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Outer Synchronization of Complex Dynamical Networks with Heterogeneous Coupling Topologies ⋆

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Abstract: This paper addresses the outer synchronization problem of complex dynamical networks with outer heterogeneous topologies. Bidirectional coupling topologies are used for network clusters, where each node is a discrete periodic oscillator. For the analysis of the implemented complex dynamical networks ensembles the diffusive coupling strategy is employed in order to achieve sufficient conditions to exhibit the collective behavior of synchronization. The objective of this work is to show a numerical analysis of the possible advantages of using heterogeneous topologies.

Keywords: Outer synchronization, Complex dynamical networks, Outer heterogeneous topologies, Network clusters, Bidirectional coupling.

1. INTRODUCTION

One of the most recognized collective behaviors in nature is synchronization, which is considered an universal behavior because it is reflected in groups of people and animals when they share simultaneous actions of coordination in an equivalent period of time. Synchronization is an emergent phenomenon caused by the interaction of two or more dynamic entities that permeate the natural world, in the case of oscillating unites synchronization behavior can be defined as the adjustment of the time evolution of common rhythms (Echenausía-Monroy et al., 2021). Being synchronized with other individuals refers to the fact that the sets of actions performed by each individual are done at the same time, in the same place, as well as in others, which is observed in a large number of intraspecific groups (Duranton and Gaunet, 2016). Many of these groups are usually represented by sets of fireflies, ants, fish, birds, among others. A common characteristic of these groups is that they exhibit social behaviors, because each individual in the population has the capacity to relate to others of the same type (homogeneous population) or even other groups (heterogeneous population).

The synchronization problem has been widely studied in recent years and has had a high impact on the development of science. Many areas of the scientific field such as mobile robotics (Herrera, 2021), secure communications (Smaoui et al., 2011), chaotic systems (Pecora and Carroll, 1990), digital electronics (Méndez-Ramírez et al., 2024), among others, have addressed the synchronization problem for different research objectives.

There are works where the synchronization problem has been explained in relation to complex dynamical networks, for example, in (Wang and Chen, 2002) the authors investigated the synchronization behavior of networks of continuous time dynamical systems with small-world connections, in (Villalobos-Aranda et al., 2023) the authors address the synchronization problem in networks with outer topology using chaotic nodes with hidden attractors using the diffusive coupling strategy, in (Arellano-Delgado et al., 2021) the authors study the small-world synchronization problem of chaotic networks where the coupling is through the use of an intermediate dynamical system, in (Barahona and Pecora, 2002) the authors quantify the dynamic interactions of the small-world phenomenon considering the generic synchronization of oscillator networks with arbitrary topologies.

Network synchronization has been a very popular topic in the scientific field due to the amount of study and research that has been done due to the number of systems that have been used for analysis purposes. However, the analysis of complex dynamical networks using heterogeneous topologies of inner-outer coupling has not been studied much, and it is the main topic of interest in this work.

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This work is organized as follows: In Sect. 2 we describe some definitions that allow to understand the synchronization of complex dynamical networks, in Sect. 3 we show the mathematical model that is used as a node in the networks, in Sect. 4 we present the synchronization behavior for a complex dynamical network with an inner-outer ring homogeneous coupling topology, in Sect. 5 we present the synchronization behavior for a complex dynamical network with an inner-outer ring heterogeneous coupling topology, finally in Sect. 6 we present the conclusions of this work.

2. PRELIMINARIES OF COMPLEX DYNAMICAL NETWORKS

In this section we mention some important preliminaries that allow to understand complex dynamical networks and how the collective behavior of synchronization is achieved in them.

A complex dynamical network is a structure formed by nodes and edges, where the nodes represent the fundamental units of these structures, which can represent individuals, groups, organizations, even places, and the edges represent the relationships between the nodes, such as friendship, economic deals, internet connections, protein interactions, among others. These structures are based on graphs and allow to carry out the analysis of a large number of research works in areas such as: Economics, Biology, Information Sciences, among others (Burguillo, 2018).

In this work we consider a set of M networks, each one of them composed with N nodes, where the inner coupling (i.e., the coupling between the nodes) is represented by a matrix $\mathbf{A_{inner}} \in \mathbb{R}^{(M\times N)\times(M\times N)}$ and the outer coupling (i.e., the coupling between the networks) is represented by a matrix $\mathbf{A_{outer}} \in \mathbb{R}^{(M\times N)\times(M\times N)}$, where a network of $M \times N$ nodes arises.

