Memorias del 2024 Congreso Nacional de Control Automático
8-11 de Octubre, 2024. Ciudad de México, México.

Synchronization of Bidirectionally Coupled Nonidentical Systems Via Output Feedback

Juan Gonzalo Barajas-Ramírez *
Hugo Gustavo González-Hernández **

* *IPICYT, División de Control y Sistemas Dinámicos, Camino a la Presa San José 2055, San Luis Potosí, 78216, S. L. P., México, (e-mail: jgbarajas@ipicyt.edu.mx.)*
** *Tecnológico de Monterrey, Escuela de Ingeniería y Ciencias, Monterrey, 64849, N. L., México.*

Abstract: We investigate the synchronization of bidirectionally coupled nonidentical chaotic systems. We propose a design of their coupling to achieve synchronization that takes into consideration the variable structure nature of the systems, and guarantees the stability of an identical synchronization manifold for nonidentical systems using output feedback linearization. We illustrate our results with numerical simulations of well-known benchmark chaotic systems.

Keywords: Synchronization, Bidirectional coupling, Nonlinear control, Relative degree.

1. INTRODUCTION

Two or more systems are said to be synchronized if through a subtle interaction between them results in their behaviors being correlated in time. From this point of view many different types of synchronized behaviors can be defined including: identical, phase, and generalized synchronization to mention but a few (Pikovsky et al., 2001; Boccaletti et al., 2002). In the simplest case, two subsystems are *unidirectionally* connected, which is usually called *drive-response* configuration (Pecora and Carroll, 1990). In this case, the interconnection can easily be interpreted, from the viewpoint of control theory, as a design problem where the coupling term in the response subsystem can be obtained using different control methodologies like robust (Almeida et al., 2006), adaptive (Hong et al., 2001) and optimal design (Pan and Yin, 1997) techniques. Alternatively to a *drive-response* configuration, two dynamical systems can be *bidirectionally* coupled. In general terms, in this configuration the synchronization problem is more complex since both subsystems depend on each other through their interactions (Boccaletti et al., 2002). The solutions to bidirectional synchronization problem has been naturally extended to the context of dynamical networks (Boccaletti et al., 2006), where problems like consensus and pinning are significant research topics (Su and Wang, 2013).

In this contribution we propose a synchronization scheme for bidirectionally coupled smooth nonlinear systems with full relative degree. Our proposal consists on a nonlinear feedback law for synchronization of bidirectionally coupled systems with parameter mismatch and non-identical possibly non-smooth components.

The remainder of this contribution is organized as follows. Section 2 contains the bidirectional synchronization problem description. Section 3 deals with the proposed synchronization strategy. In Section 4 some numerical results of applying our proposed synchronization strategy to well-known chaotic systems. Finally we conclude with a discussion and some future work.

2. SYSTEMS PRELIMINARIES

Consider two systems of the form

$$\begin{aligned} \dot{x}^j(t) &= F^j(x^j(t)) + G^j(x^j(t))u^j(t) \\ y^j(t) &= H^j(x^j(t)) \quad \text{for } j = 1, 2. \end{aligned} \tag{1}$$

where $x^j(t) \in \mathbb{R}^n$ are state vectors, for $j = 1, 2$ and vector fields $F^j(x^j(t))$ and $G^j(x^j(t))$, states $u^j(t) \in \mathbb{R}$ and $F^j(x^j(t))$ possibly describing chaotic behavior.

Definition 1. (State synchronization) For two systems of the form (1) with chaotic behavior and possibly non-smooth vector fields, that are coupled together. Synchronization is determine from the state-synchronized error

$$e(t) = x^1(t) - x^2(t) \tag{2}$$

Then, if

$$\lim_{t\to\infty} ||e(t)|| = 0 \tag{3}$$

the systems in (1) are (asymptotically) identically synchronized.

