h)] x(t) d\theta + \int_t^{t+h} x'(\theta - 2h)[\bar{N}_{422}(t,t+h,\theta) + \bar{N}_{422}'(t,\theta,t+h)] x(t-h) d\theta + \int_t^{t+h} x'(\theta - h) \times [\bar{N}_{421}(t,t+h,\theta) + \bar{N}_{421}'(t,\theta,t+h)] x(\theta - h) d\theta + \int_t^{t+h} x'(t)[\bar{N}_{421}(t,\theta,t+h) + \bar{N}_{421}'(t,t+h,\theta)] \times x(\theta - 2h) d\theta \Big\} dt. \quad (23)$$

From (22) and (23) it follows that for any $u(\cdot) \in \mathcal{A}$ we have:

$$J(u(\cdot)) = v_1(0,x(0)) + \mathbb{E}\Big\{ \int_0^{T-2h} \Big[ u(t) + \underline{R}^{-1}\Big( \bar{G}_0 + \bar{G}_1 x(t) + \sum_{i=1}^{2} \bar{G}_{2i} x(t-h_i) + \sum_{i=1}^{2} \int_t^{t+h} \bar{G}_{3i}(t,\theta) \times x(\theta - ih) d\theta \Big) \Big]' \underline{R} \Big[ u(t) + \underline{R}^{-1}\Big( \bar{G}_0 + \bar{G}_1 x(t) + \sum_{i=1}^{2} \bar{G}_{2i} x(t-ih) + \sum_{i=1}^{2} \int_t^{t+h} \bar{G}_{3i}(t,\theta) x(\theta - ih) d\theta \Big) \Big] dt \Big\} \quad (24)$$
$$+\mathbb{E}\Big\{ \mathbb{E}\Big\{ \int_{T-2h}^{T} \Big[ u(t) + \bar{R}^{-1}\Big( G_0 + G_1 x(t) + \sum_{i=1}^{2} G_{2i} \times x(t-ih) + \int_t^T \sum_{i=1}^{2} G_{3i}(t,\theta) x(\theta - ih) d\theta \Big) \Big]' \bar{R} \Big[ u(t) + \bar{R}^{-1}\Big( G_0 + G_1 x(t) + \sum_{i=1}^{2} G_{2i} x(t-ih) + \int_t^T \sum_{i=1}^{2} G_{3i}(t,\theta) x(\theta - ih) d\theta \Big) \Big] dt \Big| \mathcal{F}(T-2h) \Big\} \Big\} \geq v_1(0,x(0)).$$

This lower bound is achieved if and only if $u(t) = u^*(t)$ for $a.e.\ t \in [0,T]\ a.s.$. □

## 3. APPLICATION TO OPTIMAL INVESTMENT

As an application of Theorem 1, we solve an optimal investment problem in a market with a stochastic interest rate. A similar problem was considered in Algoulity and Gashi (2023) and Gashi and Hua (2023) where the Cox-Ingersoll-Ross (CIR) interest rate model was used. However, different from Algoulity and Gashi (2023) and Gashi and Hua (2023), here we introduce a general class of interest rate models with a factor process that is the solution to a stochastic differential equation with delay in both the drift and diffusion (which, in particular, is not fully covered by the models of Sheu et al. (2018)). Consider a market of a bank account with price $S_0$ and of a stock with price $S_1$, which are solutions to the following equations (for $t \in [0,T]$):

$$\begin{cases} dS_0(t) = S_0(t) r(t) dt, \\ dS_1(t) = S_1(t)[\mu(t)dt + \sigma(t)dW(t)] \\ S_0(0) > 0 \quad \text{and} \quad S_1(0) > 0 \quad \text{are given,} \end{cases}$$

with the interest rate $r$, the appreciation rate $\mu$, and the volatility $\sigma$ to be defined precisely below. If the investor, who has an initial wealth of $y_0$, holds $v_{S_0}(t)$ and $v_{S_1}(t)$ number of shares time $t$ in the bank account and in the stock, respectively, then his/her wealth is $y(t) := v_{S_0}(t)S_0(t) + v_{S_1}(t)S_1(t)$. We define $\tilde{u}(t) := v_{S_1}(t)S_1(t)$ and consider the self-financing portfolio with equation (for which see, for example, Korn (1997),Karatzas and Shreve (1998)):

 

$$dy(t) = [ry(t) + (\mu - r)\tilde{u}(t)]dt + \sigma\tilde{u}(t)dW. \quad (25)$$

As in Algoulity and Gashi (2023), Gashi and Hua (2023), we assume $(\mu - r)(\cdot) \in L^{\infty}(0, T; \mathbb{R})$, and $0 < \sigma(\cdot) \in L^{\infty}(0, T; \mathbb{R})$. But different from Algoulity and Gashi (2023), Gashi and Hua (2023), we define the interest rate as $r(t) := x_1(t)$, where the *factor* process $x_1$ is the solution to the following stochastic differential equation with delay (for $t \in [0, T]$):

$$dx_1(t) = \big[\alpha x_1(t) + \sum_{i=1}^{2} \alpha_i x_1(t - ih) + \beta\big]dt \quad (26)$$
$$+\big[\bar{\alpha} x_1(t) + \sum_{i=1}^{2} \bar{\alpha}_i x_1(t - ih) + \bar{\beta}\big]dW.$$