A complex dynamical network can be described as follows:

$$\boldsymbol{x}_i(k+1) = \boldsymbol{f}(\boldsymbol{x}_i(k)) + \boldsymbol{u}_i, \tag{1}$$

with $i = 1, 2, ..., M \times N$, where $\boldsymbol{f}$ is a function that represents the temporal dynamics of an isolated node, $\boldsymbol{x}_i(k) = (x_{i1}, x_{i2}, ..., x_{in})^T \in \mathbb{R}^{nxn}$ is the state vector of the node i, and $\boldsymbol{u}_i$ is the vector of input signals, which can be represented as follows:

$$\boldsymbol{u}_i = c(\boldsymbol{A} \otimes \boldsymbol{\Gamma})\boldsymbol{x}_i, \tag{2}$$

with $i = 1, 2, ..., M \times N$, where c is the coupling strength for each node, $\boldsymbol{u}_i = (u_{i1}, u_{i2}, ..., u_{in})^T \in \mathbb{R}^{nxn}$ is the vector of input signals for node i, $\mathbf{\Gamma} \in \mathbb{R}^{nxn}$ is a link matrix with values of 1 and 0 that allows the selection of node state variables for coupling the networks, $\otimes$ represents the Kronecker product, and $\mathbf{A} \in \mathbb{R}^{(M\times N)\times(M\times N)}$ is the total coupling matrix of the complex dynamical network, which represents the coupling between different inner topologies and the organization in a specific outer coupling topology.

For a set of complex dynamical networks with a homogeneous coupling topology the total coupling matrix $\mathbf{A}$ can be obtained with the following mathematical scheme:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A}_{inner} + \boldsymbol{A}_{outer} \\ \boldsymbol{A}_{inner} &= c_1(\boldsymbol{I} \otimes \boldsymbol{A}_i) \\ \boldsymbol{A}_{outer} &= c_2(\boldsymbol{A_o} \otimes \boldsymbol{\Gamma}_o), \end{aligned} \tag{3}$$

where c_1 and c_2 are the inner coupling strength and the outer coupling strength respectively, $\mathbf{I} \in \mathbb{R}^{M\times M}$ is the identity matrix, $\mathbf{A_i} \in \mathbb{R}^{N\times N}$ is the inner coupling matrix of the network nodes, $\mathbf{A_o} \in \mathbb{R}^{M\times M}$ is the outer coupling matrix of the networks, and $\mathbf{\Gamma_o} \in \mathbb{R}^{N\times N}$ is a link matrix with values of 1 and 0 that selects the nodes for coupling the networks.

Some examples of complex dynamical networks with homogeneous topologies and heterogeneous topologies are represented in Fig. 1 and in Fig. 2 respectively.

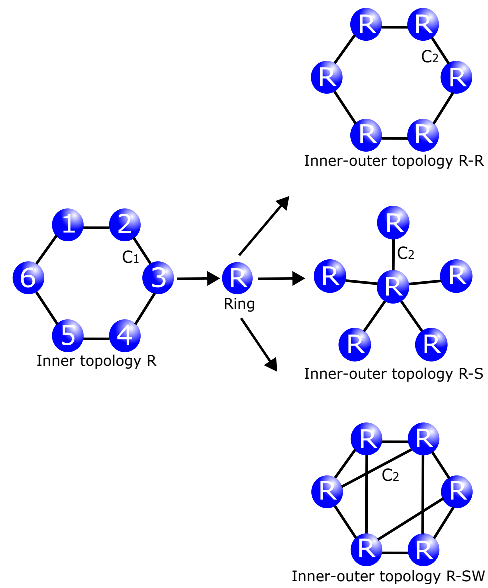

Fig. 1. Example of different inner-outer ring homogeneous coupling topologies for complex dynamical networks.

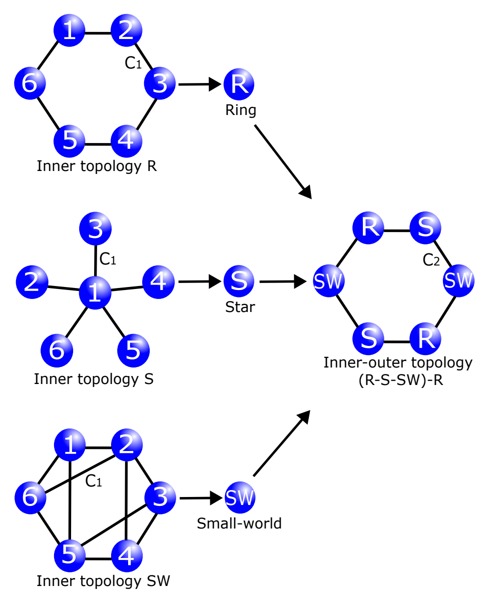

Fig. 2. Example of an inner-outer ring heterogeneous coupling topology for complex dynamical networks.