The systems under consideration can be written in normal form. In particular, if they are full relative

https://doi.org/10.58571/CNCA.AMCA.2024.003

degree a more convenient representation is possible. To this end, we give the following definition.

Definition 2. (Full relative degree) System (1) is said to have *full relative degree* if

(i) $L_G L_F^k H(x(t)) = 0$, $k = 0, 1, ..., n-2$
(ii) $L_G L_F^{n-1} H(x(t)) \neq 0$

where $L_F H(x) = \frac{\partial H(x(t))}{\partial x} F(x(t))$ denotes the Lie derivative of H(x(t)) along F(x(t)), with $L_F^0 H(x(t)) = H(x(t))$.

Assuming the systems in (1) is full relative degree they can be written as:

Theorem 1. The following system has full relative degree

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), (i = 1, ..., n-1) \qquad (4)\\ \dot{x}_n(t) &= f(x(t)) + g(x(t))u(t)\\ y(t) &= x_1(t) \end{aligned}$$

for $f(x(t))$ and $g(x(t)) \neq 0$ scalar functions and $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^\top$ the state vector.

Proof 1. The proof of Theorem 1 is straightforward from Definition 2. Let us define

$$F(x(t)) = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ f(x(t)) \end{bmatrix}; G(x(t)) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x(t)) \end{bmatrix} \qquad (5)$$

and $H(x(t)) = h(x(t)) = x_1(t)$, in order to prove condition i) from definition 2, we find

$$\begin{aligned} L_F H(x(t)) &= x_2(t)\\ L_F^2 H(x(t)) &= x_3(t)\\ &\vdots\\ L_F^{n-2} H(x(t)) &= x_{n-1}(t)\\ L_G L_F^{n-2} H(x(t)) &= 0 \end{aligned}$$

Condition ii) is proven as follows

$$\begin{aligned} L_F^{n-1} H(x(t)) &= x_n(t)\\ L_G L_F^{n-1} H(x(t)) &= g(x(t)) \neq 0 \end{aligned}$$

Let a system like (1) have full relative degree. Then it can be rewritten in its normal form through the following coordinates change $x^j = \varphi(x)$ for system j (Isidori, 1985; Nijmeijer and Van der Schaft, 1990).

$$x_i(t) = \varphi_i(x(t)) = L_F^{i-1} F H(x(t)) \qquad (6)$$

for $i = 1, 2, ..., n$. In the following sections we proposed a solution for the synchronization problem for full relative degree systems with chaotic dynamics with possibly nonsmooth components.

3. SYNCHRONIZATION STRATEGY

Consider a couple of systems of the form:

$$\begin{aligned} \dot{x}_i^j(t) &= x_{i+1}^j(t), (i = 1, ..., n-1) \qquad (7)\\ \dot{x}_n^j(t) &= f^j(x^j(t)) + g^j(x^j(t))u^j(t)\\ y^j(t) &= x_1^j(t) \end{aligned}$$

for $j = 1, 2$, $x_i(t) \in \mathbb{R}$, $i = 1, 2..., n$, and $f^j()$ and $g^j()$ are scalar functions. We define the state synchronization error as in (2), then the dynamics for the synchronization error can be expressed by

$$\dot{e}_i(t) = e_{i+1}(t), (i = 1, ..., n-1) \qquad (8)$$
$$\dot{e}_n(t) = f(x^1(t), x^2(t)) + g(x^1(t), x^2(t))u(t) \qquad (9)$$
$$y(t) = e_1(t) \qquad (10)$$

where $f(x^1(t), x^2(t)) = f^1(x^1(t)) - f^2(x^2(t))$ and $g(x^1(t), x^2(t)) = g^1(x^1(t)) - g^2(x^2(t))$.

In this sense, the synchronization error dynamics are again in normal form which facilities the design of the coupling controller such that synchronization is achieved.

