Memorias del XX Congreso Latinoamericano de Control Automático (CLCA 2025)
13-17 de Octubre, 2025. Cancún, Quintana Roo, México

Hidden Extreme Multistability in a Complex Lorenz-type Chaotic System with Stable Equilibria

Lizbeth Vargas-Cabrera, Olga G. Félix-Beltrán, Jesus M. Munoz-Pacheco

Benemérita Universidad Autónoma de Puebla, Puebla, 72570, Mexico, (e-mail: lizbeth.vargasca@alumno.buap.mx)

Abstract: Chaos phenomena have been the subject of study for decades. Even now, it remains a trending topic, as new effects are continually discovered, such as hidden attractors, fractional-order variants, and multistability, to name a few, which continue to open and expand the frontiers for chaos-based applications. Herein, this paper presents a new complex-valued chaotic system with striking characteristics, including hidden attractors that coexist with multistability and extreme multistability. In particular, the proposed system possesses two equilibrium points with positive real parts, indicating that both equilibria are stable. Surprisingly, the proposed systems generate chaos for a determined set of parameters. Analytical formulations are given to demonstrate the conditions for chaos emergence. Additionally, the chaotic behavior is numerically described using bifurcation diagrams and phase portraits.

Keywords: Lorenz system, chaos, hidden attractors, extreme multistability, complex domain.

1. INTRODUCTION

Chaos is a phenomenon that occurs in both living and non-living systems. Its main characteristic is its high sensitivity to initial conditions, which is associated with various applications in science and engineering Strogatz (2024). From a theoretical perspective, chaos has been studied within the framework of nonlinear dynamical systems, typically involving at least three differential equations in the case of continuous-time dynamical systems Devaney (2018). Furthermore, continuous-time chaotic systems, such as the well-known Lorenz, Chen, Lü, etc., are typically described in the real domain since the concept of an equilibrium point is only defined in the vector space of real numbers Yang and Chen (2014).

However, through specific algebraic manipulations, it is possible to transform a chaotic system from $\mathbb{R}^n$ to $\mathbb{C}^n$; that is, both variables and parameters are now represented as complex numbers, *i.e.*, as a two-dimensional vector space over the field of real numbers $\mathbb{R}^n$. See Jin et al. (2022); Gomez-Mont et al. (2012); Moghtadaei and Golpayegani (2012); Munoz-Pacheco et al. (2021); Yuan et al. (2015); Shoreh and Mahmoud (2024); Zhao et al. (2025). This indicates that the system in the complex domain is now expressed as an extended high-dimensional system, allowing its analysis and simulation using traditional chaos theory tools.

On the one hand, it is well known that a chaotic system is characterized in phase space by the presence of a strange attractor, which should possess sensitivity to initial conditions, transitivity, and a dense set of periodic orbits Devaney (2018). Currently, nonlinear dynamical systems can exhibit two types of attractors: self-excited and hidden Pham et al. (2017). In the former, an equilibrium point intersects the basis of attraction and is therefore easily observable. On the other hand, hidden attractors do not always intersect with their basis of attraction Leonov et al. (2011). Furthermore, hidden attractors can also be found in dynamical systems without equilibrium points, with an infinite number of equilibrium points, and even in systems with stable equilibria, which remains an open field of research.

In this framework, it is rare to find chaotic systems in the complex domain with hidden attractors whose equilibrium points are all stable. Even rarer is the emergence of diverse dynamics such as bistability, multistability, and extreme multistability in such systems. Some examples are Jin et al. (2022); Ren et al. (2023) but without stable equilibria. Bistability is understood as the presence of two attractors in phase space, resulting from a change in the initial conditions. Multistability results in different qualitative behaviors in a given nonlinear dynamical system for the same parameter values. Finally, extreme multistability is a kind of special multistability which is related to infinitely many attractors, *i.e.*, new attractors are generated as a function of very small variations in the initial conditions Muñoz-Pacheco (2019). That is, the attractors of classical systems such as the Lorenz, Chua, and Chen systems in their original versions always converge to the same attractor in phase space, even when the initial conditions are changed within the parametric chaotic region. Meanwhile, chaotic systems with hidden attractors and extreme multistability can present chaotic attractors in any area of phase space under minimal variations in their initial conditions. Those key properties, along with an increased number of system variables, can be helpful to improve security in chaos-based

https://doi.org/10.58571/CNCA.AMCA.2025.050

encryption algorithms Ayubi et al. (2023); Zhao et al. (2024)

