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An LMI-based Output Regulation of Discrete-Time Nonlinear Descriptor Systems

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Abstract: This work proposes a discrete-time output regulator for nonlinear descriptor systems. The approach consists in two parts: 1) the design of a convex nonlinear stabilizer computed by means of the so-called non-quadratic Lyapunov functions, 2) the design of a linear regulator computed through the so-called Francis equations for descriptor models; the designing conditions of both parts are in terms of linear matrix inequalities. The effectiveness of the proposal is illustrated via numerical examples.

Keywords: Output Regulation, discrete-time Descriptor Systems, Lyapunov Method, Linear Matrix Inequality.

1. INTRODUCTION

The output regulation is an important problem within the design of control systems, aiming to ensure that the output of a system asymptotically tracks a desired reference while rejecting disturbances (Isidori, 1995). The idea can be traced back to (Francis, 1977), this work presents the mathematical theory for the solution of multivariable linear systems, referred as the Francis equations. Some years after, Isidori and Byrnes (1990) extended the solution for nonlinear systems, by means of a set of nonlinear partial differential equations called Francis-Isidori-Byrnes (FIB); these conditions are difficult to solve. Therefore, several approaches tried to tackled this issue by means of Takagi-Sugeno (TS) models (Takagi and Sugeno, 1985) as they are seem as convex combination of linear models; additionally, part of the controller can be designed through linear matrix inequalities (LMIs) (Boyd et al., 1994).

In the context of TS/LMI approaches, some results have been presented for continuous-time models (Castillo et al., 2003; Bernal et al., 2012) for standard state-space configurations; Lin and Dai (1996) have results for linear descriptor systems, the extention to nonlinear descriptor systems have been done by (Poblete et al., 2022; Hernandez-Cortes et al., 2024); these works treat only TS models in continuous-time. Concerning discrete-time standard systems there are few works (Castillo et al., 1993; Castillo-Toledo and Meda-Campaña, 2004).

Generally speaking, the regulator consists on two parts: one for asymptotic stabilization at the origin and other guaranteeing asymptotic tracking of references Isidori (1995). This paper provides LMI conditions for the design of both parts.

Contribution: This paper develops LMI conditions for the output regulation of discrete-time nonlinear descriptor systems. The stabilization part of the controller is designed by means of non-quadratic approaches similar to those presented in (Guerra and Vermeiren, 2004; Estrada-Manzo et al., 2015). As for the part that guarantees tracking is borrowed from (Poblete et al., 2022) and adapted for discrete-time systems.

Organization: The rest of the paper is organized as follows: Section 2 states the problem and gives some mathematical tools for the main developments. Section 3 presents LMI conditions for the design of the nonlinear control law that guarantees asymptotic tracking of references generated by an exosystem. Section 4 illustrates the performance of the proposal via two numerical examples, and finally, Section 5 closes the paper with some conclusions and future work.

2. PRELIMINARIES AND PROBLEM STATEMENT

Consider the following discrete-time descriptor system:

$$E(x_t)x_{t+1} = f(x_t, u_t), \quad y_t = g(x_t), \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^m$ is the input vector, $y_t \in \mathbb{R}^o$ is the ouput vector, $t \in \mathbb{N}$ is the current sample. The vector field $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^o$ are assumed to be smooth and bounded for all $x \in \Omega \subset \mathbb{R}^n$, $0 \in \Omega$; $E(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, with $\text{rank}(E(x_t)) = n$, $\forall x \in \Omega$.

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The task is to design a control law such that the closed-loop performs output regulation, i.e., the output tracks a reference (generated by an exosystem).

In what follows, a methodology for obtaining an exact convex rewriting of (1) is given; such convex representation is useful in order to derive LMI conditions for the controller.