3.1 Synchronizability

A system like (8)-(10) with functions involving states from different systems should be analyzed to assure controllability. For error synchronization dynamics, let us now define:

$$F(e) = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ f(x^1, x^2) \end{bmatrix}; G(e) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x^1, x^2) \end{bmatrix} \qquad (11)$$

and extended state $x = \left[(x^1)^T \ (x^2)^T\right]^T$ Let us give some preliminary definitions (Nijmeijer and Van der Schaft, 1990)

Definition 3. (Lie Brackets) (Isidori, 1985) Consider vector fields $F(e)$ and $G(e)$ in $\mathbb{R}^n$ according to (5). Then the *Lie Bracket* operation generates a new vector field:

$$[F, G] = \frac{\partial G}{\partial x} F - \frac{\partial F}{\partial x} G \qquad (12)$$

Higher order Lie Brackets can be obtained as

$$\begin{aligned} (ad_F^1, G) &= [F, G]\\ (ad_F^2, G) &= [F, [F, G]]\\ &\vdots\\ (ad_F^k, G) &= [F, (ad_F^{k-1}, G)] \end{aligned}$$

Thus, we introduce the following proposition

Proposition 1. (Synchronizability) The system defined by (8)-(10), is locally accessible about $e = 0$ if the accessibility distribution $\mathcal{C}$ spans n space. $\mathcal{C}$ is defined by:

$$\mathcal{C} = [g, [ad_F^k g]] \qquad (13)$$

for $k = 1, 2, ...$, i.e. distribution $\mathcal{C}$ is involutive.

This will be considered as the *synchronizability condition* further on.

3.2 The synchronization law

Let us define the error vector $e = [e_1 \; e_2 \; ... \; e_n]^T$, and some constant vector $a = [-a_n \; -a_{n-1} \; ... \; -a_1]^T$, such that the polynomial $\lambda^n + a_1\lambda^{n-1} + ... + a_n$ is strictly Hurwitz. Then, provided *synchronizability condition* a synchronization control law can be stated for (8)-(10) in the following form

$$u(t) = \frac{a^T e - f(x^1, x^2)}{g(x^1, x^2)} \tag{14}$$

Application of (14) over (8)-(10) leads to the following linear, asymptotically stable dynamics:

$$\begin{aligned} \dot{e}_i &= e_{i+1}, (i = 1, ..., n-1) \\ \dot{e}_n &= a^T e \\ y &= e_1 \end{aligned} \tag{15}$$

which can be expressed as $\dot{e} = Ae$ whose equilibrium point $e = 0$ is asymptotically stable, meaning that synchronization error vanishes when $t \to \infty$.

4. SIMULATION RESULTS

In the following subsections we will describe some simulation results for synchronization law proposed applied to several well-known chaotic systems.

4.1 Continuous case

For simulation purposes we have chosen a slightly different system from one of the circuits proposed in (Sprott, 2000) which may exhibit chaotic behavior for certain set of parameters. Circuit chosen is of the form:

$$\dddot{x} = -\mu\ddot{x} + \dot{x}^2 - x + \beta u \tag{16}$$

which exhibits chaotic behavior for $\mu = -2.017$ and $\beta = 0$, Lyapunov exponents are $(0.055, 0, -2.072)$. It is easy to show that (16) has full relative degree by selecting output $y = x$, thus a couple of Sprott-like circuits can be written as follows

$$\begin{aligned} \dot{x}_1^j &= x_2^j \\ \dot{x}_2^j &= x_3^j \\ \dot{x}_3^j &= -x_1^j + (x_2^j)^2 + \mu_j x_3^j + \beta_j u \\ y_j &= x_1^j \end{aligned} \tag{17}$$

for $j = 1, 2$. Let us define the error $e_1 = x_1^1 - x_1^2 = y_1 - y_2$, then the error dynamics are depicted as

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= -e_1 + e_2(e_2 + 2x_2^2) + \mu_1 x_3^1 - \mu_2 x_3^2 + \tilde{\beta} u \end{aligned} \tag{18}$$

for $\tilde{\beta} = \beta_1 - \beta_2$ and $\beta_1 \neq \beta_2$. Systems (17) show chaotic behavior for $\mu_j = -2.017$, $\beta_j = 0$, $(j = 1, 2)$ and initial conditions $x^j(0) = [0 \; 0 \; 1]^T$.

