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Safety Control of Linear Systems with Input Delay via Observer Predictor

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Abstract: The problem of safety control for linear systems with delay in the input is addressed via an observer predictor. Under this approach, no implementation of any integral is required, avoiding potential instability phenomena that might induce violation of the imposed constraints in the state space. We provide some insights into how to select the parameters of the observer predictor scheme to guarantee that the system trajectories remain within a given set of the state space.

Keywords: Time-delay systems; safety control; observer-predictor

1. PROBLEM STATEMENT AND PRELIMINARIES

We consider systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t-\tau),\ t \geq 0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\tau \geq 0$ is the delay. Throughout the paper, we refer to system (1) with $\tau = 0$ as *delay free system*. We address the following problem:

Problem 1: Given a closed set $\mathcal{C} \subset \mathbb{R}^n$, construct a control algorithm u that ensures that if $x_0 := x(0) \in \mathcal{C}_\lambda \subset \mathcal{C}$ then $x(t, x_0) \in \mathcal{C}$ for all $t \geq 0$.

Problem 1 is commonly known in the literature as safety control problem, and its relevance is evident from a practical point of view, particularly when the state variables must remain within prescribed state-space constraints. In the last decade, the notion of Control Barrier Functions (CBF) has shown to be powerful to address this class of problems. Barrier functions were first introduced for safety verification by Prajna and Jadbabaie (2004) and then applied within the context of control design by Wieland and Allgöwer (2007). The interpretation of the control design task based on CBF as quadratic programs was then introduced by Ames et al. (2016). In this paper, we rely on this notion to approach Problem 1.

To advance the formulation of the problem, we introduce the concept of CBF in a formal framework.

1.1 Basic facts on CBF

Let us characterize the set $\mathcal{C}$ by a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned}\mathcal{C} =& \{x \in \mathbb{R}^n : h(x) \geq 0\}\\ \partial\mathcal{C} =& \{x \in \mathbb{R}^n : h(x) = 0\}\\ \mathrm{Int}(\mathcal{C}) =& \{x \in \mathbb{R}^n : h(x) > 0\}.\end{aligned}$$

Definition 1. (Ames et al., 2016) A continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a Control Barrier Function (CBF) defined on a set $\mathcal{D}$, $\mathcal{C} \subseteq \mathcal{D} \subset \mathbb{R}^n$, for delay free system (1), if there exists an extended class $\mathcal{K}$ function α [1] such that for all $x \in \mathcal{C}$ there exists u satisfying

$$\dot{h}(x) = \nabla h(x)\,(Ax + Bu(x)) \geq -\alpha(h(x)). \tag{2}$$

Hereafter, for the sake of simplicity, we consider $\alpha(h) = \alpha h$, with $\alpha > 0$.

In Ames et al. (2016), it is proved that, if h is a CBF on $\mathcal{D}$ for delay free system (1), any Lipschitz continuous function $u : \mathcal{D} \to \mathbb{R}^m$ guarantees that if $x_0 \in \mathcal{C}$ then $x(t, x_0) \in \mathcal{C}$ for all $t \geq 0$.

Thus, u such that solves Problem 1, with $\tau = 0$, can be constructed from the solution to the quadratic program (QP) (Ames et al., 2014)

$$\begin{aligned}u^\star(x) = &\underset{u\in\mathbb{R}^m}{\mathrm{argmin}}\|u - \mathrm{u_{des}}(x)\|^2\\ &\text{s.t. } \nabla h(x)\,(Ax + Bu) \geq -\alpha(h(x)),\end{aligned}$$

where

$$\mathrm{u_{des}}(x) = Kx,$$

with $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is stable. Moreover, the QP has the explicit solution (Xu et al., 2015)

$$u^\star(x) = \mathrm{u_{des}}(x) + \mathrm{u_{safe}}(x), \tag{3}$$

where

$$\mathrm{u_{safe}}(x) = \begin{cases} 0 & \text{if } \psi(x) \geq 0 \\ -\dfrac{(\nabla h(x)B)^T}{\nabla h(x)BB^T\nabla h(x)^T}\psi(x) & \text{otherwise}\end{cases} \tag{4}$$

with $\psi(x) = \nabla h(x)(Ax + B\,\mathrm{u_{des}}(x)) + \alpha h(x)$, which is locally Lipschitz for $x \in \mathcal{D}$ whenever $\nabla h(x) \neq 0$ (for all $x \in \mathcal{D}$) is locally Lipschitz. This means that for any point of $\mathcal{D}$ there is a neighbourhood $\mathcal{D}_0$ such that there exists $\delta_0 > 0$ satisfying

$$\|u^\star(x) - u^\star(y)\| \leq (\|K\| + \delta_0)\,\|x - y\|,\ \ x, y \in \mathcal{D}_0. \tag{5}$$