2.1 Exact Takagi-Sugeno models

Consider (1) and assume that

$$f(x_t,u_t) = A(x_t)x_t + B(x_t)u_t \text{ and } g(x_t) = C(x_t)x_t$$

where the entries of $A(x_t)$, $B(x_t)$, $C(x_t)$ are well-defined in the region Ω. Thus, the sector nonlinearity methodology (Ohtake et al., 2001) begins by grouping all the p non-constant terms found in $E(x_t)$, $A(x_t)$, $B(x_t)$, and $C(x_t)$ in the premise vector $z(x_t) \in \mathbb{R}^p$. Then, as each entry of $z(x_t)$ is bounded $\forall x \in \Omega$, we compute $z_i(x_t) \in [z_i^0, z_i^1]$. Now, it is time to define the so-called weighting functions

$$w_0^i(x_t) = \frac{z_i^1 - z_i(x_t)}{z_i^1 - z_i^0},\ w_1^i(x_t) = 1 - w_0^i(x_t), \tag{2}$$

which hold the convex sum property for $x \in \Omega$, that is,

$$w_0^i(x_t) + w_1^i(x_t) = 1 \text{ and } w_0^i(x_t) \geq 0, w_1^i(x_t) \geq 0.$$

Hence, the non-constant terms can be expressed as convex sums of their bounds:

$$z_i(x_t) = w_0^i(x_t)z_i^0 + w_1^i(x_t)z_i^1.$$

Thanks to convexity [1], the following scheduling functions can be constructed

$$h_i(x_t) = w_{i_1}^1(x_t)w_{i_2}^2(x_t)\cdots w_{i_p}^p(x_t),$$

$i \in \{1, 2, \cdots, 2^p\}$, $i_1, i_2, \ldots, i_p \in \{0, 1\}$ and the set of indexes $[i_1 i_2 \cdots i_p]$ is such that a p-digit binary representation of $(i-1)$ is obtained. For example, if $p = 3$, for $h_6(x_t) = w_1^1(x_t)w_1^2(x_t)w_0^3(x_t)$ we have the set [110]. By construction the scheduling functions hold the convex sum property too:

$$\sum_{i=1}^{2^p} h_i(z(x_t)) = 1 \text{ and } h_i(z(x_t)) \geq 0,\ i \in \{1, 2, ..., 2^p\}.$$

Finally, system (1) is exactly represented by TS model:

$$\begin{aligned} \sum_{i=1}^{2^p} h_i(z(x_t))E_i x_{t+1} &= \sum_{i=1}^{2^p} h_i(z(x_t))\left(A_i x_t + B_i u_t\right), \\ y_t &= \sum_{i=1}^{2^p} h_i(z(x_t))C_i x_t, \end{aligned} \tag{3}$$

where the vertex are computed such as

$$(A_i, B_i, C_i, E_i) = (A(x_t), B(x_t), C(x_t), E(x_t))\,|_{h_i=1},$$

for $i \in \{1, 2, ..., 2^p\}$.

[1] For instance, consider $z_1(x_t) = \sum_{i_1=0}^{1} w_{i_1}^1(x_t) z_1^{i_1}$ and $z_2(x_t) = \sum_{i_2=0}^{1} w_{i_2}^2(x_t) z_2^{i_2}$, it follows $z_1(x_t) + z_2(x_t) = \sum_{i_1=0}^{1} w_{i_1}^1(x_t) \sum_{i_2=0}^{1} w_{i_2}^2(x_t)(z_1^{i_1} z_2^{i_2})$ and $z_1(x_t)z_2(x_t) = \sum_{i_1=0}^{1} w_{i_1}^1(x_t) \sum_{i_2=0}^{1} w_{i_2}^2(x_t)(z_1^{i_1} z_2^{i_2})$.

2.2 Some Properties and Notation

The following results are going to be employed for the derivation of the main developments in this paper.

