Memorias del 2024 Congreso Nacional de Control Automático
8-11 de Octubre, 2024. Ciudad de México, México.

Modelling a Variable Mass Pendulum System

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Abstract: The study of pendulum-based systems has been a subject of interest, from the cart-pendulum and the Kapitza pendulum, among others, which are the most studied. However, one of the less addressed pendulums is the variable mass pendulum, which involves a difficulty not common in other types of systems. Due to this, the modelling of a variable mass system is carried out using the Euler-Lagrange formalism, thus finding a nonlinear system. The dynamic modelling of this system does not delve into the characteristics of variable mass but presents the intrinsic difficulties of such a system.

Keywords: Modelling, Lagrangian and Hamiltonian systems, Variable mass pendulum, Time-varying systems, Nonlinear systems.

1. INTRODUCTION

In the study of dynamic systems within educational institutions, nonlinear systems are often approached through the analysis of pendulums. Throughout the history of physics, this system has been studied by great pioneers such as Galileo Galilei, who laid much of the groundwork for what we now know as kinematics (Erlichson, 1999; Kossovsky, 2020). The modelling of a simple pendulum system poses a significant challenge with regard to its analytical solution. To simplify the problem, certain constraints can be introduced; however, variations of the model can also be created to study different phenomena, further increasing the system's complexity (Yu et al., 2024).

Among the wide range of pendulum model variations are the double pendulum, the spring pendulum, the pendulum on a moving cart, and others. One of the least studied systems is the variable mass pendulum, an uncommon variation generally not covered in most academic physics courses. In conventional physics courses taught at secondary education institutions, examples where mass variation is present are generally not shown (Matthews, 2000; Dandare, 2018; Femandes et al., 2022). The mass is typically assumed to be constant, leading these examples to go unnoticed by students. This oversight means students miss the opportunity to explore intriguing results that could stimulate further investigation into these or similar phenomena. Introducing variable mass pendulum systems in physics curricula could thus enhance students' understanding to engage them more deeply with the study of dynamic systems (Nanjangud and Eke, 2018).

The challenges faced when working with variable mass system models are complex. These range from the function that describes the mass variation, the function that describes the position of the system's centre of mass, and other dependent parameters, not to mention the practical implementation of the experiment (Eke and Mao, 2002). Setting a variable mass system in motion and being able to measure the parameters in a controlled environment represents a significant practical challenge (Fernández Guasti, 2007).

Another key aspect of working with such systems is the control of the system itself. There are variables that depend on others (Xie et al., 2024), and the slightest change or disturbance could affect the behaviour, potentially leading to chaos. Therefore, controlling such a system is a complex task, but one that could serve as an incentive to continue studying these kind of systems.

1.1 Pendula

The history of the pendulum dates back to around the 1600s when Galileo Galilei observed the peculiar motion of a chandelier and noted its extremely periodic oscillations (Bedini, 1992). This observation motivated him to study pendulums in depth (Teerikorpi et al., 2009). It is noteworthy that, although Galileo studied them extensively,

⋆ The authors acknowledge partial economical support by projects 20242752, 20240894, 20241163, 20241077 and 20240811, as well as EDI and PIFI grants, provided by Secretaría de Investigación y Posgrado, Instituto Politécnico Nacional.
https://doi.org/10.58571/CNCA.AMCA.2024.027

he did not discover the pendulum; rather, he dedicated time to its study, making him the first person to formalise the study of this system. However, the simple pendulum had applications long before Galileo's studies. Evidence has been found of pendulums used in primitive seismographs, which fundamentally relied on the behaviour of the pendulum (Fréchet and Rivera, 2012).