Control (3) ensures that the time derivative of a given CBF h along the solutions of the delay free system (1) satisfies (2), i.e.

$$\dot{h}(x) \geq -\alpha h(x). \tag{6}$$

[1] A function $\alpha : (-a_1, a_2) \to (-\infty, \infty)$ is said to be an extended class $\mathcal{K}$ function if it is continuous, strictly increasing and $\alpha(0) = 0$.

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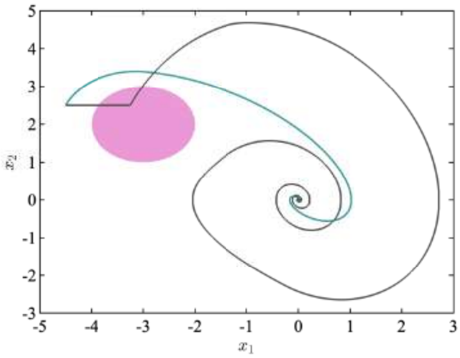

Fig. 1. System (1) with control (3) with $\tau = 0$ and with $\tau = 0.5$.

Indeed, this is precisely the kind of condition that one seeks to guarantee that the system trajectories remain within a given set. The reason is the following: if $x_0 \in \mathcal{C}$, then $h(x_0) \geq 0$, and from the solution of the differential inequality follows that $h(x(t)) \geq 0$, i.e. $x(t, x_0) \in \mathcal{C}$ for all $t \geq 0$.

Example 1. To illustrate the ideas, let us consider a double integrator, i.e. system (1) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \tau = 0.5.$$

Let the function

$$h(x) = (x_1 + 3)^2 + (x_2 - 2)^2 - 1$$

characterize the set $\mathcal{C}$. Figure 1 depicts the solution of the system on the plane (x_1, x_2) in closed-loop with constructed controller (3), $u(t) = u^\star(x(t))$, $\alpha = 0.5$, $K = -(1\ 1)$, considering control with and without input delay. The gray line depicts the first case, and the cyan-green the second. The magenta circle corresponds to the prohibitive set.

1.2 The problem with the input delay

As shown by Example 1, the input delay is detrimental to the invariance of the system trajectory. Relying on predictor theory for systems of the form (1), an immediate solution to Problem 1 is the construction of the controller $u^\star(x)$ via the predicted state $x_p(t) = x(t+\tau)$, i.e $u^\star(x_p)$. In fact, this solution was first proposed by Jankovic (2018). Nevertheless, the solution within this framework is simple only on the surface for two reasons:

(1) The implementation of the integral term in the predictor might lead to unstable response of the closed-loop system and in turn to the violation of the safety contraints; see Mondié and Michiels (2003) and the references therein.
(2) While $t < \tau$, the system trajectories evolve in open loop and must be assumed that before τ the system satisfies the state space constraints. The assumption may fail if the delay is too large or the open-loop dynamics are too fast.

Another approach that has proven effective in constructing stabilizing control for (1), yet has not been applied to Problem 1, is based on the implementation of observers (Najafi et al., 2013). This paper is devoted to exploring that direction.

The rest of the note is organized as follows. In the next section, we introduce the observer predictor scheme, and provide some guide to tune the corresponding parameters. More precisely, we show that the observation error should converge fast enough, while the gain matrix K should be selected with care and not arbitrarily. In Section 3, we revisit Example 1 and illustrate the main points discussed within Section 2. We closed the paper with some concluding remarks in Section 4.

Notation: The gradient of a function $f(x)$ is denoted by $\nabla f(x)$, while $\|\cdot\|$ stands for the Euclidian norm. The rest of notation is standard, and when necessary, it is specified throughout the paper.

2. SAFE OBSERVER PREDICTOR

The basic idea of the observer-predictor is the construction of an observer of the form

$$\dot{z}(t) = Az(t) + Bu(t) + Le(t),$$

where

$$e(t) = z(t - \tau) - x(t),$$

and $L \in \mathbb{R}^{n\times n}$ is such that the solutions of

$$\dot{e}(t) = Ae(t) + Le(t - \tau) \tag{7}$$

exponentially converge to the origin. Then, the system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau) \\ \dot{z}(t) &= Az(t) + Bu(t) + Le(t) \\ u(t) &= \nu(z(t)). \end{aligned} \tag{8}$$

If the problem consists in the stabilization of (1), then $\nu(z(t)) = \mathrm{u_{des}}(z(t))$ and it is clear that when $e(t) \equiv 0$, $x(t) = z(t - \tau)$ and $u(t - \tau) = \nu(z(t - \tau)) = Kx(t)$.