Property 1. (Bernal et al., 2022) Let be A, B of appropriate dimensions with $B = B^T > 0$. Then the following holds

$$(A-B)^T B^{-1}(A-B) \geq 0 \iff A^T B^{-1} A \geq A + A^T - B.$$

Lemma 1. (Oliveira and Skelton, 2001) Consider a vector $x \in \mathbb{R}^n$, $\mathcal{Q} = \mathcal{Q}^T \in \mathbb{R}^{n\times n}$ and $\mathcal{W} \in \mathbb{R}^{m\times n}$ such that $\text{rank}(\mathcal{W}) < n$, the following expressions are equivalent

$$1. \chi^T Q \chi < 0, \forall \chi \in \{\chi \in \mathbb{R}^n, \chi \neq 0, \mathcal{W}\chi = 0\}.$$
$$2. \exists \mathcal{M} \in \mathbb{R}^{m\times n} : \mathcal{M}\mathcal{W} + \mathcal{W}^T\mathcal{M}^T + \mathcal{Q} < 0.$$

Generally, designing controllers/observers via TS models yields inequalities involving double/triple convex-sums; thus, in order to obtain LMI conditions the scheduling functions must be removed. Several schemes exist in the literature (Wang et al., 1996; Tuan et al., 2001; Liu and Zhang, 2003). In this paper, the following adaptation of (Tuan et al., 2001) is employed:

Lemma 2. (Relaxation Lemma). Let matrices of appropriate sizes $\Upsilon_{ij}^k = (\Upsilon_{ij}^k)^T$, $(i,j,k) \in \{1, 2, \ldots, 2^p\}$ and the following inequality

$$\sum_{i=1}^{2^p}\sum_{j=1}^{2^p}\sum_{k=1}^{2^p} h_i(z(x_t))h_j(z(x_t))h_k(z(x_{t+1}))\Upsilon_{ij}^k < 0,$$

holds if the following LMIs

$$\frac{2}{2^p - 1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0, \forall (i,j,k) \in \{1, 2, \ldots, 2^p\}. \tag{4}$$

hold too.

In order to make readable the paper, the following notation is used: In matrix expressions, an asterisk $(*)$ denotes the transpose of the symmetric element, this is:

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} A & (*) \\ B & C \end{bmatrix};$$

for in-line expressions, it stands for the transpose of terms on its left side, i.e., $A+B+A^T+B^T+C = A+B+(*)+C$. As for convex expressions: single convex sums are shorten as $\Upsilon_h = \sum_{i=1}^{2^p} h_i(z(x_t))\Upsilon_i$, the inverse of a convex sum is $\Upsilon_h^{-1} = \left(\sum_{i=1}^{2^p} h_i(z(x_t))\Upsilon_i\right)^{-1}$, delayed convex sums $\Upsilon_{h^+} = \sum_{k=1}^{2^p} h_k(z(x_{t+1}))\Upsilon_k$, and double convex sum are simplified as $\Upsilon_{hh} = \sum_{i=1}^{2^p}\sum_{j=1}^{2^p} h_i(z(x_t))h_j(z(x_t))\Upsilon_{ij}$; hence (3) can be shortly written as

$$E_h x_{t+1} = A_h x_t + B_h u_t, \quad y_t = C_h x_t.$$

2.3 Output regulation theory

Consider a linearization around the origin of system (1):

$$E x_{t+1} = A x_t + B u_t, \quad y_t = C x_t \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{o\times n}$, and $E \in \mathbb{R}^{n\times n}$ are known matrices, the exosystem is

$$\omega_{t+1} = S\omega_t, \tag{6}$$

with $\omega_t \in \mathbb{R}^q$ being the state of the exosystem and $S \in \mathbb{R}^{q\times q}$ is such that it has all its eigenvalues on the imaginary axis (Isidori and Byrnes, 1990). Roughly speaking, the objective of output regulation is to design a controller that ensures that the tracking error $e_t = y_t - r_t$ holds

$$\lim_{t\to\infty} e_t = 0.$$

where $r_t = Q\omega_t$, $Q \in \mathbb{R}^{o\times q}$ is the reference output generated by the exosystem (6); this is possible under certain conditions:

- The pair (A, B) is stabilizable.
- The exosystem should be Poisson stable.