In this paper, we present a novel Lorenz-type chaotic system in the complex domain that generates hidden extreme multistability with stable equilibria. Analytical and numerical analyses demonstrate the attributes of the new system. Section 2 gives the theoretical formulation for chaos generation in the proposed complex chaotic system. Section 3 presents the numerical results, including bifurcation diagrams and the localization of chaotic attractors. Finally, the conclusion is given in Section 4.

2. DYNAMICS OF THE NEW COMPLEX CHAOTIC SYSTEM WITH HIDDEN ATTRACTORS

The Lorenz system has consistently generated considerable interest and a substantial number of investigations into three-dimensional (3D) autonomous chaotic systems featuring simple nonlinearities and distinct stability conditions. Interestingly, Ref. Yang et al. (2010) proposed an unusual 3D autonomous chaotic system (1) in $\mathbb{R}^3$, which is quite different from the other Lorenz-like systems for having two stable equilibrium points but contains, as special cases, the diffusionless Lorenz system, the Burke–Shaw system under specific transformations.

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= -by - xz, \\ \dot{z} &= xy - c. \end{aligned} \tag{1}$$

Based on the generalized Lorenz system (1) with $(x, y, z) \in \mathbb{R}^n$, a new complex chaotic system with hidden attractors is proposed in this section. Remarkably, this new system presents hidden attractors with stable equilibrium points as a function of the chosen transformation. In particular, the complex, hidden attractor chaotic system with stable equilibrium points also generates extreme multi-stability, a characteristic that is highly unusual in this kind of system. The following subsections describe the basic properties and dynamic behaviors of the new system. It is important to remark that the selection of system (1) relies on the fact that it has only stable equilibrium points. In this manner, we could select any other system with similar properties to transform it to the complex domain without loss of generality.

2.1 Lorenz-type chaotic system with two stable equilibrium points and extreme multistability

In this paper, we propose the complex extension of system (1) as follows:

$$\begin{aligned} \dot{z}_1 &= \frac{a}{2}(z_2 + \bar{z}_2) - az_1, \\ \dot{z}_2 &= -bz_2 - z_1 z_3, \\ \dot{z}_3 &= z_1 z_2 - c, \end{aligned} \tag{2}$$

where $z_1 = u_1$ is the real state variable, $z_2 = u_2 + ju_3$ and $z_3 = u_4 + ju_5$ are complex state variables, a and b are positive constant parameters, and $c = c_r + jc_i$ is a complex constant parameter. The overbar $\bar{z}_2$ denotes the complex conjugate of z_2. System (2) can be recast as a real-variable system $f(u_k)$ of the form:

$$\begin{aligned} \dot{u}_1 &= a(u_2 - u_1), \\ \dot{u}_2 &= -bu_2 - u_1 u_4, \\ \dot{u}_3 &= -bu_3 - u_1 u_5, \\ \dot{u}_4 &= u_1 u_2 - c_r, \\ \dot{u}_5 &= u_1 u_3 - c_i, \end{aligned} \tag{3}$$

where u_k with $k = 1, \ldots, 5$ represents the state-variables of the expanded system obtained from complex system (2). The divergence of vector fields of states of the system (3) can be described as

$$\nabla V = \sum_{k=1}^{5} \frac{\partial \dot{u}_k}{\partial u_k} = -(a + 2b), \tag{4}$$

