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Optimal growth rate tracking in an incomplete market with delays

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Abstract: We consider the problem of optimally tracking a given growth rate for investor's wealth over a finite-horizon in a market with *distributed and discrete delays*. This represents an example of a stochastic linear quadratic control problem with state-delay, and with additive-multiplicative noise. We derive an explicit closed-form solution to this problem as a state-feedback control law.

Keywords: Optimal control, stochastic systems, state-delay, optimal tracking.

1. INTRODUCTION AND PROBLEM FORMULATION

In classical portfolio optimization models, the problem is typically framed as an optimal stochastic control problem, where the investor selects an optimal strategy to maximize the expected utility of wealth. To establish the foundational framework for this class of models, we introduce the most basic *optimal investment problem* with expected utility from terminal wealth in continuous-time is as follows. Consider a market of a *bond* with price $\bar{S}_0$ and of a *stock* with price $\bar{S}$ that are solutions to following equations:

$$\begin{cases} d\bar{S}_0(t) = \bar{S}_0(t)\bar{r}(t)dt, \qquad t\in[0,T] \\ d\bar{S}(t) = \bar{S}(t)[\mu(t)dt + \bar{\sigma}'(t)d\bar{W}(t)], \qquad t\in[0,T] \\ \bar{S}_0(0)>0, \quad \bar{S}(0)>0, \qquad \text{are given.}\end{cases} \quad (1)$$

Here $\bar{r}$ is the interest rate, μ is the appreciation rate of the stock, $\bar{\sigma}$ is the volatility vector of the stock, and $\bar{W}$ is an n-dimensional standard Brownian motion. The market coefficients $\bar{r}$, μ, and $\bar{\sigma}$, are random processes and *unbounded* in general and must be such that equations (1) have unique strong solutions. Further, consider an *investor* with initial wealth $\bar{x}_0 > 0$ that holds $v_{\bar{S}_0}(t)$ number of shares in the bond and $v_{\bar{S}}(t)$ number of shares in the stock at time t. The value of the investor's portfolio, i. e. the investor's wealth, at time t is thus:

$$\bar{x}(t) := v_{\bar{S}_0}(t)\bar{S}_0(t) + v_{\bar{S}}(t)\bar{S}(t), \quad t\in[0,T]. \quad (2)$$

This portfolio is called *self-financing* if it has the following dynamics (see, for example, Korn (1997)):

$$d\bar{x}(t) = v_{\bar{S}_0}(t)d\bar{S}_0(t) + v_{\bar{S}}(t)d\bar{S}(t), \quad t\in[0,T]. \quad (3)$$

If we substitute the differentials of $\bar{S}(t)$ and $\bar{S}_0(t)$ from (1) into (3), and further knowing that $v_{\bar{S}_0}(t)\bar{S}_0(t) = \bar{x}(t) - v_{\bar{S}}(t)\bar{S}(t)$, which follows from (2), we obtain (for $t\in[0,T]$):

$$d\bar{x}(t) = [\bar{r}(t)\bar{x}(t) + (\mu(t)-\bar{r}(t))u(t)]dt + u(t)\bar{\sigma}'(t)d\bar{W}(t), \quad (4)$$

where $u(t) := v_{\bar{S}}(t)\bar{S}(t)$, $t\in[0,T]$. The self-financing portfolio (4) is thus an example of a *linear* stochastic control system with multiplicative noise with the investor's wealth $\bar{x}$ being the state of the system and the amount of wealth invested in the stock u being the control variable. The optimal investment problem with the expected utility from terminal wealth as the criterion, is the following optimal stochastic control problem:

$$\begin{cases} \max_{u(\cdot)\in\mathcal{D}} \mathbb{E}\left[U(\bar{x}(T))\right], \\ \text{s.t. } (4), \end{cases} \quad (5)$$

for some suitable admissible set of controls $\mathcal{D}$ and utility function U. Problem (5) represents a typical example in which past information is irrelevant and decisions are based only on current information. Extensive research has been conducted on models with such settings (see, e.g., Korn (1997), Karatzas and Shreve (1998), for textbook accounts).

In real-world financial markets the stock price process is affected by past information. Consequently, investors frequently rely on past stock performance when making decisions, leading to memory effects that Markov models cannot capture. To formulate such a framework, we begin by introducing the following: let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. We further consider the filtration $(\mathcal{F}(t),\ t\in[0,T])$, where $\mathcal{F}(t)$ is the augmentation of $\sigma\{B(s)\colon 0\le s\le t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$, where $(B(t),\ t\ge 0)$ is defined as an m_1-dimensional standard Brownian motion. Moreover, we assume that r, μ_1, μ_2, and μ_3 are positive constants, and that σ is a vector with strictly positive constant components such that $\sigma'\sigma > 0$. Further, let $I(t)$ be defined as the investment rate on the stock at time t. More precisely, let $L^2_{\mathcal{F}}(0,T;E)$ be the set of E valued square-integrable adapted processes (with E being an Euclidean space). Now, consider a fi-