We consider the control in (3):

$$u(t) = \nu(z(t)) = \mathrm{u_{des}}(z(t)) + \mathrm{u_{safe}}(z(t)). \tag{9}$$

Once again, if $e(t) \equiv 0$, then the closed loop system is

$$\dot{x}(t) = Ax(t) + B\nu(x(t)),$$

which by construction of ν satisfies (2). This is not the case while $e(t) \neq 0$. The main results are presented next.

Let

$$\mathcal{C}_\lambda := \{(\varphi, x) \in PC_\tau \times \mathcal{D} : \eta(\varphi, x) \geq 0\},$$

where φ is the initial condition of system (7), PC_τ denotes the space of $\mathbb{R}^n$-valued piecewise continuous functions defined on $[-\tau, 0]$, and

$$\eta(\varphi, x) := -v^{\frac{1}{2}}(\varphi) + \lambda h(x),$$

with λ some positive number and v the complete type functional (A.1) of system (7).

Theorem 1. Assume that there exists γ_L such that

$$\gamma_L = \max_{z\in\mathcal{Z}} \|\nabla h(z)L\|,$$

where $\mathcal{Z} \subset \mathcal{D}$ and let

$$\lambda_z = \frac{(\sigma - \alpha)\beta}{\gamma_L},$$

where $\alpha > 0$ is given, and σ and $\beta = \sqrt{\beta_1}$ are computed from (A.4) and (A.2), respectively. If $\sigma > \alpha$ and $(\varphi, z_0) \in \mathcal{C}_{\lambda_z}$, then $z(t, z_0)$ satisfying (8) remains within $\mathcal{C}$ for all $t \geq 0$.

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Proof. It is enough to prove that the time derivative of η along the solutions of (8) satisfies

$$\dot{\eta}(e_t, z) \geq -\alpha\eta(e_t, z)$$

for some $\alpha > 0$. Indeed, this implies that if $(z_0, \varphi) \in \mathcal{C}_{\lambda_z}$, then $\eta(e_t, z(t)) \geq 0$, and in turn that, $h(z(t)) \geq \frac{1}{\lambda} v^{\frac{1}{2}}(e_t) \geq 0$ for all $t \geq 0$.

By construction of u,

$$\dot{h}(z) = \nabla h(z)\,(Az + Bu + Le) \geq -\alpha h(z) + \nabla h(z) Le.$$

Then, from (A.3) (see the Appendix),

$$\begin{aligned}
&\dot{\eta}(e_t, z) + \alpha\eta(e_t, z)\\
&\quad = -\frac{1}{2} v^{-\frac{1}{2}}(e_t)\frac{d}{dt} v(e_t) + \lambda\dot{h}(z) + \alpha\eta(e_t, z)\\
&\quad \geq \sigma v^{\frac{1}{2}}(e_t) - \alpha(\lambda h(z) - \eta(e_t, z)) + \lambda\nabla h(z) Le(t)\\
&\quad = (\sigma - \alpha) v^{\frac{1}{2}}(e_t) + \lambda\nabla h(z) Le(t)\\
&\quad \geq ((\sigma - \alpha)\beta - \lambda\gamma_L)\,\|e(t)\|,\ t \geq 0,
\end{aligned}$$

where the last inequality follows from the lower bound (A.2). The proof is concluded by considering $\lambda = \lambda_z$. □

Remark 2. The parameter σ corresponds to an estimate of the convergence rate of the observer error.

Since $z(t-\tau) = x(t) + e(t)$, Theorem 1 implies that if $e(t)$ is sufficiently small for $t \geq \tau$ then $x(t) \in \mathcal{C}$ for $t \geq \tau$. This is achieved by a suitable choice of L.

The next result is focused on $x(t)$.