Here $\alpha, \alpha_i, \bar{\alpha}, \bar{\alpha}_i, \beta, \bar{\beta} \in \mathbb{R}$ and $x_1(s) = \xi(s)$, $s \in [-2h, 0]$, with $\xi$ being a one-dimensional continuous function. Some special cases of this interest rate model appear in Sheu et al. (2018). The *optimal investment* problem with *logarithmic utility* is:

$$\begin{cases} \max\limits_{\tilde{u}(\cdot) \in \widetilde{\mathcal{A}}} \mathbb{E}[\log(y(T))], \\ s.t. \ (25), \end{cases} . \quad (27)$$

where $\widetilde{\mathcal{A}}$ is the space of control processes $\tilde{u}$ under which (25) has a unique and positive strong solution. If we define $v(t) := \tilde{u}(t)/y(t)$, and $x_2(t) := y(t) + \int_0^t 0.5\sigma^2 v^2(s)ds$, then it can be shown (similarly to Algoulity and Gashi (2023), Gashi and Hua (2023)) that (27) is equivalent to the problem of minimizing

$$\widetilde{J}(v(\cdot)) := \mathbb{E}\left[\int_0^T 0.5\sigma^2 v^2(s)ds - x_2(T)\right], \quad (28)$$

with respect to $v$ and subject to (3) and

$$dx_2 = [x_1(t) + (\mu - r)v(t)]dt + \sigma v(t)dW, \quad x_2(0) = \log y_0.$$

However, this is an example of optimal regulator problem (6) with $n = 2$, $m = 1$,

$$\zeta(s) = \begin{bmatrix} \xi(s) \\ \log y_0 \end{bmatrix}, \ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \ A = \begin{bmatrix} \alpha & 0 \\ 1 & 0 \end{bmatrix},$$
$$A_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \mu - r \end{bmatrix}, \ C = \begin{bmatrix} \beta \\ 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} \bar{\alpha} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\bar{A}_i = \begin{bmatrix} \bar{\alpha}_i & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}, \ \bar{C} = \begin{bmatrix} \bar{\beta} \\ 0 \end{bmatrix},$$
$$H = Q = Q_{ij} = F = F_i = L_{1i} = L_{2i} = Z = S = b_i = 0,$$
$$R = 0.5\sigma^2, \ f' = [0 \ -1].$$

If we further assume, for example, that $2h < T < 3h$, then in this case the solutions to equations (7)-(13) on the intervals $t \in [T - 2h, T]$, $\theta \in [t, T]$, $s \in [t, T]$ are: $P(t) = 0$, $q(t) = e^{A'(T-t)}f$, $k(t) = \int_t^T (q'C - 0.25q'BR^{-1}B'q)ds$, $N'_{1i}(t, \theta) = q'(\theta)A_i$, $N_{2i}(t, \theta) = 0$, $N_{3ij}(t, \theta) = 0$, $N_{4ij}(t, s, \theta) = 0$. The solutions to equations (14)-(20) on the intervals $t \in [0, T - 2h]$, $\theta \in [t, t + h]$, $s \in [t, t + h]$ are: $\bar{P}(t) = 0$, $\bar{q}(t) = e^{A'(T-2h-t)}q(T - 2h) + \int_{t+h}^{T} e^{-A'(t-s+h)}A'_i\bar{q}ds$, $\bar{k}(t) = k(T - 2h) + \int_t^{T-2h}(-\bar{q}'C + 0.25\bar{q}'BR^{-1}B'\bar{q})ds$, $\bar{N}'_{1i}(t, \theta) = A'_i\bar{q}(\theta)$, $\bar{N}_{2i}(t, \theta) = 0$, $\bar{N}_{3ij}(t, \theta) = 0$, $\bar{N}_{4ij}(t, s, \theta) = 0$. As Assumption 1 holds, it follows from Theorem 1 that the control process that minimizes (28) is $v^*(t) = -0.5R^{-1}B'\bar{q}$ for $t \in [0, T - 2h]$, and $v^*(t) = -0.5R^{-1}B'q$ for $t \in [T - 2h, T]$. The solution to problem (27) is therefore $\tilde{u}^*(t) = v^*(t)y(t)$ for $t \in [0, T]$.

## 4. CONCLUSIONS

We have solved a general version of the optimal regulator problem for linear stochastic systems with state-delay and the quadratic-linear criterion. The unique solution is obtained in an explicit closed-form as an affine feedback on the system state, its delayed value, and its past values. Its applicability is illustrated with an example from optimal investment. The consideration of systems with multiple *input* delays in our setting would generalise a result of Liang et al. (2018), and it is thus an interesting future problem.

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