The study of oscillating systems is vast, with the simple pendulum being the quintessential model for initiating the analysis of these systems. However, it is a highly idealised model that lacks the ability to explain a large number of involved phenomena. The wide variation in the behaviours of different types of pendulums led to the development of new systems and mathematical models that more closely resemble observed phenomena (Domadzra et al., 2024). The second most common example is the inverted pendulum, which, despite behaving very similarly to the simple pendulum, has distinguishing characteristics that make it mathematically incompatible with the simple model. The list of pendulum types is extensive, including the Foucault pendulum, double pendulum, triple pendulum, inverted pendulum, and spring pendulum (Shinbrot et al., 1992; Awrejcewicz et al., 2007; Diniz, 2023; Szuminski and Maciejewski, 2024), among others. However, there is a much unknown and non-studied model: the variable mass pendulum, which is the focus of this study.

1.2 Variable mass systems

Most physical systems typically have the characteristic of maintaining certain quantities constant throughout their behaviour, such as height, volume, moments of inertia, and length. In most cases, the commonest constant quantity in the majority of systems is mass, as it usually does not vary over time, at least not within the period of study during which the systems are analysed, or due to the insignificance of the change. When referring to variable mass systems, it encompasses all systems in which the mass does not remain constant, changing as a function of time. This leads to mathematical complications regarding the description of the system's behaviour (Eke, 1998; Eke and Mao, 2002; Gui-cheng, 2010). The following systems are examples of variable mass models:

- Airplanes
- Rockets
- Ships
- Automobiles
- Helicopters

2. DESCRIPTION OF THE MODEL

The system under consideration models a pendulum consisting of a rigid bar and a cylindrical container filled with fluid, where the bottom has a hole allowing the fluid (in this case water) to exit. The length of the cylinder is significantly greater than its diameter, designed to prevent the fluid surface inside from becoming parallel to the horizontal. Instead, the goal is to maintain the water surface as parallel as possible to the circular faces of the cylinder, as shown in Figure 2. This preserves cylindrical symmetry, crucial for ensuring that the fluid and the centre of mass of the cylinder (when full - Figure 3, half-full - Figure 2, empty - Figure 1) move along the cylinder's axis of symmetry.

The pendulum is suspended by an inextensible cable of length "l" attached to the geometric centre of the non-perforated circular face. It is assumed that the cable does not bend at any point during the motion.

Fig. 1. Cross-sectional view of the empty model

Fig. 2. Cross-sectional view of the model at half capacity

Fig. 3. Cross-sectional view of the model at full capacity.

3. MODELLING USING THE EULER-LAGRANGE FORMALISM

The system under consideration will be modelled through the Euler-Lagrange formalism. The system consists of the following parameters: the length of the cable (from which the cylinder hangs and is considered constant), denoted as l; the variable distance from the upper face of the cylinder to the centre of mass, denoted as r (thus, the amount

https://doi.org/10.58571/CNCA.AMCA.2024.027

of material inside the cylinder will change and hence the moment of inertia); and the angle of rotation of the cable relative to the vertical axis. Thus, the generalised coordinates of the system are the rotation angle θ and the distance r from the cable to the centre of mass, as depicted in Figure 4.

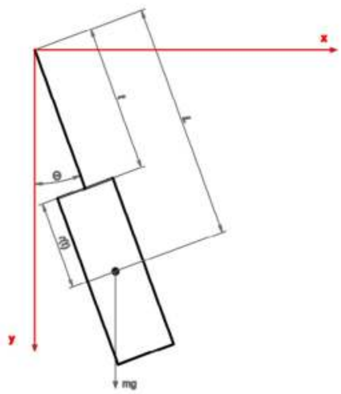

Fig. 4. Model parameters.

In Figure 5, the projections of the centre of mass position onto the Cartesian coordinate system are shown along the X-Y axes.

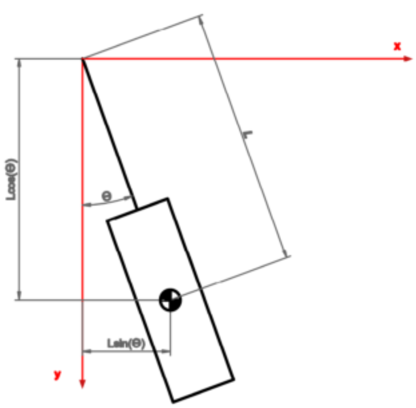

Fig. 5. Position projections with respect to the angle θ.