as long as $a + 2b > 0$, the system is dissipative. Thus, the volume of the system trajectory converges exponentially to the origin with an exponential contraction rate of $-(a + 2b)$. Additionally, system (3) has rotational symmetry with respect to the u_4, and u_5-axes, due to its invariance under the coordinates transform from $(u_1, u_2, u_3, u_4, u_5)$ to $(-u_1, -u_2, -u_3, u_4, u_5)$. Since the rotational symmetry can lead to attractors with symmetric scrolls, it can generate many identical attractors located symmetrically in phase space. In security applications, rotational symmetry can be useful for key space expansion, as it provides an additional control parameter. The corresponding Jacobian matrix of system (3) is expressed by:

$$J = \begin{bmatrix} -a & a & 0 & 0 & 0 \\ -u_4^* & -b & 0 & -u_1^* & 0 \\ -u_5^* & 0 & -b & 0 & -u_1^* \\ u_2^* & u_1^* & 0 & 0 & 0 \\ u_3^* & 0 & u_1^* & 0 & 0 \end{bmatrix}. \tag{5}$$

From $f(u_k) = 0$, the equilibrium points EP are:

$$\begin{aligned} EP_{1,2} &= (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) \\ &= \left(\pm\sqrt{c_r}, \pm\sqrt{c_r}, \mp\left(\frac{c_i}{\sqrt{c_r}}\right), -b, \frac{bc_i}{c_r} \right). \end{aligned} \tag{6}$$

The equilibrium point exists if the real part of the complex parameter c is not zero, i.e, $c_r \neq 0$. To analyze the stability of equilibrium points, we state the following theorem.

Theorem 1. *Suppose the parameters of system (2) are positive, $a, b, c_r > 0$. Then, the stability of its EP_j with $j = 1, 2$ has the following property:*

- *If $b > a$, system (2) is stable with two stable node-foci equilibrium points, and it may produce hidden attractors.*

Proof. *By considering that the characteristic equation of (5) evaluated at EP_j with $j = 1, 2$, is:*

Copyright© AMCA, ISSN: 2594-2492 https://doi.org/10.58571/CNCA.AMCA.2025.050

$$s^5 + (a + 2b)s^4 + (ab + b^2 + 2c_r)s^3 + (3ac_r + 2bc_r)s^2 + (2abc_r + c_r^2)s + 2ac_r^2 = 0. \quad (7)$$

Applying the tabular approach of the Routh-Hurwitz stability criterion, we obtain the elements of the first column. As is well known, the number of sign changes in the first column will be the number of non-negative roots. Therefore, the complex system (2), is stable if and only if satisfies the following conditions.

$$\begin{cases} i. & a + 2b > 0, \\ ii. & \phi_1 + \phi_6 > 0, \\ iii. & \dfrac{\phi_2 + \phi_7}{\phi_3 + \phi_8} > 0, \\ iv. & -\dfrac{b(-b + a)(\phi_4 + \phi_9)}{\phi_5 + \phi_{10}} > 0, \\ v. & 2ac_r^2 > 0. \end{cases} \quad (8)$$

Since the parameters $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$ are positive, it is easy to demonstrate that $(\phi_6, \phi_7, \phi_8, \phi_{10}) > 0$ and $\phi_9 > 0$, when $b > \frac{a}{2}$ and $b > a$, respectively. In this manner, the stability of system (2) is only controlled by the numerator of condition (iv), that is $[-(-b+a)]$. As a result, if $b > a$, there are no sign changes, and all roots have a negative real part, and thus, $EP_{1,2}$ are stable. □

3. HIDDEN CHAOTIC ATTRACTOR LOCALIZATION

As is well known, the most overwhelming chaotic systems are from unstable saddle points. However, further studies have shown that the self-excited periodic and chaotic oscillations do not provide exhaustive information about the possible types of oscillations, *i.e.*, hidden oscillations and hidden attractors.