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3. DYNAMICAL MATHEMATICAL MODEL

In this section we present the discrete mathematical model that will be used as a node in the complex dynamical networks for the purpose of constructing different coupled networks. The mathematical model is described as follows (Arellano-Delgado et al., 2015):

$$\boldsymbol{x}_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} = \begin{bmatrix} \omega_{i1} \\ x_{i2} \end{bmatrix} = \begin{bmatrix} 2\pi f_i \\ sin(\omega_{i1} t(k)) \end{bmatrix}, \tag{4}$$

where i is a simple node, f is the frequency in Hertz of the oscillator, $\omega(k)$ is the angular frequency in radians per second, and $t(k)$ is the discrete time vector in which the oscillator is in dynamic state. The time vector has the following arrangement:

$$t(k+1) = t(k) + \frac{\pi}{1000}. \tag{5}$$

An importan thing to keep in mind is that we do not consider cases where the nodes feedback on themselves or cases where two or more edges are connected to the same pair of nodes.

A simple discrete periodic oscillator is represented in Fig. 3, where (a) is the sinusoidal dynamic behavior and (b) is the dynamic behavior of its angular frequency. The complex dynamical networks we will use are composed by this kind of discrete periodic oscillator as a node.

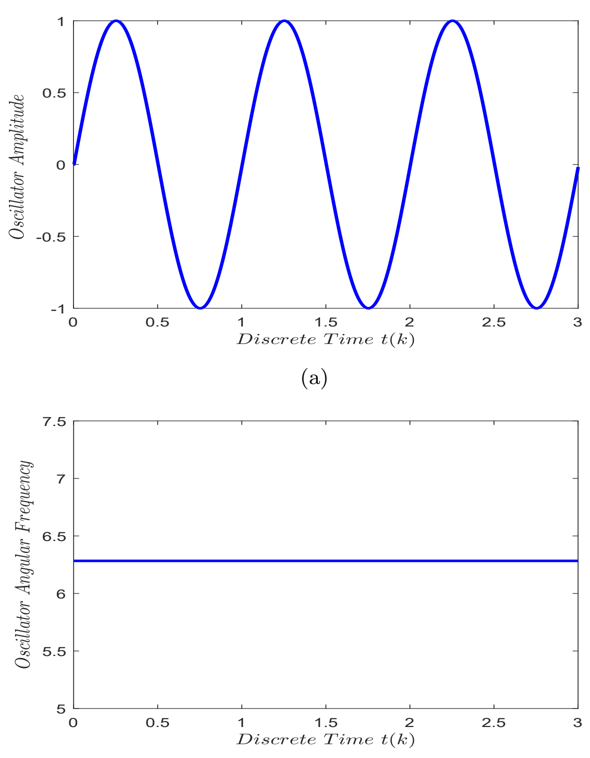

Fig. 3. Behavior of a simple oscillator: (a) temporal sinusoidal dynamics, (b) angular frequency dynamics.

4. HOMOGENEOUS COMPLEX DYNAMICAL NETWORKS

In this section we describe the mathematical analysis to obtain the synchronization of a set of dynamic networks organized in a specific topology, for this particular case we use the ring topology.

We first choose a set of M networks, each one of them with a number of N nodes (i.e., each complex network has the same number of nodes), in this work we select equivalently the sets of networks with the sets of nodes (i.e., $M = N$) where $M = N = 6$ for analysis purposes. Initially we tested the inner-outer ring homogeneous coupling topology shown as in Fig. 4.

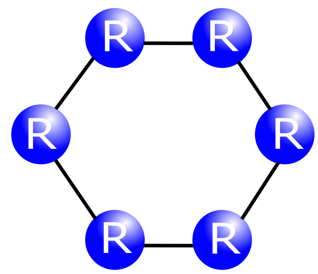

Fig. 4. Inner-outer ring homogeneous coupling topology for a complex network.