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nancial market consisting of a *bond* with amount S_0 and a stock with amount S, which are modeled as solutions to following stochastic differential system with delay (for $t \in [s,T],\ s \in [0,T]$, see Chang et al. (2011) and Pang and Hussain (2017)):

$$\begin{cases} dS_0(t) = [rS_0(t) - I(t)]dt, \\ dS(t) = \{S(t)[\mu_1 + \mu_2 Y(t) + \mu_3 Z(t)] \\ +I(t)\}dt + \sigma S(t)dB(t). \end{cases} \tag{6}$$

Further, we assume that the differential stochastic equation of S depends on two delay variables Y and Z which are defined as follows (for $\forall t \in [s,T],\ s \in [0,T]$):

$$Y(t) := \int_{-h}^{0} e^{\lambda\theta} x(t+\theta)d\theta, \quad Z(t) := x(t-h),$$

where $\lambda > 0$ is a constant and $h > 0$ is the delay coefficient. Further, the total wealth $x(t)$ is defined as the sum of the bond and stock amounts, i.e., $x(t) = S_0(t) + S(t)$. Consequently, the corresponding wealth equation is as follows:

$$\begin{cases} dx(t) = \{[\mu_1 + \mu_2 Y(t) + \mu_3 Z(t)]S(t) + rS_0(t)\}\, dt \\ +\sigma' S(t)dB(t) \quad , t \in [0,T], \\ x(t) = \zeta(t-h), \quad t \in [s-h,s], \quad s \in [0,T]. \end{cases} \tag{7}$$

where $\zeta : [-h,0] \to \mathbb{R}$ is a given continuous function.
The optimal investment problem with an expected utility from terminal wealth as a criterion, is the following stochastic control problem:

$$\begin{cases} \max\limits_{S(\cdot)\in\mathcal{C}} \mathbb{E}[U(x(T)], \\ s.t.(7), \end{cases} \tag{8}$$

for some suitable admissible set of controls $\mathcal{C}$. This problem has been previously studied in the context of a finite-horizon and constant market coefficients, as in Chang et al. (2011) and for the infinite-horizon case (see Pang and Hussain (2016)). In addition, several researchers have studied similar stochastic models that incorporate delayed information structures: examples of such models include stochastic portfolio optimization Pang and Hussain (2017), stochastic investment and consumption optimization Chang et al. (2011), and infinite-horizon optimization Pang and Hussain (2015). Note that in equation (7) the control process $I(t)$ does not appear. As $x(t) = S_0(t) + S(t)$, by defining the control process $u(t) := S(t)$, we have that $S_0(t) = x(t) - u(t)$, and the wealth equation (7) becomes (for $t \in [s,T],\ s \in [0,T]$):

$$\begin{cases} dx(t) = \left[rx(t) + \mu_3 x(t-h) + \mu_2 \int_{-h}^{0} e^{\lambda\theta} x(t+\theta)d\theta \right. \\ \left. + (\mu_1 - r)\, u \right] dt + \sigma' u dB, \\ x(t) = \zeta(t-h), \quad t \in [s-h,s]\,. \end{cases} \tag{9}$$

In this paper, we take a different approach to the problem of optimal investment as compared to the above cited literature. Instead of maximising the expected utility from terminal wealth as in (8), we consider the problem of *tracking* a given *growth rate* for the wealth, i.e. the investor chooses a desired growth rate for his wealth, and then selects a trading strategy that minimises the tracking error. This approach to optimal investment is well-known in the markets *without* delay (see, for example, Yao et al. (2006), Gashi and Date (2012), Algoulity and Gashi (2023)), however, it has not been applied previously to the markets with delay as here. More precisely, consider the following *financial benchmark* equation that the investor aims to reach (track):

$$\begin{cases} dF(t) = bF(t)dt, \quad t \geq 0, \\ F(\tilde{\theta}) = \bar{\phi}(\tilde{\theta}) > 0, \ \tilde{\theta} \in [-h,0], \end{cases} \tag{10}$$

where $b > 0$ and $b > r$. As $F(t) = F(0)e^{bt}$ for $t \geq 0$, this benchmark specifies the desired growth rate for the wealth through the specification of $F(0)$ and b. We define the *tracking error* at time t, denoted by $y(t)$, as the difference between the wealth $x(t)$ and the value of the benchmark:

$$y(t) := x(t) - F(t), \quad t \in [-h,T].$$

The differential of the tracking error is $dy(t) = dx(t) - dF(t)$. Thus by taking the difference between equations (9) and (10) we obtain:

$$\begin{aligned} dy(t) &= dx(t) - dF(t) \\ &= \left[rx + \mu_3 x\,(t-h) + \mu_2 \int_{-h}^{0} e^{\lambda\theta} x(t+\theta)d\theta \right. \\ &\quad \left. + (\mu_1 - r)\, u \right] dt + u\sigma' dB - bF(t)dt \\ &= \left\{ r(x - F + F) + \mu_3[x(t-h) - F(t-h) + F(t-h)] \right. \\ &\quad +\mu_2 \int_{-h}^{0} e^{\lambda\theta}[x(t+\theta) - F(t+\theta) + F(t+\theta)]d\theta + (\mu_1 - r) \\ &\quad \left. \times u - bF \right\} dt + u\sigma' dB \\ &= \left\{ ry + \mu_3 y(t-h) + \mu_2 \int_{-h}^{0} e^{\lambda\theta} y(t+\theta)d\theta + (\mu_1 - r)u \right. \\ &\quad \left. +(r-b)F + \mu_3 F(t-h) + \mu_2 \int_{-h}^{0} e^{\lambda\theta} F(t+\theta)d\theta \right\} dt \\ &\quad +u\sigma' dB. \end{aligned} \tag{11}$$