Definition 1. *An attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of an equilibrium point; otherwise, it is called a hidden attractor. It also includes dynamical systems with no equilibrium points, an infinite number of equilibrium points, and stable equilibrium points Pham et al. (2017); Leonov et al. (2011).*

A few rare cases of complex-valued systems with hidden attractors and extreme multistability have been introduced in the literature. The necessary conditions for chaotic behavior in the complex Lorenz-type system (2) are Eq. (4) and Theorem 1. For the case of the hidden attractor with two stable equilibrium points, we must select $b > a$; thus, $a = 10$ and $b = 10.1$, with $c_r = 100$, $c_i = 0$, and initial conditions $(1, 1 + j2000, 0.1 + j2000)$. For all simulations, the time step is $h = 0.01$. It is important to remark that in the proposed system (2), z_1, z_2, z_3 are complex variables composed by their real (u_1, u_2, u_4) and imaginary (u_3, u_5) parts. Therefore, in our simulations, we set the initial conditions for the real and imaginary parts of the complex-valued system (2). It is worth noting that if we choose $u_3 = u_5 = 0$ for the imaginary parts, we recover the original system (1).

3.1 Hidden extreme multistability

Under the mentioned system parameters, we have the following equilibrium points $EP_{1,2} = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (\pm 10, \pm 10, 0, -10.1, 0)$. So, the eigenvalues are given by $\lambda_1 = -20.08$, $\lambda_{2,3} = -0.01 \pm j9.98$, $\lambda_{4,5} = -5.05 \pm j8.63$, which means $EP_{1,2}$ are stable node-foci equilibrium points. Similarly, we have the equilibria $EP_{1,2} = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (\pm 10, \pm 10, \mp 10, -10.1, 10.1)$, for the case where the parameter c is also considered as complex number, *i.e.*, $c_r = 100$, $c_i = -100$. As a result, the eigenvalues are $\lambda_1 = -20.08$, $\lambda_{2,3} = -0.01 \pm j9.98$, $\lambda_{4,5} = -5.05 \pm j8.63$, which means $EP_{1,2}$ are also stable node-foci equilibrium points.

Figure 1(a)-(b) presents the phase portraits of the hidden chaotic attractors for real (u_1, u_2, u_4) and complex variables (u_3, u_5) for the proposed complex-valued Lorenz-type chaotic system (2). To demonstrate its sensitivity to initial conditions, we perform a slight variation in the initial condition of the real variable u_2 and compute the time series of the complex variable u_3 as shown in Figure 1(c).

In addition, the phenomenon of extreme multistability is found by changing the initial condition of the complex variable u_3, which produces a shrinking and stretching movement in the chaotic attractor as can be seen in Figure 1(d).

Definition 2. *A chaotic system is said to converge to extreme multistability if its response leads to entirely different qualitative behavior for the same parameter values with minimal variation in its initial conditions. That is, extreme multistability is a special type of multistability associated with infinite hidden attractors. Therefore, new attractors are generated based not only on minimal variations in the initial conditions but also in the observation time Bao et al. (2017); Zhang and Li (2019); Fang et al. (2019); Lin et al. (2020).*

Figure 2 evidences the symmetry of the hidden chaotic attractor with initial conditions $(1, 1 + j2000, 0.1 + j2000)$ and $(1, 1 - j2000, 0.1 + j2000)$, respectively. Figure 3 shows the bifurcation diagram for the parameter a of the proposed complex chaotic system (2). We observed that a cascade of reverse double-period bifurcations leads to the hidden chaotic attractor since $b > a$. On the other hand, Figure 4 demonstrates the extreme multistability of the hidden chaotic attractor in the proposed complex chaotic system (2). For any change in the complex variable u_3, we have a scaling in the amplitude of the complex variable u_5.