Theorem 3. Let $\mathcal{D}_0$ be a neighborhood of any point from $\mathcal{C} \subset \mathcal{D}$,

$$\gamma = \max_{x \in \mathcal{D}_0} \|\nabla h(x) B\|,$$

and

$$\lambda = \frac{(\sigma - \alpha)\beta}{\gamma(\|K\| + \delta_0)},$$

where $\alpha > 0$ is given, σ and $\beta = \sqrt{\beta_1}$ are computed from (A.4) and (A.2), respectively, and δ_0 is the estimate in (5). If $\sigma > \alpha$ and $(\varphi, x_0) \in \mathcal{C}_\lambda$, then $x(t, x_0)$ satisfying (8) remains within $\mathcal{C}$ locally in time.

Proof. The proof follows the same arguments from those of Theorem 1. Since we rely on (5), the statement holds only locally in time.

Notice first that by construction of (9),

$$\|\nu(x) - \nu(y)\| \leq (\|K\| + \delta_0)\|x - y\|,\ \ x, y \in \mathcal{D}_0$$

and

$$\nabla h(x)\,(Ax + B\nu(x)) \geq -\alpha h(x).$$

Hence,

$$\begin{aligned}
\dot{h}(x) &= \nabla h(x)\dot{x}(t)\\
&= \nabla h(x)\,(Ax(t) + B\nu(z(t-\tau)))\\
&\geq -\nabla h(x) B\,(\nu(x(t)) - \nu(z(t-\tau))) - \alpha h(x(t))\\
&\geq -\gamma\,(\|K\| + \delta_0)\,\|x(t) - z(t-\tau)\| - \alpha h(x(t))\\
&= -\gamma\,(\|K\| + \delta_0)\,\|e(t)\| - \frac{\alpha}{\lambda}(v^{\frac{1}{2}}(e_t) + \eta(e_t, x)),
\end{aligned}$$

and, from (A.3),

$$\begin{aligned}
&\dot{\eta}(e_t, x) + \alpha\eta(e_t, x)\\
&\quad = -\frac{1}{2} v^{-\frac{1}{2}}(e_t)\frac{d}{dt} v(e_t) + \lambda\dot{h}(x) + \alpha\eta(e_t, x)\\
&\quad \geq (\sigma - \alpha) v^{\frac{1}{2}}(e_t) - \lambda\gamma\,(\|K\| + \delta_0)\,\|e(t)\|\\
&\quad \geq ((\sigma - \alpha)\beta - \lambda\gamma\,(\|K\| + \delta_0))\,\|e(t)\|.
\end{aligned} \tag{10}$$

The proof is finished by considering λ as in the statement of the theorem. □

Let us point out some remarks on the above theorem. First, the locality in Theorem 3 can be removed by assuming that the system trajectories evolve on a compact set $\mathcal{D}$ containing $\mathcal{C}$. In this case, the constants γ and δ_0 can be computed over $\mathcal{D}$, albeit the constant λ might be too small, depending on the definition of $\mathcal{D}$.

Second, it offers insight into how to tune the design parameters of (8) so that $x(t, x_0)$ stays within $\mathcal{C}$. Indeed, L should be such that σ is as bigger as possible, while K should not be arbitrarily large. The above ensures a sufficiently large λ to guarantee that the initial condition x_0 satisfies

$$\lambda h(x_0) \geq v^{\frac{1}{2}}(\varphi). \tag{11}$$

Finally, for system (7), the initial condition is defined as

$$\varphi(\theta) = z(\theta - \tau) - x(\theta),\ \ \theta \in [-\tau, 0].$$

If

$$z(t-\tau) = \begin{cases} z_0 & t = 0,\\ 0, & t \in [-\tau, 0) \end{cases}$$

and $x(t) = 0$ for $t \in [-\tau, 0)$, then

$$\varphi(\theta) = \begin{cases} z_0 - x_0, & \theta = 0,\\ 0, & \theta \in [-\tau, 0) \end{cases} \tag{12}$$

Hence, the set $\mathcal{C}_\lambda$ reduces to

$$\mathcal{C}_\lambda = \Big\{(\varphi, x_0) \in PC_\tau \times \mathcal{D}_0 :\\ -\sqrt{\varphi(0)V(0)\varphi(0)} + \lambda h(x_0) \geq 0\Big\},$$

where $V(0)$ is the delay Lyapunov matrix presented in the Appendix. Consistently with Theorem 1, this expression makes evident that for error function sufficiently small at the initial time, even small values of λ might guarantee that (11) holds, and that the trajectories remain within $\mathcal{C}$.

Remark 4. If the estimate σ is not too conservative, Theorems 1 and 3 might be used to obtain a region around any initial condition for which safety of the closed-loop trajectory is guaranteed, or to compute suitable gains L and K.