Let y denote the first state variable, i.e. $x_1(t) := y(t)$. The term

$$\mu_2 \int_{-h}^{0} e^{\lambda\theta} y(t+\theta)d\theta$$

can be rewritten in the following more convenient form:

$$\begin{aligned} \mu_2 \int_{-h}^{\theta} e^{\lambda\theta} y(t+\theta)d\theta &= \mu_2 \int_{t-h}^{t} e^{\lambda(\tau - t)} y(\tau)d\tau \\ &= \mu_2 e^{-\lambda t} \int_{t-h}^{t} e^{\lambda\tau} y(\tau)d\tau. \end{aligned}$$

The second state variable x_2 is defined as:

$$x_2(t) := \int_{t-h}^{t} e^{\lambda\tau} y(\tau)d\tau, \quad \forall t \geqslant 0.$$

The differential of x_2 is:

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$$dx_2(t) = \left[e^{\lambda t}y(t) + e^{\lambda(t-h)}y(t-h)\right]dt$$
$$= \left[e^{\lambda t}x_1(t) + e^{\lambda(t-h)}x_1(t-h)\right]dt.$$

The equation of y in (11) in terms of x_1 and x_2 is:

$$dx_1(t) = \left[rx_1(t) + \mu_3 x_1(t-h) + \mu_2 e^{-\lambda t}x_2(t) + (\mu_1 - r)\,u + \tilde{F}(t)\right]dt + u\sigma' dB,$$

where $\tilde{F}(t)$ for $t \in [0,T]$ is defined as:

$$\tilde{F}(t) := (r-b)F(t) + \mu_3 F(t-h) + \mu_2 \int_{-h}^{0} e^{\lambda\theta}F(t+\theta)d\theta.$$

Now, we define the state vector $X(t) := [x_1(t) \quad x_2(t)]'$ with its equation being:

$$d\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left\{ \begin{bmatrix} r & \mu_2 e^{-\lambda t} \\ e^{\lambda t} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \mu_3 & 0 \\ e^{\lambda(t-h)} & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t-h) \\ x_2(t-h) \end{bmatrix} + \begin{bmatrix} \mu_1 - r \\ 0 \end{bmatrix} u + \begin{bmatrix} \tilde{F} \\ 0 \end{bmatrix} \right\} dt + u \begin{bmatrix} \sigma' \\ 0 \end{bmatrix} dB.$$

This can also be written as:

$$dX(t) = [A(t)X(t) + B(t)X(t-h) + C(t)u(t) + D(t)]dt + u(t)\tilde{C}(t)dB(t), \tag{12}$$

where

$$A(t) := \begin{bmatrix} r & \mu_2 e^{-\lambda t} \\ e^{\lambda t} & 0 \end{bmatrix}, \quad B(t) := \begin{bmatrix} \mu_3 & 0 \\ e^{\lambda(t-h)} & 0 \end{bmatrix},$$
$$C(t) := \begin{bmatrix} \mu_1 - r \\ 0 \end{bmatrix}, \quad D(t) := \begin{bmatrix} \tilde{F}(t) \\ 0 \end{bmatrix}, \quad \tilde{C}(t) := \begin{bmatrix} \sigma'(t) \\ 0 \end{bmatrix}.$$

As we aim to minimise the tracking error x_1, the optimality criterion is the following finite-horizon quadratic functional:

$$J(u(\cdot)) = \mathbb{E}\left\{ \int_0^T X'(t)N_1X(t)dt + X'(T)Q_1X(T) \right\},$$

where

$$N := \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix},$$

with N_1 and Q_1 being non-negative scalars. The *optimal growth rate tracking problem* to be solved is:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ s.t.(12), \end{cases} \tag{13}$$

where the admissible set of controls is defined as $\mathcal{A} := L^2_{\mathcal{F}}(0,T;\mathbb{R})$, and this in particular ensures the existence of a unique strong solution to (12) (see for example, Mao (2007)). Note that (13) is an optimal stochastic control problem, or more precisely a linear-quadratic (LQ) stochastic control problem with both *discrete and distributed delays*. Although there is a large recent literature on stochastic LQ control problem with delay under different settings (see, Chen and Zhang (2023), Pang and Hussain (2015), Pang and Hussain (2017), Zhang et al. (2021) Huang et al. (2012), Chen and Wu (2010), Liang et al. (2018)), Alasmi and Gashi (2024b), Alasmi and Gashi (2024a)), the problem (13) cannot be solved in a state-feedback form by directly applying such results. Instead, we have extended the existing approaches that rely on the completion of squares method to obtain the solution to problem (13) in an explicit closed-form as a state-feedback control law. The coefficients of such a control law are determined by a system of coupled linear ordinary and partial differential equations. Note that in what follows we have omitted the argument t whenever convenient for notational simplicity.