Clearly, if $\mu_1 = \mu_2$ and $x^1(0) = x^2(0)$, circuits are perfectly synchronized. In order to prove our

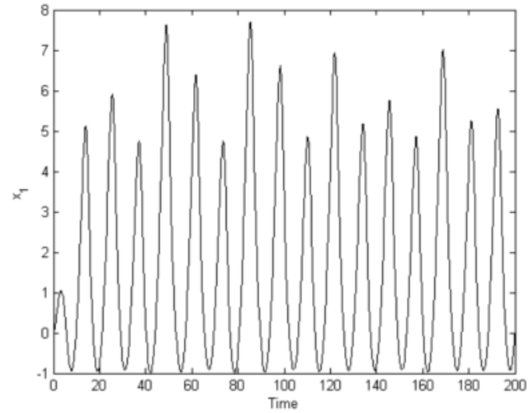

Fig. 1. Time evolution for x_1 variable of system (17) for $\mu_1 = -2.017$

proposal, we use $\mu_1 = -2.017$ and $\mu_2 = -2.02$. Fig. 1 shows solution for x_1 for system 16. Although this parameter mismatch appears to be non significant, time responses for both systems diverge considerably as shown in Fig. 2.

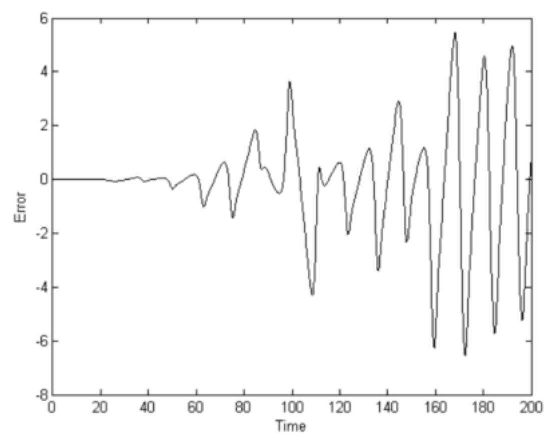

Fig. 2. Error for x_1 of systems (17) with respect to time considering parameter mismatch

In order to synchronize the states (and outputs) of both systems $e \to 0$, we propose the synchronization law:

$$u = \frac{1}{\tilde{\beta}}[a^T e + e_1 - e_2(e_2 + 2x_2^2) - \mu_1 x_3^1 + \mu_2 x_3^2] \tag{19}$$

For demonstration purposes, let us now choose $a = [-6 \; -11 \; -6]^T$ for behavioral modes located at $\lambda = -1, -2, -3$. For this synchronization law, an asymptotically stable equilibrium point is expected for the closed loop system.

Synchronization law was applied at $t = 70s$ and its time evolution is depicted in Fig. 3. Synchronization error is shown in Fig. 4.

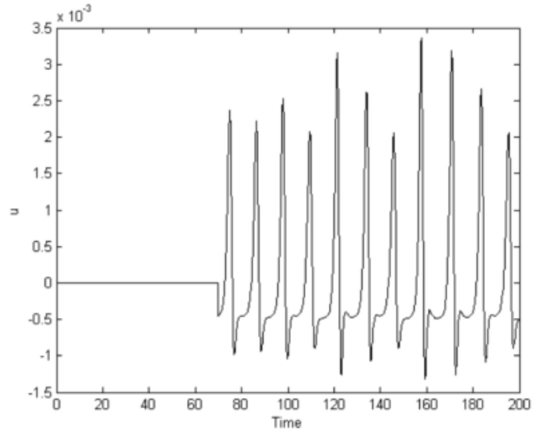

Fig. 3. Synchronization law for (17)