4. CONCLUSION

A novel complex-valued Lorenz-type chaotic system has been introduced. By Theorem 2 and numerical analysis, we have demonstrated that the system generates a hidden chaotic attractor and extreme multistability as a function of the minimal variations in its initial conditions, which is very rare since the complex system possesses only stable equilibrium points. Potential applications are in the chaos-based data encryption field, such as random number generation, image encryption, and authentication protocols.

https://doi.org/10.58571/CNCA.AMCA.2025.050

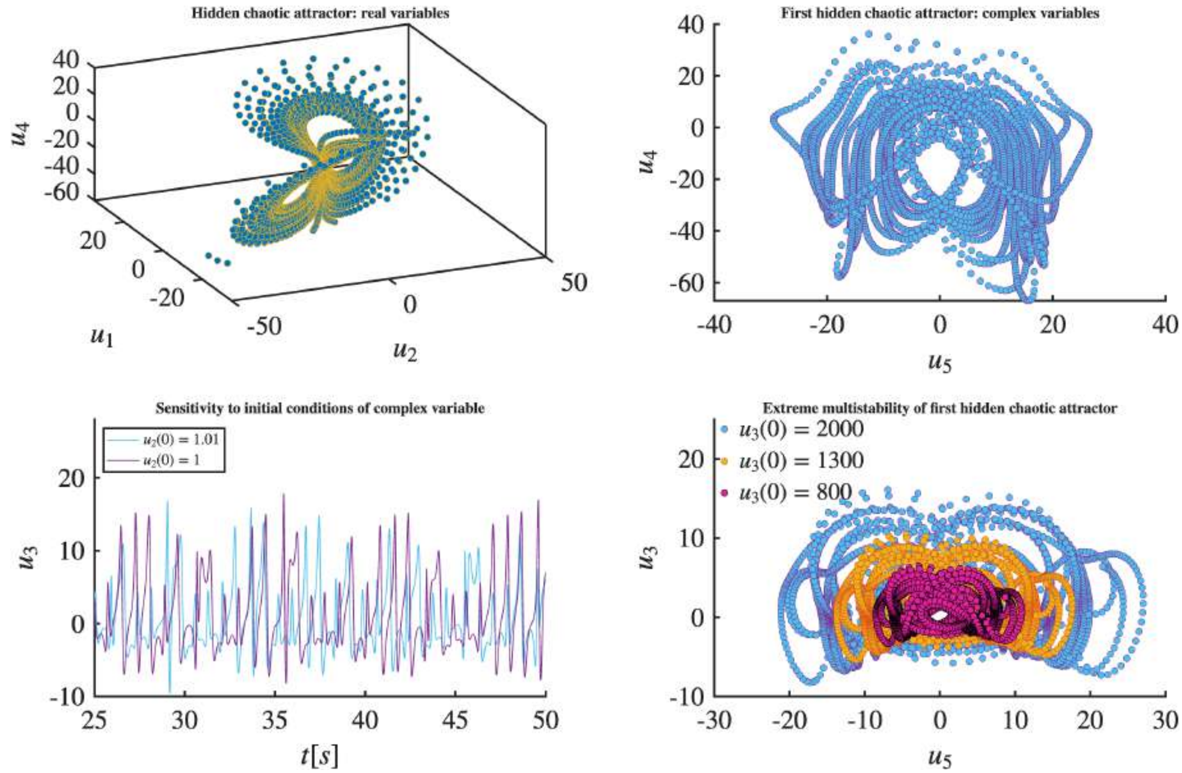

Fig. 1. (a)-(b) Hidden chaotic attractor for real (u_1, u_2, u_4) and complex variables (u_3, u_5) with $a = 10$, $b = 10.1$, $c = 100$, and initial conditions $(1, 1 + j2000, 0.1 + j2000)$. (c) Time series of the complex variable u_3 for a minimal change in initial conditions fos $z_2 = (u_2 + ju_3)$ with $(1 + j2000)$ and $(1.01 + j2000)$, respectively, (d) Extreme multistability of the hidden chaotic attractor as a function of the complex variable u_3 with initial conditions $(1, 1 + j2000, 0.1 + j2)$, $(1, 1 + j1300, 0.1 + j2)$, and $(1, 1 + j800, 0.1 + j2)$.