Hence, the task it to find gains $F \in \mathbb{R}^{m\times n}$, $\Pi \in \mathbb{R}^{n\times q}$, and $\Gamma \in \mathbb{R}^{m\times q}$ such that

- $A + BF$ is asymptotically stable.
- $x_{ss}(t) = \Pi\omega(t)$ and $u_{ss}(t) = \Gamma\omega(t)$.

Following the results given by Lin and Dai (1996), the traditional FIB equations are adapted for the descriptor system (5), therefore the output regulation problem has solution if and only if the following holds:

$$\begin{aligned} E\Pi S &= A\Pi + B\Gamma \\ 0 &= C\Pi - Q. \end{aligned} \tag{7}$$

If the previous set of equations is satisfied, the control law is

$$u_t = Fx_t - F\Pi\omega_t + \Gamma\omega_t. \tag{8}$$

The next section states the main results of the paper, that is, to provide LMI conditions for the design of a nonlinear gain $\mathcal{F}(x_t)$ (which generalizes the linear one F) and the linear mappings Π and Γ.

3. MAIN RESULTS

The task is to design a nonlinear control law

$$u_t = \mathcal{F}(x_t)x_t - \mathcal{F}(x_t)\Pi\omega_t + \Gamma\omega_t, \tag{9}$$

which is a generalization of (8) in the sense that stabilizing part is nonlinear. Thus, the design is carried out in two parts: the first one is concerned with the stabilization part $\mathcal{F}(x_t)$ while the second involves the linear mappings Π and Γ. An scheme summarizing the approach is depicted in Figure 1.

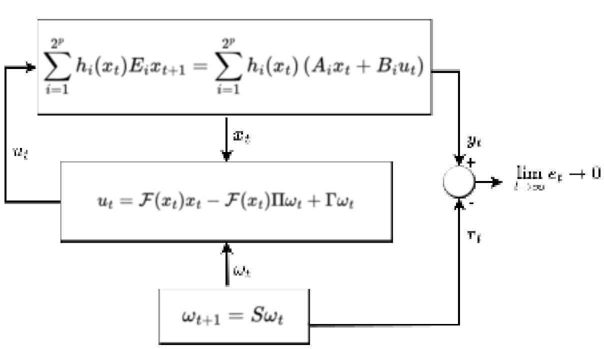

Fig. 1. Diagram of the proposal.

For the stabilization part, let us resume the convex descriptor model (3), for which the following nonlinear control law is to be designed:

$$u_t = \mathcal{F}(x_t)x_t, \quad \mathcal{F}(x_t) = F_h H_h^{-1}, \tag{10}$$

where

$$F_h = \sum_{j=1}^{2^p} h_j(z(x_t))F_j \quad \text{and} \quad H_h^{-1} = \left(\sum_{j=1}^{2^p} h_j(z(x_t))H_j\right)^{-1},$$

with $F_j \in \mathbb{R}^{m\times n}$, $H_j \in \mathbb{R}^{n\times n}$, $j \in \{1, 2, \ldots, 2^p\}$; note that F_h and H_h are convex and depend on the same scheduling functions of (3). The proposed control law is the so-called non-PDC (Estrada-Manzo et al., 2014; Bernal et al., 2022). The closed-loop system yields

$$E_h x_{t+1} = (A_h + B_h F_h H_h^{-1})x_t,$$

which can be conveniently expressed as

$$\begin{bmatrix} A_h + B_h F_h H_h^{-1} & -E_h \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} = 0; \tag{11}$$

note that it has the form $\mathcal{W}\chi = 0$ in Lemma 1. With this in mind, the following result can be stated:

Theorem 3. The origin $x = 0$ of the system (1), under the control law (10), is asymptotically stable if there exist matrices $P_j = P_j^T \in \mathbb{R}^{n\times n}$, $H_j \in \mathbb{R}^{n\times n}$, $F_j \in \mathbb{R}^{m\times n}$, $j \in \{1, 2, \ldots, 2^p\}$ such that $P_j > 0$ and LMIs (4) are satisfied with

$$\Upsilon_{ij}^k = \begin{bmatrix} P_j - H_j^T - H_j & (*) \\ A_i H_j + B_i F_j & P_k - E_i P_k - P_k E_i^T \end{bmatrix}, \tag{12}$$

for $(i, j, k) \in \{1, 2, \ldots, 2^p\}$.

Proof. Consider a Lyapunov function candidate:

$$V(x_t) = x_t^T P_h^{-1} x_t, \quad P_h > 0, \tag{13}$$

$P_j = P_j^T \in \mathbb{R}^{n\times n}$, $j \in \{1, 2, \ldots, 2^p\}$. Its variation is

$$\Delta V(x_t) = x_{t+1}^T P_{h^+}^{-1} x_t - x_t^T P_h^{-1} x_t < 0. \tag{14}$$

In order to use Lemma 1, the above inequality can be expressed as

$$\Delta V(x_t) = \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix}^T \begin{bmatrix} -P_h^{-1} & 0 \\ 0 & P_{h^+}^{-1} \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} < 0. \tag{15}$$

Once again notice that it is in the form of $\chi^T \mathcal{Q}\chi < 0$ as in Lemma 1. Thus, (11) and (15) can be put together by means of Lemma 1:

$$\mathcal{M}\begin{bmatrix} A_h + B_h F_h H_h^{-1} & -E_h \end{bmatrix} + (*) + \begin{bmatrix} -P_h^{-1} & 0 \\ 0 & P_{h^+}^{-1} \end{bmatrix} < 0, \tag{16}$$

where $\mathcal{M} \in \mathbb{R}^{2n\times n}$ is a free matrix to be pre-selected such that a set of LMIs arises. Using the congruence property, that is, pre-multiplying (16) with $\begin{bmatrix} H_h^T & 0 \\ 0 & P_{h^+} \end{bmatrix}$ and post-multiplying by its transpose together with $\mathcal{M} = \begin{bmatrix} 0 \\ P_{h^+}^{-1} \end{bmatrix}$ yields

$$\begin{bmatrix} -H_h^T P_h^{-1} H_h & (*) \\ A_h H_h + B_h F_h & P_{h^+} - E_h P_{h^+} - P_{h^+} E_h^T \end{bmatrix} < 0, \tag{17}$$

applying **Property 1** gives:

$$\begin{bmatrix} P_h - H_h^T - H_h & (*) \\ A_h H_h + B_h F_h & P_{h^+} - E_h P_{h^+} - P_{h^+} E_h^T \end{bmatrix} < 0.$$

Finally by means of the relaxation Lemma 2 the desired result appears. □

The LMIs of Theorem 3 can be improved in terms of decay rate, that is, the convergence speed (Tanaka and Wang, 2001).The following result provides LMIs for such task.

Corollary 4. The origin $x = 0$ of the system (1), under the control law (10), is asymptotically stable with decay rate $\alpha < 1$, $0 < \alpha^2 < 1$ if there exist matrices $P_j = P_j^T \in \mathbb{R}^{n\times n}$, $H_j \in \mathbb{R}^{n\times n}$, $F_j \in \mathbb{R}^{m\times n}$, $j \in \{1, 2, \dots, 2^p\}$ such that $P_j > 0$ and LMIs (4) are satisfied with

$$\Upsilon_{ij}^k = \begin{bmatrix} -\alpha^2 H_j^T - \alpha^2 H_j + \alpha^2 P_j & (*) \\ A_i H_j + B_i F_j & P_k - E_i P_k - P_k E_i^T \end{bmatrix}, \tag{18}$$

for $(i, j, k) \in \{1, 2, \dots, 2^p\}$.