The rotational kinetic energy obtained is shown in Equation 1.

$$T = \frac{1}{2} I \dot{\theta}^2 \tag{1}$$

Using the theorem of parallel axes with respect to the axis of rotation and considering the radial velocity due to the displacement of the centre of mass, the system's kinetic energy is described by Equation (2).

$$T = \frac{1}{2}\dot{\theta}^2 (I + m(l+r)^2) + \frac{1}{2} m \dot{r}^2 \tag{2}$$

Equation (3) represents the potential energy of the system.

$$U = -mg(l+r)\cos\theta \tag{3}$$

Recalling that the Lagrangian of a mechanical system is defined as the difference between kinetic energy and potential energy, as shown in Equation (4).

$$\mathcal{L} = T - U \tag{4}$$

The Lagrangian of the system is finally described by Equation (5).

$$\mathcal{L} = \frac{1}{2}\dot{\theta}^2 (I + m(l+r)^2) + \frac{1}{2} m \dot{r}^2 + mg(l+r)\cos\theta \tag{5}$$

The Euler-Lagrange equation for conservative systems is shown in Equation (6).

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0 \tag{6}$$

Applying the Euler-Lagrange equation with respect to the parameters θ and $\dot{\theta}$ yields Equation (7).

$$\begin{aligned} &\ddot{\theta}[I + m(l+r)^2] + \dot{\theta}[\dot{I} + 2m\dot{r}(l+r) + \\ &\dot{m}(l+r)^2] + mg(l+r)\sin\theta = 0 \end{aligned} \tag{7}$$

Similarly, applying the Euler-Lagrange equation with respect to the parameters r and $\dot{r}$ results in Equation (8).

$$m\ddot{r} + \dot{m}\dot{r} - \dot{\theta}^2 m(l+r) - mg\cos\theta = 0 \tag{8}$$

Expressing the equations of the system in matrix form gives Equation (9).

$$\begin{aligned} &\begin{bmatrix} I + m(l+r)^2 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} + \\ &\begin{bmatrix} \dot{I} + m\dot{r}(l+r) + \dot{m}(l+r)^2 & m\dot{\theta}(l+r) \\ -m\dot{\theta}(l+r) & \dot{m} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} + \\ &\begin{bmatrix} mg\sin\theta(l+r) \\ -mg\cos\theta \end{bmatrix} = 0 \end{aligned} \tag{9}$$

Using Equation (10),

$$\left[\dot{D} - 2C\right] = -\left[\dot{D} - 2C\right]^T \tag{10}$$

results in Equation (11).

$$\left[\dot{D} - 2C\right] = \begin{bmatrix} -\dot{I} - \dot{m}(l+r)^2 & -2m\dot{\theta}(l+r) \\ 2m\dot{\theta}(l+r) & -\dot{m} \end{bmatrix} \tag{11}$$

Finally, it can be observed that the matrix obtained in the last equation does not satisfy the property shown in Equation (10). However, it should be noted that if the model had a constant mass, the following expression would be obtained for Equation (11):

$$\left[\dot{D} - 2C\right] = \begin{bmatrix} 0 & -2m\dot{\theta}(l+r) \\ 2m\dot{\theta}(l+r) & 0 \end{bmatrix} \tag{12}$$

Equation (12) is an expression that does satisfy the property of mechanical systems.