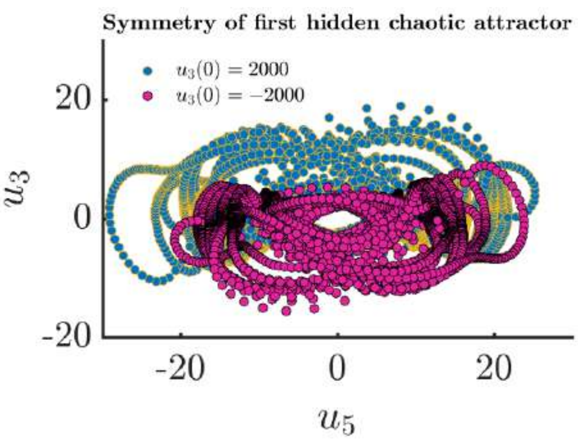

Fig. 2. (a) Symmetry of the hidden chaotic attractor with $a = 10$, $b = 10.1$, $c = 100$, and initial conditions $(1, 1 + j2000, 0.1 + j2000)$ and $(1, 1 - j2000, 0.1 + j2000)$, respectively. (b) Localization of the second hidden chaotic attractor using the complex parameter $c = 100 \pm j100$ and $a = 10$ and $b = 10.1$, with initial conditions $(1, 1 + j2, 0.1 + j2)$.

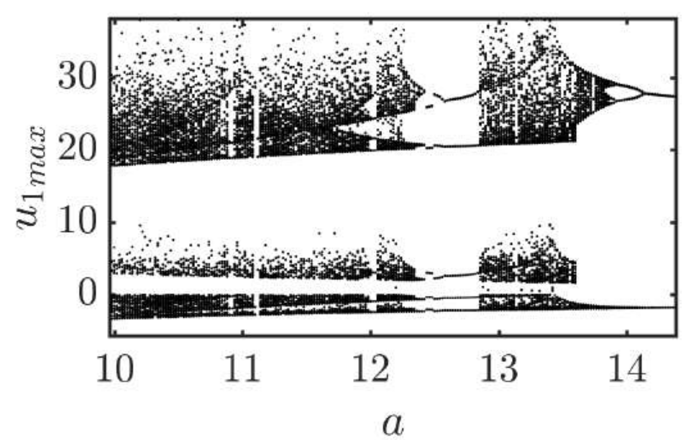

Fig. 3. Bifurcation diagram for the parameter a with $b = 10.1$, $c = 100$, and initial conditions $(1, 1 + j2000, 0.1 + j2000)$ in the complex chaotic system (2).

ACKNOWLEDGEMENTS

The first author thanks SECIHTI-Mexico for the support through the Ph.D. scholarship CVU 105600. The authors acknowledge BUAP for its support via the VIEP 2025 project.

REFERENCES

Ayubi, P., Setayeshi, S., and Rahmani, A.M. (2023). Chaotic complex hashing: A simple chaotic keyed hash function based on complex quadratic map. *Chaos, Solitons & Fractals*, 173, 113647.

Bao, B., Bao, H., Wang, N., Chen, M., and Xu, Q. (2017). Hidden extreme multistability in memristive hyperchaotic system. *Chaos, Solitons & Fractals*, 94, 102–111.

https://doi.org/10.58571/CNCA.AMCA.2025.050

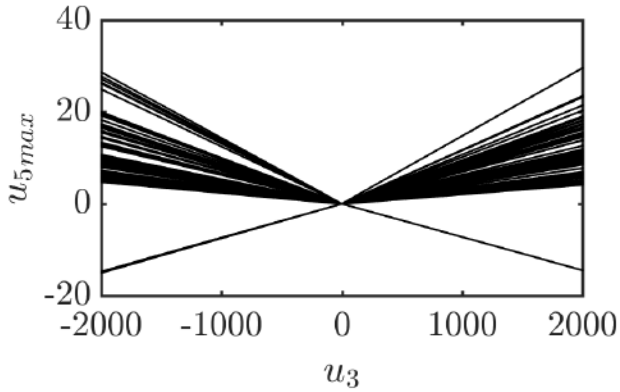

Fig. 4. Bifurcation diagram for extreme multistability under the initial condition u_3 with $a = 10$, $b = 10.1$, $c = 100$, and initial conditions $(1, 1 + j2000, 0.1 + j2000)$ in the complex chaotic system (2).