2. SOLUTION TO THE OPTIMAL GROWTH RATE TRACKING PROBLEM

In order to state the solution to optimal growth rate tracking problem (13), we introduce two set of linear ordinary differential equations that are coupled with a system of partial differential equations for $t \in [0, T-h]$, $\bar{\theta} \in [t, t+h]$, and $\bar{\xi} \in [t, t+h]$:

$$\begin{cases} \dot{p}_1'(t) + p_1'(t)A(t) + 2D'(t)p_2(t) + p_3(t,0) \\ -2R^{-1}p_1'(t)C(t)C'(t)p_2(t) = 0, \\ R(t) := tr(\tilde{C}'(t)p_2(t)\tilde{C}(t)), \\ p_1(T-h) = \bar{p}_1(T-h), \end{cases} \tag{14}$$

$$\begin{cases} \dot{p}_2(t) + A'(t)p_2(t) + p_2(t)A(t) + 2p_4(t,0) - 2R^{-1}p_2(t) \\ \times C(t)C'(t)p_2(t) + N = 0, \quad R(t) > 0, \\ p_2(T-h) = \bar{p}_2(T-h), \end{cases} \tag{15}$$

$$\begin{cases} \dfrac{\partial p_3(t, \bar{\theta}-h-t)}{\partial t} + 2D'(t)p_4(t, \bar{\theta}-h-t) \\ -2R^{-1}p_1'(t)C(t)C'(t)p_4(t, \bar{\theta}-h-t) = 0, \\ -e^{-\lambda h}p_3(t,-h) + p_1'(t)B(t) = 0, \\ p_3(T-h, z-h-t) = \bar{p}_3(T-h, z-h-t), \\ z \in [T-h, T], \end{cases} \tag{16}$$

$$\begin{cases} \dfrac{2\partial p_4(t, \bar{\theta}-h)}{\partial t} + 2A'(t)p_4(t, \bar{\theta}-h-t) + p_5'(t, \bar{\theta}-h-t, 0) \\ +p_5(t, 0, \bar{\theta}-h-t) - 2R^{-1}p_2(t)C(t)C'(t) \\ \times p_4(t, \bar{\theta}-h-t) = 0, \\ -e^{-\lambda h}p_4(t,-h) + p_2(t)B(t) = 0, \\ p_4(T-h, z-h-t) = \bar{p}_4(T-h, z-h-t), \\ z \in [T-h, T], \end{cases} \tag{17}$$

$$\begin{cases} \dfrac{\partial p_5(t, \bar{\theta}-h-t, \bar{\xi}-h-t)}{\partial t} - 2R^{-1}p_4'(t, \bar{\theta}-h-t) \\ \times C(t)C'(t)p_4(t, \bar{\xi}-h-t) = 0, \\ -e^{-\lambda h}p_5'(t, \bar{\theta}-h-t, -h) - e^{-\lambda h}p_5(t, -h, \bar{\theta}-h-t) \\ +2B(t)p_4(t, \bar{\theta}-h-t) = 0, \\ p_5(T-h, z-h-t, \bar{z}-h-t) \\ = \bar{p}_5(T-h, z-h-t, \bar{z}-h-t), \ z \in [T-h, T] \\ \text{and } \bar{z} \in [T-h, T] \end{cases} \tag{18}$$

For $t \in [T-h, T]$, $\bar{\theta} \in [t, T]$, and $\bar{\xi} \in [t, T]$:

$$\begin{cases} \dot{\bar{p}}_1'(t) + \bar{p}_1'(t)A(t) + 2D'(t)\bar{p}_2(t) - 2\tilde{R}^{-1}\bar{p}_1'(t)C(t)C'(t) \\ \times \bar{p}_2(t) = 0, \\ \bar{p}_1'(T) = 0, \end{cases} \tag{19}$$