The coupling matrix for a ring topology with a number of $N = 6$ nodes is defined as follows:

$$\boldsymbol{A_{Ring}} = \boldsymbol{A_i} = \boldsymbol{A_o} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}, \tag{6}$$

while the identity matrix is defined as follows:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

It is important to mention that the identity matrix must be a symmetric matrix of the same dimensions as the number of complex networks used.

Now, it is important to know how to couple the complex dynamical networks, in this work we use the last node (node 6) of each network to couple the sets of them, for this we use the link matrix $\boldsymbol{\Gamma_o}$, this matrix is defined as follows:

$$\boldsymbol{\Gamma_o} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

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We use $c_1 = 0.1$ and $c_2 = 0.1$ for the inner and outer coupling strengths.

In Fig. 5 we show the synchronization behavior for a complex dynamical network with inner-outer ring homogeneous coupling topology.

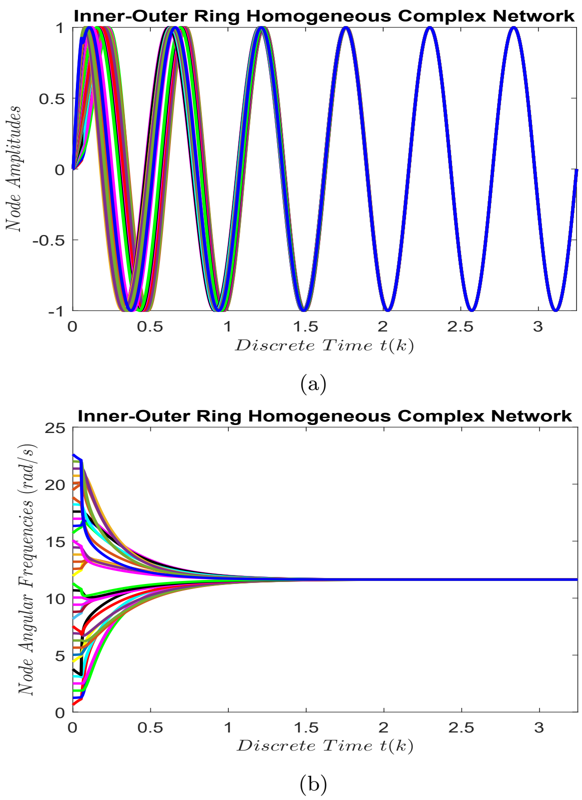

Fig. 5. Synchronization for a complex dynamical network with inner-outer ring homogeneous coupling topology: (a) temporal sinusoidal dynamics, (b) angular frequency dynamics.

The synchronization collective behavior is obtained for each of the state variables of the complex dynamical network, this behavior is thanks to the coupling matrix $\mathbf{A}$ where each row is an input signal that allows the coupling with the nodes that maintain an interaction relation in the complex network.

An interesting thing to note is that only one inner coupling matrix $\mathbf{A_i}$ was used because each node took the role of a ring coupling topology.

In the following section we will show how to obtain the coupling matrix $\mathbf{A}$ for a complex dynamical network with inner-outer ring heterogeneous coupling topology.

5. HETEROGENEOUS COMPLEX DYNAMICAL NETWORKS

To obtain the synchronization behavior in a complex dynamical network with a heterogeneous coupling topology it is necessary first to obtain the inner and outer coupling matrices, i.e., $\mathbf{A_{inner}} \in \mathbb{R}^{(M\times N)\times(M\times N)}$ and $\mathbf{A_{outer}} \in \mathbb{R}^{(M\times N)\times(M\times N)}$ respectively, which will form the total coupling matrix $\mathbf{A} \in \mathbb{R}^{(M\times N)\times(M\times N)}$ of the network. These three matrices are obtained with the following mathematical scheme:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A_{inner}} + \boldsymbol{A_{outer}} \\ \boldsymbol{A_{inner}} &= c_1(\boldsymbol{\Gamma_{ij}} \otimes \boldsymbol{A_{ij}}) \\ \boldsymbol{A_{outer}} &= c_2(\boldsymbol{A_o} \otimes \boldsymbol{\Gamma_o}), \end{aligned} \tag{9}$$

with $j = 1, 2, ..., M$ where M is the number of complex dynamical networks used, for this particular case $M = 6$, each network will be composed of a number of $N = 6$ nodes, c_1 and c_2 are the inner and outer coupling strengths respectively, $\mathbf{\Gamma_i} \in \mathbb{R}^{N\times N}$ is a link matrix with values of 1 and 0 that allows coupling the internal networks used, $\mathbf{A_i} \in \mathbb{R}^{N\times N}$ is the inner coupling matrix of nodes used in a network, $\mathbf{A_o} \in \mathbb{R}^{M\times M}$ is the outer coupling matrix of the networks, and $\mathbf{\Gamma_o} \in \mathbb{R}^{N\times N}$ is a link matrix with values of 1 and 0 that allows the selection of the nodes for coupling the networks used.