Devaney, R. (2018). *An introduction to chaotic dynamical systems*. CRC press.

Fang, S., Li, Z., Zhang, X., and Li, Y. (2019). Hidden extreme multistability in a novel no-equilibrium fractional-order chaotic system and its synchronization control. *Brazilian Journal of Physics*, 49(6), 846–858.

Gomez-Mont, X., Flores-Godoy, J.J., and Fernandez-Anaya, G. (2012). The attractors in the complex lorenz model. *IFAC Proceedings Volumes*, 45(12), 87–92.

Jin, M., Sun, K., and Wang, H. (2022). Hyperchaos, extreme multistability, and hidden attractors in the novel complex nonlinear system and its adaptive hybrid synchronization. *Nonlinear Dynamics*, 110(4), 3853–3867.

Leonov, G., Kuznetsov, N., and Vagaitsev, V. (2011). Localization of hidden chua's attractors. *Physics Letters A*, 375(23), 2230–2233.

Lin, H., Wang, C., and Tan, Y. (2020). Hidden extreme multistability with hyperchaos and transient chaos in a hopfield neural network affected by electromagnetic radiation. *Nonlinear Dynamics*, 99(3), 2369–2386.

Moghtadaei, M. and Golpayegani, M.H. (2012). Complex dynamic behaviors of the complex lorenz system. *Scientia Iranica*, 19(3), 733–738.

Muñoz-Pacheco, J.M. (2019). Infinitely many hidden attractors in a new fractional-order chaotic system based on a fracmemristor. *The European Physical Journal Special Topics*, 228(10), 2185–2196.

Munoz-Pacheco, J.M., Volos, C., Serrano, F.E., Jafari, S., Kengne, J., and Rajagopal, K. (2021). Stabilization and synchronization of a complex hidden attractor chaotic system by backstepping technique. *Entropy*, 23(7), 921.

Pham, V.T., Volos, C., and Kapitaniak, T. (2017). *Systems with hidden attractors: from theory to realization in circuits*. Springer.

Ren, L., Li, S., Banerjee, S., and Mou, J. (2023). A new fractional-order complex chaotic system with extreme multistability and its implementation. *Physica Scripta*, 98(5), 055201.

Shoreh, A.A. and Mahmoud, G.M. (2024). A novel adaptive synchronization algorithm for a general class of fractional-order complex-valued systems with unknown parameters, and applications to circuit realization and color image encryption. *Physica Scripta*, 99(2), 025212.

Strogatz, S.H. (2024). *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. Chapman and Hall/CRC.

Yang, Q. and Chen, Y. (2014). Complex dynamics in the unified lorenz-type system. *International Journal of Bifurcation and Chaos*, 24(04), 1450055.

Yang, Q., Wei, Z., and Chen, G. (2010). An unusual 3d autonomous quadratic chaotic system with two stable node-foci. *International Journal of Bifurcation and Chaos*, 20(04), 1061–1083.

Yuan, X., Li, C., and Huang, T. (2015). Complete synchronization and projective lag synchronization of complex-valued hyperchaotic lorenz system and lü system via active control. In *2015 5th International Conference on Information Science and Technology (ICIST)*, 339–344. IEEE.

Zhang, X. and Li, Z. (2019). Hidden extreme multistability in a novel 4d fractional-order chaotic system. *International Journal of Non-Linear Mechanics*, 111, 14–27.

Zhao, L., Zhao, L., Cui, F., and Sun, T. (2024). Satellite image encryption based on rna and 7d complex chaotic system. *The Visual Computer*, 40(8), 5659–5679.

Zhao, L., Liu, X., and Jin, J. (2025). A novel adaptive parameter zeroing neural network for the synchronization of complex chaotic systems and its field programmable gate array implementation. *Measurement*, 242, 115989.

https://doi.org/10.58571/CNCA.AMCA.2025.050