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$$\begin{cases} \dot{\bar{p}}_2(t) + A'(t)\bar{p}_2(t) + \bar{p}_2(t)A(t) - 2\tilde{R}^{-1}\bar{p}_2(t)C(t)C'(t) \\ \times\bar{p}_2(t) + N = 0, \\ \tilde{R}(t) := tr(\tilde{C}'(t)\bar{p}_2(t)\tilde{C}(t)), \\ p_2(T) = Q, \end{cases} \tag{20}$$

$$\begin{cases} \dfrac{\partial\bar{p}_3'(t,\bar{\theta}-h-t)}{\partial t} + 2D'(t)\bar{p}_4(t,\bar{\theta}-h-t) - 2\tilde{R}^{-1}\bar{p}_1'(t) \\ \times C(t)C'(t)\bar{p}_4(t,\bar{\theta}-h-t) = 0, \\ -e^{-\lambda h}\bar{p}_3(t,-h) + \bar{p}_1'(t)B(t) = 0, \end{cases} \tag{21}$$

$$\begin{cases} \dfrac{2\partial\bar{p}_4'(t,\bar{\theta}-h-t)}{\partial t} + 2A(t)\bar{p}_4(t,\bar{\theta}-h-t) - 2\tilde{R}^{-1}\bar{p}_2(t)C(t) \\ \times C'(t)\bar{p}_4'(t,\bar{\theta}-h-t) = 0, \\ -e^{-\lambda h}\bar{p}_4(t,-h) + \bar{p}_2(t)B(t) = 0, \end{cases} \tag{22}$$

$$\begin{cases} \dfrac{\partial\bar{p}_5(t,\bar{\theta}-h-t,\bar{\xi}-h-t)}{\partial t} - 2\tilde{R}^{-1}\bar{p}_4'(t,\bar{\theta}-h-t)C(t) \\ \times C'(t)\bar{p}_4(t,\bar{\xi}-h-t) = 0, \\ -e^{-\lambda h}\bar{p}_5'(t,\bar{\theta}-h-t,-h) - e^{-\lambda h}\bar{p}_5(t,-h,\bar{\theta}-h-t) \\ +2B'(t)\bar{p}_4'(t,\bar{\theta}-h-t) = 0, \end{cases} \tag{23}$$

Assumption 1. *The system of coupled equations (14)-(23) has a unique solution.*

Theorem 1. *There exist a unique solution u^* to the optimal growth rate tracking problem (13). If $T-h \le 0$, then the solution is given by:*

$$u^* := -2R^{-1}\Big[0.5C'(t)p_1(t) + C'(t)p_2(t)X(t) + C'(t) \times \int_t^T e^{\lambda(\bar{\theta}-h-t)}p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta}\Big], \quad t\in[0,T], \tag{24}$$

If $T-h>0$, then the solution is given by:

$$u^* := -2R^{-1}M = -2R^{-1}\Big[0.5C'(t)p_1(t) + C'(t)p_2(t)X(t) + C'(t) \times \int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta}\Big],\ t\in[0,T-h],$$

where $M := 0.5C'(t)p_1(t) + C'(t)p_2(t)X(t) + C'(t) \times \int_t^{t+h} e^{\lambda(\theta-h-t)}p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta},\ t\in[0,T]$.

$$u^* := -2\tilde{R}^{-1}\bar{M} = -2\tilde{R}^{-1}\Big[0.5C'(t)\bar{p}_1(t) + C'(t)\bar{p}_2(t)X(t) + C'(t) \times \int_t^T e^{\lambda(\bar{\theta}-h-t)}\bar{p}_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta}\Big], \quad t\in[T-h,T],$$

where $\bar{M} := 0.5C'(t)\bar{p}_1(t) + C'(t)\bar{p}_2(t)X(t) + C'(t) \times \int_t^T e^{\lambda(\bar{\theta}-h-t)}\bar{p}_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta},\ t\in[T-h,T]$.

Proof. We only consider the case of $T-h>0$, as the case of $T-h\le 0$ is very similar and simpler. For $t\in[0,T-h]$, we define the process v_1 as:

$$v_1 := p_1'(t)X(t) + X'(t)p_2(t)X(t) + \int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)} \times p_3'(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta} + 2X'(t)\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)} \times p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta} + \int_t^{t+h}\int_t^{t+h} X'(\bar{\theta}-h) \times e^{\lambda(\bar{\theta}-h-t)}e^{\lambda(\bar{\xi}-h-t)}p_5(t,\bar{\theta}-h-t,\bar{\xi}-h-t)X(\bar{\xi}-h)d\bar{\theta}d\bar{\xi}.$$

By applying Itô's formula, the differential of v_1 is:

$$\begin{aligned} dv_1 &= \dot{p}_1'(t)X(t)dt + p_1'(t)[A(t)X(t) + B(t) \\ &\times X(t-h) + C(t)u(t) + D(t)]dt + p_1'(t)u(t)\tilde{C}(t)dB(t) \\ &+[A(t)X(t) + B(t)X(t-h) + C(t)u(t) + D(t)]'p_2(t) \\ &\times X(t)dt + [u(t)\tilde{C}(t)dB(t)]'p_2(t)X(t) + X'(t)\dot{p}_2(t)X(t)dt \\ &+X'(t)p_2(t)[A(t)X(t) + B(t)X(t-h) + C(t)u(t) \\ &+D(t)]dt + X'(t)p_2(t)u(t)\tilde{C}(t)dB(t) + 0.5u^2(t) \\ &\times tr(\tilde{C}'(t)p_2(t)\tilde{C}(t))dt + \int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}\frac{\partial p_3'(t,\bar{\theta}-h-t)}{\partial t} \\ &\times X(\bar{\theta}-h)d\bar{\theta} + p_3'(t,0)X(t)dt - e^{-\lambda h}p_3'(t,-h)X(t-h)dt \\ &+2X'(t)\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}\frac{\partial p_4(t,\bar{\theta}-h-t)}{\partial t}X(\bar{\theta}-h)d\bar{\theta} \\ &+2\left[A(t)X(t) + B(t)X(t-h) + C(t)u(t) + D(t)\right]' \\ &\times\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta}dt + 2[u(t) \\ &\times\tilde{C}(t)dB(t)]'\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta} \\ &+2X'(t)p_4(t,0)X(t)dt - 2e^{-\lambda h}X'(t)p_4\left(t,-h\right)X(t-h)dt \\ &+\int_t^{t+h}\int_t^{t+h} X'(\bar{\theta}-h)e^{\lambda(\bar{\theta}-h-t)}e^{\lambda(\bar{\xi}-h-t)} \\ &\times\frac{\partial p_5(t,\bar{\theta}-h-t,\bar{\xi}-h-t)}{\partial t}X(\bar{\xi}-h)d\bar{\theta}d\bar{\xi} \\ &+X'(t)\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}p_5(t,0,\bar{\theta}-h-t) \\ &\times X(\bar{\theta}-h)d\bar{\theta}dt - X'(t-h)e^{-\lambda h}\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)} \\ &\times p_5(t,-h,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta}dt + \int_t^{t+h} X'(\bar{\theta}-h) \\ &\times e^{\lambda(\bar{\theta}-h-t)}p_5(t,\bar{\theta}-h-t,0)X(t)d\bar{\theta}dt \\ &-e^{-\lambda h}\int_t^{t+h} X'(\bar{\theta}-h)e^{\lambda(\bar{\theta}-h-t)}p_5(t,\bar{\theta}-h-t,-h) \\ &\times X(t-h)d\bar{\theta}dt \end{aligned} \tag{25}$$

Now, we define $G(u)$ as the collection of all terms in (25) that depend explicitly on the control u. In other words,

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$$G(u) := u^2(t)\left[0.5tr(\tilde{C}'(t)p_2(t)\tilde{C}(t))\right] + 2u(t)\Bigg[0.5C'(t)p_1(t)$$
$$+C'(t)p_2(t)X(t) + C'(t)\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}p_4(t,\bar{\theta}-h-t)$$
$$\times X(\bar{\theta}-h)d\bar{\theta}\Bigg]$$
$$= 0.5u^2(t)R(t) + 2u(t)M \quad (26)$$

By the completion of square, we rewrite $G(u)$ in (26) as:

$$G(u) = 0.5R(t)\left[u^2(t) + 4u(t)R^{-1}M + (2R^{-1})^2M'M\right.$$
$$\left.-\left(2R^{-1}\right)^2 M'M\right]$$
$$= 0.5R(t)\left[u + 2R^{-1}M\right]'\left[u + 2R^{-1}M\right] - 2R^{-1}M'M$$
$$= 0.5R(t)\left[u + 2R^{-1}M\right]'\left[u + 2R^{-1}M\right] - 2R^{-1}$$
$$\times\Bigg[0.5C'(t)p_1(t) + C'(t)p_2(t)X(t) + C'(t)$$
$$\times\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta}\Bigg]'$$
$$\times\Bigg[0.5C'(t)p_1(t) + C'(t)p_2(t)X(t) + C'(t)\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}$$
$$\times p_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\theta\Bigg]$$

By integrating both sides of the equation (25) from 0 to $T-h$, then taking the expectation, we obtain:

$$\mathbb{E}[J_1(u(\cdot)) + v_1(T-h)] = v_1(0)$$
$$+\mathbb{E}\Bigg\{\int_0^{T-h}\Bigg\{p_1'(t)D(t) - 0.5R^{-1}p_1'(t)C(t)C'(t)p_1(t)$$
$$+X'(t)[\dot{p}_2(t) + A'(t)p_2(t) + p_2(t)A(t) + 2p_4(t,0) - 2R^{-1}$$
$$\times p_2(t)C(t)C'(t)p_2(t) + N]X(t) + [\dot{p}_1'(t) + p_1'(t)A(t)$$
$$+2D'(t)p_2(t) + p_3'(t,0) - 2R^{-1}(t)p_1'(t)C(t)C'(t)p_2(t)]X(t)$$
$$+\int_t^{t+h} e^{-\lambda(\bar{\theta}-h-t)}\left[\frac{\partial p_3(t,\bar{\theta}-h-t)}{\partial t} + 2D'(t)\right.$$
$$\left.\times p_4(t,\bar{\theta}-h-t) - 2R^{-1}p_1'(t)C(t)C'(t)p_4(t,\bar{\theta}-h-t)\right]$$
$$\times X(\bar{\theta}-h)d\bar{\theta} + X'(t)\int_t^{t+h} e^{\lambda(\bar{\theta}-h-t)}\left[\frac{2\partial p_4(t,\bar{\theta}-h-t)}{\partial t}\right.$$
$$-2R^{-1}p_2(t)C(t)C'(t)p_4(t,\bar{\theta}-h-t) + 2A'(t)p_4(t,\bar{\theta}-h-t)$$
$$\left.+p_5'(t,\bar{\theta}-h-t,0) + p_5(t,0,\bar{\theta}-h-t)\right]X(\bar{\theta}-h)d\bar{\theta}$$
$$+\int_t^{t+h}\int_t^{t+h} X'(\bar{\theta}-h)e^{\lambda(\bar{\theta}-h-t)}e^{\lambda(\bar{\xi}-h-t)}$$
$$\times\left[\frac{\partial p_5(t,\bar{\theta}-h-t,\bar{\xi}-h-t)}{\partial t} - 2R^{-1}p_4'(t,\bar{\theta}-h-t)C(t)\right.$$
$$\left.\times C'(t)p_4(t,\bar{\xi}-h-t)\right]X(\bar{\xi}-h)d\bar{\theta}d\bar{\xi}$$
$$+0.5R\left[u + 2R^{-1}M\right]'\left[u + 2R^{-1}M\right]\Bigg\}dt\Bigg\}$$
$$= J_1(u^*(\cdot)) + \mathbb{E}\int_0^{T-h} 0.5R\left[u+2R^{-1}M\right]'\left[u+2R^{-1}M\right]dt, \quad (27)$$