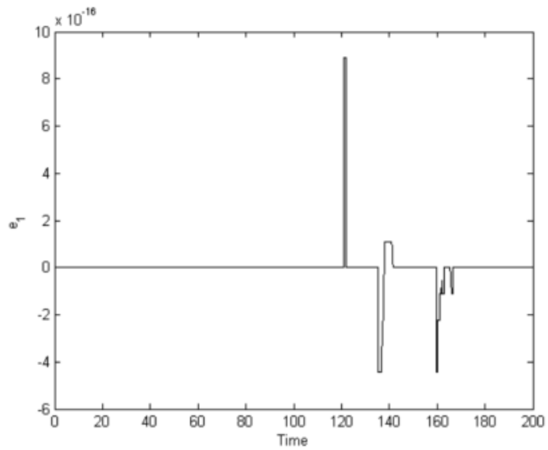

Fig. 4. Synchronization error for systems (17)

4.2 Non-identical systems

In this section we deal with the synchronization of a couple of different systems applying the proposed strategy. We choose a normal-form Lorenz system and a Sprott system. Lorenz dynamics can be described as

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 + u \\ y &= x_1\end{aligned} \tag{20}$$

It is well known that for parameters $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$ it shows chaotic behavior. Clearly, system (20) is not in the form (**??**), but it can be expressed in normal form by using coordinates transformation shown in (**??**), i.e. $x^1 = \varphi(x)$. Thus, for the Lorenz system, new coordinates are:

$$\begin{aligned}\varphi_1(x) &= = x_1 \\ \varphi_2(x) &= \sigma(x_2 - x_1) \\ \varphi_3(x) &= -\sigma^2(x_2 - x_1) + \sigma(\rho x_1 - x_2 - 20 x_1 x_3)\end{aligned} \tag{21}$$

leading to the following dynamics

$$\begin{aligned}\dot{x}_1^1 &= x_2^1 \\ \dot{x}_2^1 &= x_3^1 \\ \dot{x}_3^1 &= f^1(x^1) + g^1(x^1)u \\ y &= x_1^1\end{aligned} \tag{22}$$

for

$$\begin{aligned}f^1(x^1) = {}& (\rho - 1)\sigma x_2^1 - \\ & (\sigma + 1)x_3^1 - (x_1^1)^2(x_2^1 + \sigma x_1^1) - \\ & (\beta x_1^1 - x_2^1) \\ & \left[\frac{\sigma(1-\rho)x_1^1 + (\sigma+1)x_2^1 + x_3^1}{x_1^1}\right]\end{aligned} \tag{23}$$

and

$$g^1(x^1) = -\sigma x_1^1 \tag{24}$$

System (22) is now in normal form. Let us now consider another system to synchronize with:

$$\dddot{x} = -\mu\ddot{x} - \dot{x} + x - x^3 + \beta u \tag{25}$$

Equation (25) represent an electronic circuit described in Sprott (2000) and exhibits chaotic behavior for $\mu = 0.39$ and $\beta = 0$. Normal form for the system is shown below

$$\begin{aligned}\dot{x}_1^2 &= x_2^2 \\ \dot{x}_2^2 &= x_3^2 \\ \dot{x}_3^2 &= -\mu x_3^2 - x_2^2 + x_1^2 - (x_1^2)^3 + \beta u \\ y^2 &= x_1^2\end{aligned} \tag{26}$$

Synchronization error is defined as $e_1 = y^1 - y^2 = x_1^1 - x_1^2$, leading to the following synchronization error dynamics

$$\begin{aligned}\dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= f(x^1, x^2) + g(x^1, x^2)u\end{aligned} \tag{27}$$

with $f(x^1, x^2) = f^1(x^1) - f^2(x^2)$ and $g(x^1, x^2) = \beta - \sigma x_1^1$. Output error between both systems is shown in Fig. 5. Thus, a synchronization law can be stated in