Now we show the configuration for a complex dynamical network with inner-outer ring heterogeneous coupling topology, which we use to achieve the collective behavior of synchronization. The ring topology that we tested is shown in Fig. 6.

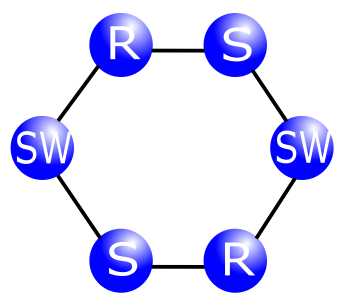

Fig. 6. Inner-outer ring heterogeneous coupling topology for a complex network.

In Fig. 6 $\mathbf{R}$ is a complex dynamical network with ring topology (network 1 and network 4), $\mathbf{S}$ is a complex dynamical network with star topology (network 2 and network 5), and $\mathbf{SW}$ is a complex dynamical network with small-world topology (network 3 and network 6), while the outer topology of the networks is a ring topology.

To obtain the total coupling matrix $\mathbf{A}$ it is important to define firstly the inner coupling matrices of the networks (i.e., the coupling matrices between the nodes), which will be three, one for the ring topology $\mathbf{A_R} \in \mathbb{R}^{N\times N}$, one for the star topology $\mathbf{A_S} \in \mathbb{R}^{N\times N}$, and one for the small-world topology $\mathbf{A_{SW}} \in \mathbb{R}^{N\times N}$.

The coupling matrix for the ring topology is defined as follows:

$$\boldsymbol{A_R} = \boldsymbol{A_{i1}} = \boldsymbol{A_{i4}} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{10}$$

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The coupling matrix for the star topology is the following:

$$\boldsymbol{A_S} = \boldsymbol{A_{i2}} = \boldsymbol{A_{i5}} = \begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \quad (11)$$

The coupling matrix for the small-world topology is the following:

$$\boldsymbol{A_{SW}} = \boldsymbol{A_{i3}} = \boldsymbol{A_{i6}} = \begin{bmatrix} -3 & 1 & 0 & 0 & 1 & 1 \\ 1 & -4 & 1 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 1 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 0 & 1 & -3 \end{bmatrix}. \quad (12)$$

Now we define the link matrices $\mathbf{\Gamma_i}$, the matrix $\mathbf{\Gamma_{i1}}$ is:

$$\boldsymbol{\Gamma}_{i1} = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}. \quad (13)$$

The matrix $\mathbf{\Gamma_{i2}}$ is defined as follows:

$$\boldsymbol{\Gamma}_{i2} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}. \quad (14)$$

The matrix $\mathbf{\Gamma_{i3}}$ is defined as follows:

$$\boldsymbol{\Gamma}_{i3} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}. \quad (15)$$

The matrix $\mathbf{\Gamma_{i4}}$ is defined as follows:

$$\boldsymbol{\Gamma}_{i4} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}. \quad (16)$$

The matrix $\mathbf{\Gamma_{i5}}$ is defined as follows:

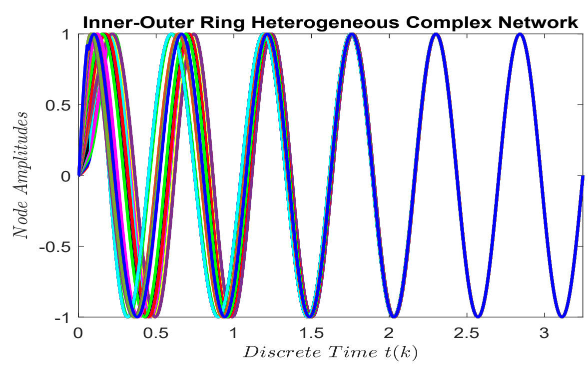
$$\boldsymbol{\Gamma}_{i5} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \end{bmatrix}. \quad (17)$$

The matrix $\mathbf{\Gamma_{i6}}$ is defined as follows:

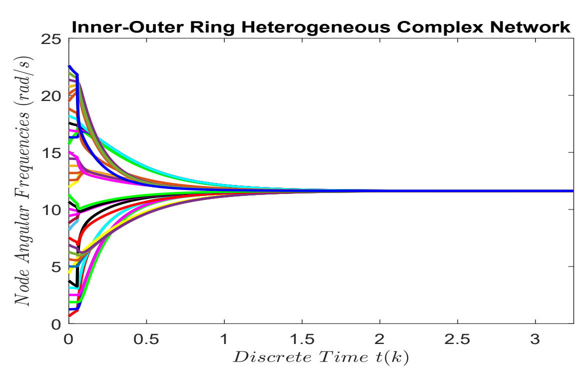
$$\boldsymbol{\Gamma}_{i6} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{bmatrix}. \quad (18)$$

On the other hand, we use $c_1 = 0.1$ and $c_2 = 0.1$ for the inner and outer coupling strengths respectively, the matrix $\mathbf{A_o}$ is the same as that represented by the matrix (10), while the matrix $\mathbf{\Gamma_o}$ is the same as that represented by the matrix (18).

In Fig. 7 we show the synchronization behavior for a complex dynamical network with inner-outer ring heterogeneous coupling topology. We show the synchronization behavior for each state variable of every node used in the network.



(a)



(b)

Fig. 7. Synchronization for a complex dynamical network with inner-outer ring heterogeneous coupling topology: (a) temporal sinusoidal dynamics, (b) angular frequency dynamics.

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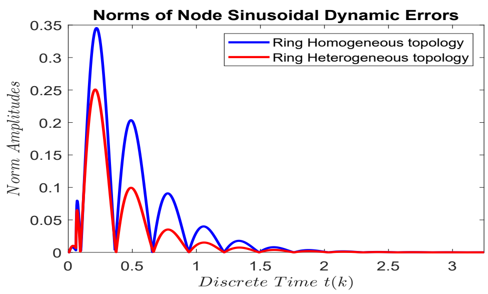
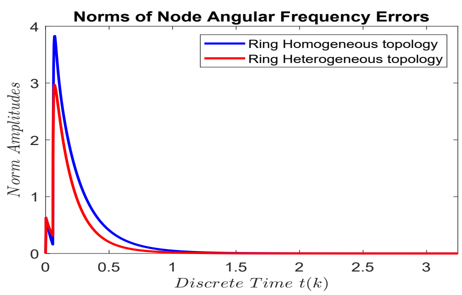
Now we show the error norms between the complex dynamical network with inner-outer ring homogeneous coupling topology and the complex dynamical network with inner-outer ring heterogeneous coupling topology.



(a)



(b)

Fig. 8. Error norms for complex dynamical networks with inner-outer ring homogeneous coupling topology and inner-outer ring heterogeneous coupling topology: (a) temporal sinusoidal dynamics, (b) angular frequency dynamics.

The error norm for the temporal sinusoidal dynamics is calculated with the following expression:

$$||\boldsymbol{e}_x|| = \sqrt{\sum_{i=1}^{M\times N} \sum_{j=1,j\neq i}^{M\times N} (x_i - x_j)^2}. \tag{19}$$

The error norm for the angular frequency dynamics is calculated with the following expression:

$$||\boldsymbol{e}_\omega|| = \sqrt{\sum_{i=1}^{M\times N} \sum_{j=1,j\neq i}^{M\times N} (\omega_i - \omega_j)^2}. \tag{20}$$

In Fig. 8 shows that the effect of using a ring heterogeneous coupling topology synchronizes more efficiently than a ring homogeneous coupling topology, this is due to the multiple interactions generated by the different inner topologies in the composition of an outer heterogeneous coupling topology.

6. CONCLUSION

In this work we addressed the problem of outer synchronization in complex dynamical networks with heterogeneous coupling topologies, specifically for a ring topology, for a mathematical model that represents a discrete time system, we provide a new methodology to explore the problem of outer synchronization in relation to the total coupling matrix for an outer heterogeneous coupling topology. Our results offer an improvement in the reduction of the transient time for a network with ring heterogeneous coupling topology and a comparison of the error norms of each state variable with respect to a network with a ring homogeneous coupling topology.

For future work it would be interesting to explore the problem of outer synchronization of networks using another type of outer coupling topology, such as the star or small-world topologies, as well as using a different coupling scheme, and also to perform an experimental implementation to check if the theories used in this work are fulfilled.

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