where

$$J_1(u^*(\cdot)) := p_1'(0)X(0) + X'(0)p_2(0)X(0) + \int_0^h e^{\lambda(\bar{\theta}-h)}$$
$$\times p_3'(0,\bar{\theta}-h)X(\bar{\theta}-h)d\bar{\theta} + 2X'(0)\int_0^h e^{\lambda(\bar{\theta}-h)}p_4(0,\bar{\theta}-h)$$
$$\times X(\bar{\theta}-h)d\bar{\theta} + \int_0^h\int_0^h X'(\bar{\theta}-h)e^{\lambda(\bar{\theta}-2h+\bar{\xi})}$$
$$\times p_5(0,\bar{\theta}-h,\bar{\xi}-h)X(\bar{\xi}-h)d\bar{\theta}d\bar{\xi} + \int_0^{T-h}[p_1'(t)D(t)$$
$$-0.5R^{-1}p_1'(t)C(t)C'(t)p_1(t)]d.$$

For $t \in [T-h, T]$, we define the process v_2 as:

$$v_2 := \bar{p}_1'(t)X(t) + X'(t)\bar{p}_2(t)X(t) + \int_t^T e^{\lambda(\bar{\theta}-h-t)}$$
$$\times\bar{p}_3'(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta} + 2X'(t)\int_t^T e^{\lambda(\bar{\theta}-h-t)}$$
$$\times\bar{p}_4(t,\bar{\theta}-h-t)X(\bar{\theta}-h)d\bar{\theta} + \int_t^T\int_t^T X'(\bar{\theta}-h)$$
$$\times e^{\lambda(\bar{\theta}-h-t)}e^{\lambda(\bar{\xi}-h-t)}\bar{p}_5(t,\bar{\theta}-h-t,\bar{\xi}-h-t)$$
$$\times X(\bar{\xi}-h)d\bar{\theta}d\bar{\xi}.$$

By proceeding in a similar way with v_2 as we did for the case of v_1, i.e. by applying Itô's formula to v_2, completing the square for the terms that depend explicitly on u, integrating both sides, and then taking the conditional expectation, we obtain:

$$J_2(u(\cdot)) := v_2(T-h) + \mathbb{E}\Bigg\{\int_{T-h}^T\Bigg\{-0.5\tilde{R}^{-1}\bar{p}_1'(t)C(t)$$
$$\times C'(t)\bar{p}_1(t) + \bar{p}_1'(t)D(t) + [\dot{\bar{p}}_1'(t) + \bar{p}_1'(t)A(t) + 2D'(t)$$
$$\times\bar{p}_2(t) - 2\tilde{R}^{-1}\bar{p}_1'(t)C(t)C'(t)\bar{p}_2(t)]X(t) + [\bar{p}_1'(t)B(t)$$
$$-e^{-\lambda h}\bar{p}_3'(t,-h)\big]X(t-h) + X'(t)[\dot{\bar{p}}_2(t) + A'(t)\bar{p}_2(t)$$
$$+\bar{p}_2(t)A(t) - 2\tilde{R}^{-1}\bar{p}_2(t)C(t)C'(t)\bar{p}_2(t) + N\big]X(t)$$
$$+X'(t)\left[2\bar{p}_2(t)B(t) - 2e^{-\lambda h}\bar{p}_4(t,-h)\right]X(t-h) + X'(t)$$
$$\times\int_t^T e^{\lambda(\bar{\theta}-h-t)}\left[-4\tilde{R}^{-1}\bar{p}_2(t)C(t)C'(t)\bar{p}_4(t,\bar{\theta}-h-t)\right.$$
$$\left.+2A'(t)\bar{p}_4(t,\bar{\theta}-h-t) + \frac{2\partial\bar{p}_4(t,\bar{\theta}-h-t)}{\partial t}\right]X(\bar{\theta}-h)d\bar{\theta}$$
$$+\int_t^T e^{\lambda(\bar{\theta}-h-t)}\left[-2\tilde{R}^{-1}\bar{p}_1'(t)C(t)C'(t)\bar{p}_4(t,\bar{\theta}-h-t)\right.$$
$$\left.+2D'(t)\bar{p}_4(t,\bar{\theta}-h-t) + \frac{\partial\bar{p}_3'(t,\bar{\theta}-h-t)}{\partial t}\right]$$
$$\times X(\bar{\theta}-h)d\bar{\theta} + X'(t-h)\int_t^T e^{\lambda(\bar{\theta}-h-t)}\Big[e^{-\lambda h}$$