3. ILLUSTRATION VIA NUMERICAL SIMULATION

We illustrate the highlighted points derived from Theorem 1 and Theorem 3 with Example 1 from the introduction. Here, according to the definition of $\mathcal{C}$ in Section 1.1, we refer to the region of the state space where $h(x) < 0$ as unsafe region, the magenta circle in the figures. We focus on showing the system trajectories on the plane (x_1, x_2) and omit graphs of them w.r.t. time as they are not relevant to the discussion.

Figure 2 displays system trajectories of system (8) with control (9) for different initial conditions, with

$$K = -10\,(1\ 1),\ \text{and}\ L = -0.75 I_{2\times 2}.$$

They clearly avoids the unsafe region in the plane.

However, as discussed in the previous section, the choice of the gains is not trivial. We study three scenarios, each of them aiming at illustrating respectively the discussed points. The illustrated trade-off between gains is provided for illustrative purposes only and should not be interpreted as a basis for general conclusions.

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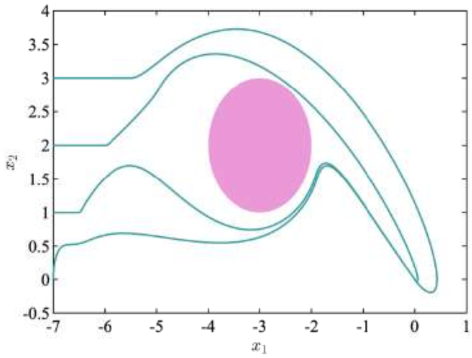

Fig. 2. System trajectories for different initial conditions avoiding an unsafe set. The initial condition of the first state variable is equal for all the cases, $x_{1_0} = -7$.

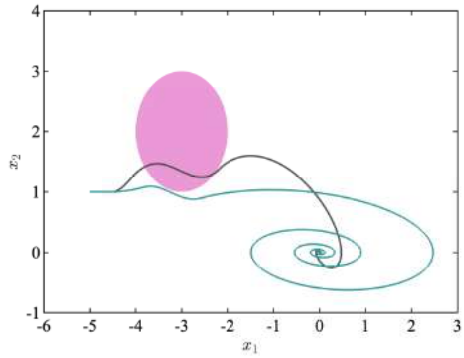

Fig. 3. First case: System trajectories for different values of the gain K.

For the first scenario we set the observer gain $L = -I_{2\times 2}$ and gains K

$$K_1 = -\,(1\ 1) \text{ and } K_2 = -\frac{1}{10}\,(1\ 1)\,.$$

The results are depicted in Figure 3. The system solution with K_1 is depicted by the gray line, while the solution with K_2 by cyan-green. Gain K_2 provides a slower response of the closed-loop system, but keeps the system trajectories within the safe set.

For the second scenario, we set the controller gain $K = K_1$ and test two observer gains. The outcomes are presented in Figure 4, where the gray line depicts the trajectory corresponding to

$$L_1 = -2.9I_{2\times 2},$$

while the cyan-green to

$$L_2 = -I_{2\times 2}.$$

From the estimates (A.4), it is clear that σ with $L = L_2$ is bigger than with $L = L_1$. Though barely noticeable, the simulation indicates constraint violation due to slow observer convergence when $L = L_2$.

Finally, Figure 5 displays the results from the third scenario, where we set $K = K_2$ and $L = L_2$ and consider initial conditions

$$x_0 = [-5\ \ 1.2]^T$$

and

$$x_0 = [-5\ \ 1]^T.$$

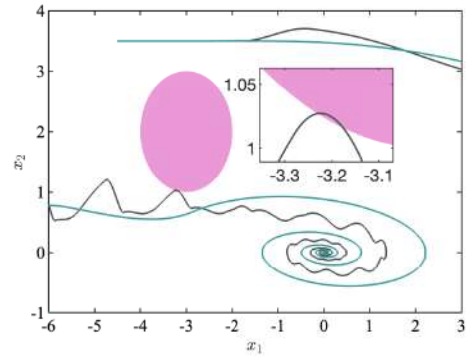

Fig. 4. Second case: System trajectories for different values of the gain L.

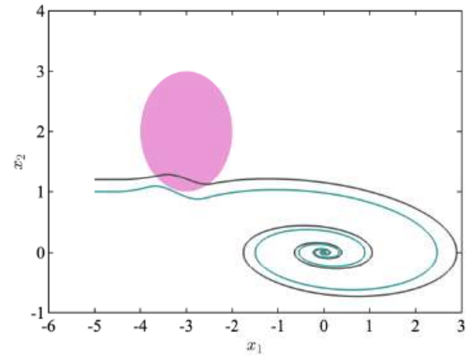

Fig. 5. Third case: System trajectories for different initial conditions.