Proof. It follows similar lines as the one in Theorem 3 but considering $\Delta V(x_t) \leq (\alpha^2 - 1)V(x_t)$. □

The stabilization part of the full control law (9) has been just developed, now let us provide LMI conditions for the design of Π and Γ, these gains are going to be designed by means of the linear approximation (5) together with the exosystem (6). To this end, let us recall a previous result given in (Bernal et al., 2012).

Remark 5. Equations can be approximated by an element-wise minimization problem, i.e., $M(x) - N(x) = 0$, $M(x)$ and $N(x)$ being continuously differentiable linear matrix functions of the decision variable vector x, can be solved as a minimization problem: $\min \varepsilon > 0 : -\varepsilon \prec M(x) - N(x) \prec \varepsilon$, that is

$$\min \quad \varepsilon > 0 : \begin{cases} M(x) - N(x) - \varepsilon \prec 0 \\ M(x) - N(x) + \varepsilon \succ 0 \end{cases},$$

where $\prec$ and $\succ$ stand for element-wise lower-than and greater-than, respectively.

Remark 6. The solution of the FIB equations (7) can be approximated by an LMI optimization problem

$$\begin{aligned} &\min \quad \varepsilon > 0 \ \text{ such that:} \\ &-\varepsilon \prec \begin{bmatrix} A\Pi + B\Gamma - E\Pi S & 0 \\ 0 & C\Pi - Q \end{bmatrix} \prec \varepsilon. \end{aligned} \tag{19}$$

The following result provides conditions for the output regulation problem of discrete-time nonlinear descriptor (1):

Theorem 7. The output of the nonlinear system (1) tracks a desired signal $y_r(t)$ if there exist a nonlinear control law (9) where the stabilization gain $\mathcal{F}(x)$ is computed from LMI conditions in Theorem 3 and gains Π and Γ are obtained from the LMI minimization problem in Remark (6).

4. NUMERICAL EXAMPLES

The proposed results are illustrated with numerical examples. For both examples, let us consider the harmonic oscillator as the exosystem:

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad Q = [1 \ \ 0]. \tag{20}$$

Example 8. Consider a nonlinear descriptor system (1) with:

$$E(x_t) = \begin{bmatrix} 0.9 & 0.1 + 0.4\cos x_1 \\ 0.1 + 0.4\cos x_1 & 1.4 \end{bmatrix},$$

$$A(x_t) = \begin{bmatrix} 0.2 + 0.12\cos x_1 & 1.6 \\ -0.8 & 0 \end{bmatrix}, B(x_t) = \begin{bmatrix} 0.1 \\ -2 - 1.04\sin x_1 \end{bmatrix},$$

with output $y_t = x_1$. The nonlinear terms are $z_1 = \cos x_1 \in [-1, 1]$, $z_2 = \sin x_1 \in [-1, 1]$, the bounds have been computed within the region $\Omega = \{x \in \mathbb{R}^n : |x_1| \leq \pi\}$. The convex weighting functions are $w_0^1(\cdot) = (z_1^1 - \cos x_1) / (z_1^1 - z_1^0)$, $w_1^1(\cdot) = 1 - w_0^1$, $w_0^2 = (z_2^1 - \sin x_1) / (z_2^1 - z_2^0)$, $w_1^2 = 1 - w_0^2$. LMI conditions in Corollary 4 are feasible for $\alpha^2 = 0.99$ yielding gains