https://doi.org/10.58571/CNCA.AMCA.2024.027

A further discussion on the non-compliance of the mechanical modelling property is given in the conclusions. Now, representing the model in state-space form, the following state variables are assigned:

$$x_1 = \theta \tag{13}$$

$$x_2 = \dot{\theta} \tag{14}$$

$$x_3 = r \tag{15}$$

$$x_4 = \dot{r} \tag{16}$$

From Equations (9), (13), (14), (15), and (16), the following functions are obtained:

$$\dot{x}_1 = x_2 \tag{17}$$

$$\begin{aligned}\dot{x}_2 = &\left[\frac{-\dot{I} - \dot{m}(l + x_3)^2 - m x_4 (l + x_3)}{I + m(l + x_3)^2}\right] x_2 \\ &- \frac{m x_2 x_4 (l + x_3)}{I + m(l + x_3)^2} - \frac{m g \sin x_1 (l + x_3)}{I + m(l + x_3)^2}\end{aligned} \tag{18}$$

$$\dot{x}_3 = x_4 \tag{19}$$

$$\dot{x}_4 = x_2^2 (l + x_3) - \frac{\dot{m} x_4}{m} + g \cos x_1 \tag{20}$$

After obtaining Equations (17), (18), (19), and (20), we proceed to calculate the equilibrium points of the system

$$x_{1_\delta} = 0 \tag{21}$$

$$x_{2_\delta} = 0 \tag{22}$$

$$x_{3_\delta} = \bar{x}_3 \tag{23}$$

$$x_{4_\delta} = 0 \tag{24}$$

Linearising around the equilibrium points described in Equations (21), (22), (23), and (24) yields the matrix which describes the autonomous system, as shown in Equation (25).

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{-mg(l + x_3)}{I + m(l + x_3)^2} & \dfrac{-\dot{I} - \dot{m}(l + x_3)^2}{I + m(l + x_3)^2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\dfrac{\dot{m}}{m} \end{bmatrix} \tag{25}$$

Finally, after performing the Taylor Series linearisation, the following state-space model is obtained in Equation (26).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{-mg(l + x_3)}{I + m(l + x_3)^2} & \dfrac{-\dot{I} - \dot{m}(l + x_3)^2}{I + m(l + x_3)^2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\dfrac{\dot{m}}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mu \tag{26}$$

And the output of the system is defined through the Equation (27).

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{27}$$

4. CONCLUSIONS

In this work, the dynamic modelling of a variable mass pendulum was addressed, finding unusual issues in constant mass systems. A verification of the model through the skew-symmetric property of the inertia and Coriolis matrices is performed.

It was observed that the final matrix obtained did not satisfy the property described in Equation (11). This discrepancy may be attributed to the non-constant nature of the mass and moment of inertia. If these quantities were indeed constant, the terms in the first row - first column and second row - second column of the matrix would be null, thereby fulfilling the aforementioned property. The dynamic behaviour of the system, particularly with respect to the fluid motion and its impact on the overall system dynamics, warrants further investigation. This could involve the development of more sophisticated fluid dynamics models and their integration into the existing framework.

Some potential areas for improvement include deriving the mathematical expression governing the fluid variation within the cylinder, thus enabling the incorporation of time-dependent parameters such as mass and moment of inertia, among others. Furthermore, additional phenomena could be taken into account, such as centrifugal acceleration and its influence on the expulsion of mass from the cylinder.

Future work in this area should focus on implementing control strategies for the system, leveraging advanced techniques specifically designed for highly nonlinear systems. This may include adaptive control methods, robust control algorithms, or model predictive control approaches that can accommodate the system's time-varying parameters and complex dynamics.

Additionally, experimental validation of the theoretical model would be beneficial to assess its accuracy and identify any discrepancies between predicted and observed behaviour. This empirical data could then be used to refine and improve the model, potentially leading to more accurate simulations and more effective control strategies.

REFERENCES

Awrejcewicz, J., Kudra, G., and Wasilewski, G. (2007). Experimental and numerical investigation of chaotic regions in the triple physical pendulum. *Nonlinear*

Dynamics, 50, 755–766. doi:10.1007/S11071-007-9235-0.

Bedini, S. (1992). Historic scientific relic hidden in the palazzo vecchio. *Interdisciplinary Science Reviews*, 17, 382–384. doi:10.1179/ISR.1992.17.4.382.

Dandare, K. (2018). A study of conceptions of preservice physics teachers in relation to the simple pendulum. *Physics Education*, 53. doi:10.1088/1361-6552/aac92f.