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$$\times\bar{p}_5(t,-h,\bar{\theta}-h-t)+2B'(t)\bar{p}_4(t,\bar{\theta}-h-t)-e^{-\lambda h}$$

$$\times\bar{p}_5'(t,\bar{\theta}-h-t,-h)\Big]X(\bar{\theta}-h)d\bar{\theta}+\int_t^T\int_t^T e^{\lambda(\bar{\theta}-h-t)}$$

$$\times e^{\lambda(\bar{\xi}-h-t)}x'(\bar{\theta}-h)\left[\frac{\partial\bar{p}_5(t,\bar{\theta}-h-t,\bar{\xi}-h-t)}{\partial t}-2\tilde{R}^{-1}\right.$$

$$\left.\times\bar{p}_4'(t,\bar{\theta}-h-t)C(t)C'(t)\bar{p}_4(t,\bar{\xi}-h-t)\right]X(\bar{\xi}-h)d\bar{\theta}d\bar{\xi}$$

$$+0.5R\left[u+2\tilde{R}^{-1}\bar{M}\right]'\left[u+2\tilde{R}^{-1}\bar{M}\right]\Bigg\}dt\Big|\mathcal{F}(T-h)\Bigg\}$$

$$=v_2(T-h)+\mathbb{E}\Bigg\{\int_{T-h}^T\Bigg\{-0.5\tilde{R}^{-1}\bar{p}_1'(t)C(t)C'(t)\bar{p}_1(t)$$

$$+\bar{p}_1'(t)D(t)+0.5\tilde{R}\left[u+2\tilde{R}^{-1}\bar{M}\right]'$$

$$\times\left[u+2\tilde{R}^{-1}\bar{M}\right]\Bigg\}dt\Big|\mathcal{F}(T-h)\Bigg\}$$

$$=v_2(T-h)+J_2(u^*(\cdot))+\mathbb{E}\Bigg\{\int_{T-h}^T 0.5\tilde{R}$$

$$\times\left[u+2\tilde{R}^{-1}\bar{M}\right]'\left[u+2\tilde{R}^{-1}\bar{M}\right]\Bigg\}dt\Big|\mathcal{F}(T-h)\Bigg\} \qquad (28)$$

where $J_2(u^*(\cdot)) := \mathbb{E}\Bigg\{\int_{T-h}^T\Bigg\{-0.5\tilde{R}^{-1}\bar{p}_1'(t)C(t)C'(t)$

$$\times\bar{p}_1(t)+\bar{p}_1'(t)D(t)\Bigg\}dt\Big|\mathcal{F}(T-h)\Bigg\}.$$

Note that $v_1(T-h)=v_2(T-h)$. Now, from (27) and (28) it follows that for any $u(\cdot)\in\mathcal{A}$ we have:

$$J(u(\cdot))=J_1(u^*(\cdot))+\mathbb{E}\Bigg\{\int_0^{T-h}0.5R\left[u+2R^{-1}M\right]'$$

$$\times\left[u+2R^{-1}M\right]dt\Bigg\}+J_2(u^*(\cdot))+\mathbb{E}\Bigg\{\mathbb{E}\Bigg\{\int_{T-h}^T 0.5\tilde{R}$$

$$\times\left[u+2\tilde{R}^{-1}\bar{M}\right]'\left[u+2\tilde{R}^{-1}\bar{M}\right]dt\Big|\mathcal{F}(T-h)\Bigg\}\Bigg\}$$

$$\geq J_1(u^*(\cdot))+J_2(u^*(\cdot)),$$

This lower bound is achieved if and only if $u(t)=u^*(t)$ for *a.e.* $t\in[0,T]$ *a.s.*. □

3. CONCLUSIONS

We have considered the problem of optimal growth rate tracking problem in an incomplete market with delays. By deriving a certain completion of squares method, along with a system of coupled ordinary and partial differential equations, we successfully obtained a unique and explicit closed-form solution to such an optimal stochastic control problem. A set of challenging extensions that may be explored in future studies is the optimal benchmark tracking problem in more general market models and benchmarks, such as those that include random and unbounded coefficients with delays, the inclusion of consumption, or the incorporation of borrowing constraints with delays.

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