The gray line corresponds to the first and the cyan-green line to the second. In both cases we are considering φ as in (12) with $z_0 = 0$.

Unfortunately, for this example, using Theorem 1 and 3 to tune gains or compute a safety region in terms of the observer error is impractical, as the estimate σ provided by (A.4) is too small.

4. CONCLUDING REMARKS

A partial solution to Problem 1 is provided by Theorem 1 and Theorem 3. The first one characterizes safety of the system trajectories through the predicted state and the corresponding error, while the second do it locally in time. The results shed light on how to tune the gains of the observer predictor scheme.

In contrast with the reported proposals in the literature, we do not cope with implementation problems of the integral nor the open loop response during the interval $[0, \tau]$, while keeping simplicity of the scheme at the theoretical level. A downside of adopting the presented approach is the complexity in tuning the controller and observer gains, as well as the computation of $\mathcal{C}_\lambda$. The latter principally due to the class of functional used in the analysis.

Ongoing research work is focused in achieving better estimates for $\mathcal{C}_\lambda$ and incorporating sequential subpredictors to deal with larger delays.

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Appendix A. FACTS ON COMPLETE TYPE FUNCTIONAL

The proof of the main result of the paper relies on the complete type functional for system (7). We provide some basic facts on it; see (Kharitonov, 2013, Chapter 2). A complete type functional for system (7) with initial condition $\varphi \in PC_\tau$ is given by

$$v(\varphi) = v_0(\varphi) + \int_{-\tau}^{0} \varphi^T(\theta)\left(W_1 + (\tau+\theta)W_2\right)\varphi(\theta)d\theta, \tag{A.1}$$

where

$$v_0(\varphi) = \varphi^T(0)V(0)\varphi(0) + 2\varphi^T(0)\int_{-\tau}^{0} V(-\tau-\theta)A_1\varphi(\theta)d\theta + \int_{-\tau}^{0}\varphi^T(\theta_1)A_1^T\int_{-\tau}^{0}V(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2 d\theta_1,$$

and V is the delay Lyapunov matrix of system (7) associated with $W = W_0 + W_1 + \tau W_2$, with $W_j > 0$, $j = 0, 1, 2$. The delay Lyapunov matrix is solution of a boundary value problem and can be computed from the following expression:

$$\mathrm{vec}(V(\theta)) = (I\ \ 0)\, e^{L\theta}(M + Ne^{L\tau})^{-1}\begin{pmatrix} 0 \\ -\mathrm{vec}(W)\end{pmatrix},$$

where $\theta \in [0, \tau]$,

$$M = \begin{pmatrix} I & 0 \\ I\otimes A_0^T + A_0^T \otimes I & A_1^T \otimes I\end{pmatrix},$$

$$N = \begin{pmatrix} 0 & -I \\ I\otimes A_1^T & 0\end{pmatrix}$$

and

$$L = \begin{pmatrix} A_0^T \otimes I & A_1^T \otimes I \\ -I\otimes A_0^T & -I\otimes A_1^T\end{pmatrix}.$$

Here, the symbol $\otimes$ denotes the Kronecker product and $\mathrm{vec}(\cdot)$ stands for the vectorization of a matrix.

If system (7) is exponentially stable, the complete type functional satisfies the following:

Property 1:

$$v(\varphi) \geq \beta_1 \|\varphi(0)\|^2 \tag{A.2}$$

where $\beta_1 > 0$ is such that

$$\beta_1 \begin{pmatrix} A_0^T + A_0 & A_1 \\ A_1^T & 0\end{pmatrix} + \begin{pmatrix} W_0 & 0 \\ 0 & W_1\end{pmatrix} > 0.$$

Property 2: The time derivative along the solutions of (7) satisfies

$$\frac{d}{dt}v(e_t) + 2\sigma v(e_t) \leq 0, \tag{A.3}$$

for some $\sigma > 0$, which can be estimated from the inequalities

$$2\sigma\delta_1 \leq \lambda_{\min}(W_0),\ \ 2\sigma\delta_2 \leq \lambda_{\min}(W_2), \tag{A.4}$$

where

$$\delta_1 = \|U(0)\|(1 + \|L\|\tau),$$

and

$$\delta_2 = \|L\|\delta_1 + \left(\|W_1\| + \tau\|W_2\|\right).$$

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