Diniz, E.M. (2023). On the foucault pendulum. *Revista Brasileira de Ensino de Física*. doi:10.1590/1806-9126-RBEF-2022-0325.

Domadzra, Y., Bhandari, M., and Hasan, M. (2024). Seismic response of base-isolated buildings: exploring isolator properties. *Asian Journal of Civil Engineering*, 25(5), 4197–4209. doi:10.1007/s42107-024-01041-9. Cited by: 0.

Eke, F. and Mao, T. (2002). On the dynamics of variable mass systems. *International Journal of Mechanical Engineering Education*, 30, 123 – 137. doi: 10.7227/IJMEE.30.2.4.

Eke, F.O. (1998). Dynamics of variable mass systems. NASA Contractor Report NASA/CR-208246, University of California, Davis, Davis, CA. Final Technical Report, NASA Research Grant No. NAG-2-4003.

Erlichson, H. (1999). Galileo's pendulum. *The Physics Teacher*, 37, 478–479. doi:10.1119/1.880380.

Femandes, H., Dias, F., Henriques, R., Obaldia, E., Rossa, P., and Santos, M.B. (2022). The hidden physics behind a pendulum adding up physics studies on a pendulum remote experiment. *2022 8th International Engineering, Sciences and Technology Conference (IESTEC)*, 299–302. doi:10.1109/IESTEC54539.2022.00052.

Fernández Guasti, M. (2007). Péndulo de longitud variable: Experimentos. *Revista Mexicana de Fisica E*, 53(1), 120 – 126. Cited by: 0.

Fréchet, J. and Rivera, L. (2012). Horizontal pendulum development and the legacy of ernst von rebeur-paschwitz. *Journal of Seismology*, 16, 315–343. doi: 10.1007/s10950-011-9272-5.

Gui-cheng, S. (2010). Essential analysis on variable mass. *Journal of Shanxi Normal University*.

Kossovsky, A.E. (2020). Galileo's discoveries about pendulum motion. 117–125. doi:10.1007/978-3-030-51744-1_25.

Matthews, M.R. (2000). Teaching and learning about time and pendulum motion. 275–292. doi:10.1007/978-94-011-3994-6_11.

Nanjangud, A. and Eke, F.O. (2018). Angular momentum of free variable mass systems is partially conserved. *Aerospace Science and Technology*, 79, 1–4. doi: https://doi.org/10.1016/j.ast.2018.03.003.

Shinbrot, T., Grebogi, C., Wisdom, J., and Yorke, J. (1992). Chaos in a double pendulum. *American Journal of Physics*, 60, 491–499. doi:10.1119/1.16860.

Szuminski, W. and Maciejewski, A.J. (2024). Dynamics and non-integrability of the double spring pendulum. *Journal of Sound and Vibration*, 589. doi: 10.1016/j.jsv.2024.118550. Cited by: 0.

Teerikorpi, P., Valtonen, M., Lehto, K., Lehto, H., Byrd, G., and Chernin, A. (2009). *Galileo Galilei and His Successors*, 67–78. Springer New York, New York, NY. doi:10.1007/978-0-387-09534-9_7. URL `https://doi.org/10.1007/978-0-387-09534-9_7`.

Xie, Z., Wang, Y., Luo, X., Arpenti, P., Ruggiero, F., and Siciliano, B. (2024). Three-dimensional variable center of mass height biped walking using a new model and nonlinear model predictive control. *Mechanism and Machine Theory*, 197. doi: 10.1016/j.mechmachtheory.2024.105651. Cited by: 0.

Yu, Y., Ma, J., Shi, X., Wu, J., Cai, S., Li, Z., Wang, W., Wei, H., and Wei, R. (2024). Study on the variable length simple pendulum oscillation based on the relative mode transfer method. *PLoS ONE*, 19(4). doi:10.1371/journal.pone.0299399. Cited by: 1; All Open Access, Gold Open Access.