$$F_1 = [-0.5231 \ \ 0.1130], F_2 = [-0.3400 \ \ 0.0856],$$
$$F_3 = [-0.3577 \ \ 0.0510], F_4 = [-0.3390 \ \ 0.0474],$$

$$H_1 = \begin{bmatrix} 0.3446 & -0.0027 \\ -0.0006 & 0.0698 \end{bmatrix}, H_2 \begin{bmatrix} 1.3241 & -0.1868 \\ -0.2631 & 0.1448 \end{bmatrix},$$
$$H_3 = \begin{bmatrix} 0.2653 & -0.1082 \\ -0.0288 & 0.0750 \end{bmatrix}, H_4 = \begin{bmatrix} 1.3772 & -0.2963 \\ -0.1876 & 0.1291 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0.3042 & -0.0500 \\ -0.0500 & 0.0719 \end{bmatrix}, P_2 = \begin{bmatrix} 0.3723 & -0.0736 \\ -0.0736 & 0.0903 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 0.3390 & -0.0722 \\ -0.0722 & 0.0810 \end{bmatrix}, P_4 = \begin{bmatrix} 0.3489 & -0.0715 \\ -0.0715 & 0.0833 \end{bmatrix}.$$

On the other hand, LMIs in Remark 5 are feasible with $\varepsilon = 6.8976 \times 10^{-12}$ and gains

$$\Pi = \begin{bmatrix} 1 & 0 \\ -0.3381 & 0.4577 \end{bmatrix} \text{ and } \Gamma = [-0.0796 \ \ -0.0134].$$

The simulation results for initial conditions $x(0) = [0.5 \ \ -0.5]^T$ and $\omega(0) = [-0.1 \ \ 0.1]^T$ have been run. Figure 2 shows the output of the system and the reference signal while Figure 3 plots the tracking error. It can be seen that regulation is achieved.

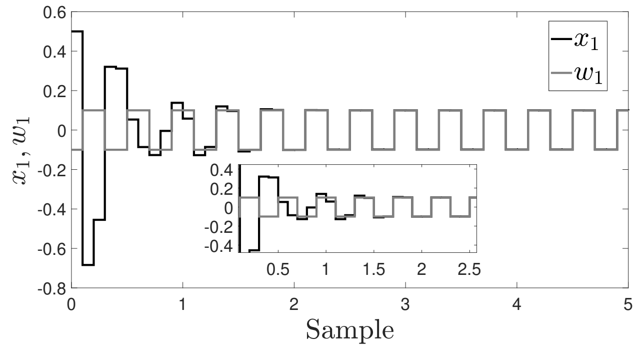

Fig. 2. Output x_1 versus reference w_1 in Example 8.

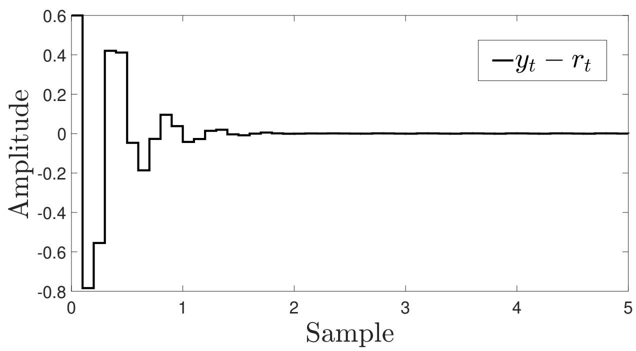

Fig. 3. Tracking error $e_1 = x_1 - w_1$ for Example 8.

Example 9. Consider the descriptor (1) with matrices

$$E(x_t) = \begin{bmatrix} 0.8 & 0.1 + 0.7\eta \\ 0.1 - 0.7\eta & 0.9 \end{bmatrix},$$

$$A(x_t) = \begin{bmatrix} -0.4 & 0.83 \\ 1.01\dfrac{\sin x_1}{x_1} & 0.2 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

with $\eta = (1 + x_2^2)^{-1}$, the output is $y_t = x_1$. Within the region $\Omega = \{x \in \mathbb{R}^n : |x_1| \leq \pi, |x_2| \leq 1\}$ the nonlinear terms are bounded as follows $z_1 = \eta \in [0.5, 1]$ and $z_2 = \frac{\sin x_1}{x_1} \in [0.990, 1]$. The weighting functions are computed as in (2). The exosystem is the same as before, with values (20). Once again, the stabilizer is computed with LMIs in Corollary 4 with $\alpha^2 = 0.8$ and gains