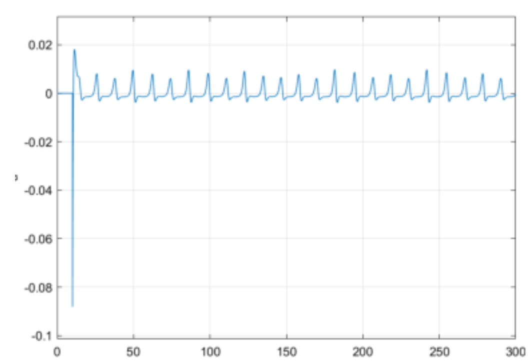

Fig. 5. Time evolution for error between Lorenz and Sprott systems in open loop

the form

$$u = \frac{a^T e - f(x^1, x^2)}{g(x^1, x^2)} \tag{28}$$

For demonstration purposes, we apply synchronization law at $t = 100s$, Fig. 6 shows time response for e_1.

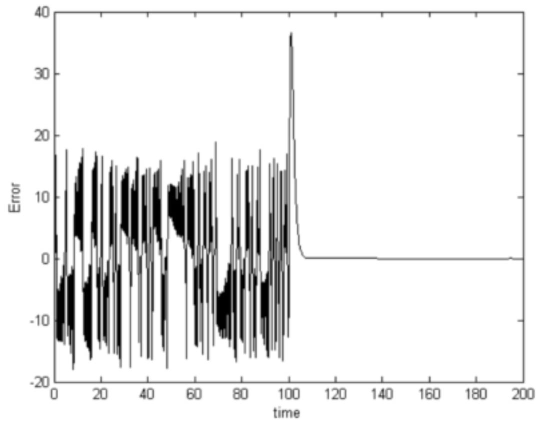

Fig. 6. Error e_1 between (22) and (26) with synchronization law applied at $t = 100s$.

5. CONCLUDING REMARKS AND FUTURE WORK

The proposed strategy has shown successful synchronicity for a class of chaotic systems and it can be implemented for systems depicting some nonsmoothness features. It has been implemented for systems not in normal form but fully linearizable with coordinates transformation. Future work is focused in synchronization for more than two systems, and on robust synchronization using Internal model controller.

REFERENCES

Almeida, D.I.R., Alvarez, J., and Barajas, J.G. (2006). Robust synchronization of sprott circuits using sliding mode control. *Chaos, Solitons & Fractals*, 30(1), 11–18.

Boccaletti, S., Kurths, J., Osipov, G., Valladares, D., and Zhou, C. (2002). The synchronization of chaotic systems. *Physics reports*, 366(1-2), 1–101.

Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., and Hwang, D.U. (2006). Complex networks: Structure and dynamics. *Physics reports*, 424(4-5), 175–308.

Hong, Y., Qin, H., and Chen, G. (2001). Adaptive synchronization of chaotic systems via state or output feedback control. *International Journal of Bifurcation and Chaos*, 11(04), 1149–1158.

Isidori, A. (1985). *Nonlinear control systems: an introduction*. Springer.

Nijmeijer, H. and Van der Schaft, A. (1990). *Nonlinear dynamical control systems*, volume 464. Springer.

Pan, S. and Yin, F. (1997). Optimal control of chaos with synchronization. *International Journal of Bifurcation and Chaos*, 7(12), 2855–2860.

Pecora, L.M. and Carroll, T.L. (1990). Synchronization in chaotic systems. *Physical review letters*, 64(8), 821.

Pikovsky, A., Rosenblum, M., Kurths, J., and Synchronization, A. (2001). A universal concept in nonlinear sciences. *Self*, 2, 3.

Sprott, J.C. (2000). Simple chaotic systems and circuits. *American Journal of Physics*, 68(8), 758–763.

Su, H. and Wang, X. (2013). *Pinning control of complex networked systems: Synchronization, consensus and flocking of networked systems via pinning.* Springer Science & Business Media.