$$P_1 = \begin{bmatrix} 0.0401 & -0.0053 \\ -0.0053 & 0.0419 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0225 & -0.0054 \\ -0.0054 & 0.0314 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.0406 & -0.0053 \\ -0.0053 & 0.0421 \end{bmatrix}, P_4 = \begin{bmatrix} 0.0237 & -0.0054 \\ -0.0054 & 0.0323 \end{bmatrix},$$

$$F_1 = [0.0259 \ -0.0619], F_2 = [0.0352 \ -0.0495],$$

$$F_3 = [0.0274 \ -0.0612], F_4 = [0.0394 \ -0.0493],$$

$$H_1 = \begin{bmatrix} 0.0330 & -0.0065 \\ 0.0006 & 0.0354 \end{bmatrix}, H_2 = \begin{bmatrix} 0.0235 & -0.0011 \\ -0.0052 & 0.0300 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0.0333 & -0.0064 \\ 0.0005 & 0.0355 \end{bmatrix}, H_4 = \begin{bmatrix} 0.0248 & -0.0008 \\ -0.0051 & 0.0305 \end{bmatrix}.$$

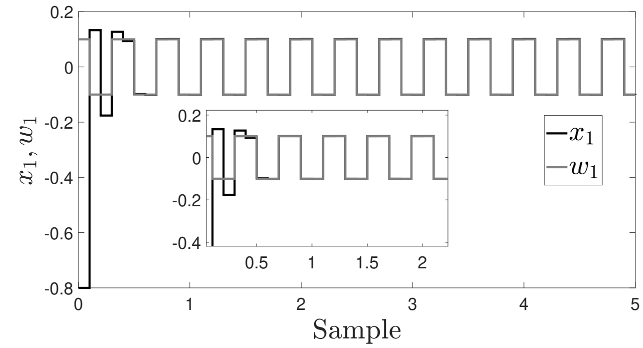

Fig. 4. Output x_1 vs reference w_1 in Example 9.

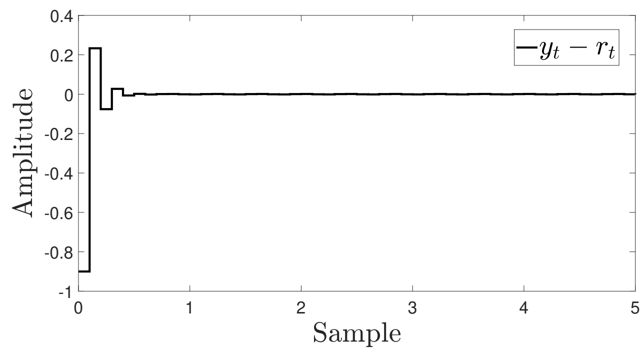

Fig. 5. Tracking error $e_1 = x_1 - w_1$ for Example 9.

The mappings are computed by means of Remark 5, the minimization problem therein is feasible with $\varepsilon = 1.4511 \times 10^{-12}$ and matrices:

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0.3976 & -1.2106 \end{bmatrix} \text{ and } \Gamma = [2.0768 \ 4.2458].$$

With initial conditions $x(0) = [-0.8 \ 0.8]^T$ and $\omega(0) = [0.1 \ \ -0.1]^T$, simulation results are shown in figures 4 and 5. As expected, output regulation takes place.

5. CONCLUSION

It has been presented the design of a nonlinear regulator for discrete-time nonlinear descriptor systems; thus, the resulting closed-loop system asymptotically tracks a reference generated by an exosystem. The designing conditions are in terms of LMIs, which can be efficiently solved by convex optimization techniques already available in commercial software. Two numerical examples illustrate the performance of the proposal. As future work, a full nonlinear regulator is to be designed.

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