Memorias del Congreso Nacional de Control Automático
ISSN: 2594-2492

A class of observers for the observable part of linear systems with commensurate delays

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***Abstract*—The present congressarticle deals with the observer design conditions for two particular classes of linear systems with commensurate delay. The first one is for systems with known inputs and the second one for systems affected by unknown inputs. The use of a Kalman-like decomposition is proposed, as well as the application of a Luenberger-like observer only for the observable part of the system decomposed. The conditions proposed for the observers are considered based on the definition of invariant factors of the original system.**

I. Introduction

Linear systems with delays (also called hereditary systems, equation with deviation in the argument, dead time or post-effect) are systems that include certain information of the past state in the process of the mathematical modeling of the physical systems studied. A survey of chemical and biological applications have been included in [7]. In the terms of mechanical, electrical and aeronautical systems a number of examples using these post-effects is shown in [1] and [6]. The design of a Luenberger-like observer is a problem widely studied and solved in the literature [9]. This consists on generating a virtual copy of the studied system, which allows to have direct access to the state. If the original system and the observer are subjected to the same initial conditions, it is expected, as time progresses, that both systems will behave the same.

The design of observers for linear systems with delays became quite complex, because those delays could appear in the state equations, the inputs and the system outputs. The main problem with these deviated argument equations is the difficulty to ensure the convergence between the original system and the virtual one. An important work of study has been carried out for the case with known inputs in [5], and for the case with unknown inputs in [4], [8] and [10]; the last one considering systems where the output is also affected by unknown inputs. Both of the previous mentioned works have studied systems that satisfy the observability condition, so the aim of this paper is to show the necessary conditions for the design of observers in linear systems with commensurate delays that do not satisfy such condition but can be worked with a Kalman-like decomposition. First, separating the system in its corresponding observable and unobservable part, occupying the properties previously developed in [5] and [10], respectively. And then, designing the observer only for the observable part of the system. The present work adopts matrix polynomial methods based on ring theory in order to allow an approach to existing techniques developed in linear systems that do not consider delay.

A. Notation

The following notation will be used: Let $G(\delta) \in \mathbb{R}[\delta]^{n\times m}$ be a polynomial matrix. It is defined $G(\delta)_R^{-1}$ as a right inverse (provided it exists) of $G(\delta)$ (i.e. $G(\delta)G(\delta)_R^{-1} = I$), and $G(\delta)_L^{-1}$ as a left inverse (provided it exists) of $G(\delta)$ (i.e. $G(\delta)_L^{-1}G(\delta)_L = I$). If $G \in \mathbb{R}[\delta]^{n\times n}$ has an inverse $G(\delta)^{-1}$ ($G(\delta)^{-1}G(\delta) = G(\delta)G(\delta)^{-1} = I$) then, it is called unimodular. For $J(\delta) \in \mathbb{R}[\delta]^{n\times m}$ with rank equal to r, we define $J(\delta)^{\perp} \in \mathbb{R}[\delta]^{n-r\times n}$ as a matrix achieving $J(\delta)^{\perp}J(\delta) = 0$ and $\operatorname{rank} J(\delta) = n - r$.

Let $G(\delta)$ be a polynomial matrix of $n\times m$ dimension with rank equal to r (taking into account that $r \le \min\{n, m\}$). There exist two invertible matrices $U(\delta)$ and $Z(\delta)$ over $\mathbb{R}[\delta]$ such that $G(\delta)$ is reduced to its Smith form, i.e.,

$$U(\delta)G(\delta)Z(\delta) = \begin{bmatrix} \operatorname{diag}(\psi_1(\delta)\cdots\psi_r(\delta)) & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$

where the $\{\psi_i(\delta)\}'s$ are monic nonzero polynomials satisfying

$$\psi_i(\delta)|\psi_{i+1}(\delta) \qquad ; \qquad d_i(\delta) = d_{i-1}(\delta)\psi_i(\delta)$$

where $d_i(\delta)$ is the gcd of all $i\times i$ minors of $G(\delta)$ ($d_0 = 1$). The $\{\psi_i(\delta)\}'s$ are called invariant factors, and the $\{d_i(\delta)\}'s$ determinant divisors. Thus, $G(\delta) \in \mathbb{R}[\delta]^{n\times m}$ has a right inverse if and only if it has n constant invariant factors, and $G(\delta) \in \mathbb{R}[\delta]^{n\times m}$ has a left inverse if and only if it has m constant invariant factors. Hence, a square matrix $G(\delta) \in \mathbb{R}[\delta]^{n\times n}$ is unimodular if and only if it has n constant invariant factors.

II. Formulation of the problem. CASE I

The mathematical model is presented as follows:

$$\begin{aligned} \dot{x}(t) &= \textstyle\sum_{i=0}^{k_a} A_i x(t-ih) + \sum_{i=0}^{k_b} B_i u(t-ih) \\ y(t) &= \textstyle\sum_{i=0}^{k_c} C_i x(t-ih) + \sum_{i=0}^{k_d} D_i u(t-ih) \end{aligned} \quad (2)$$

Where:
$x(t) \in \mathbb{R}[\delta]^n$, is vector of the system trajectories of the state.

$u(t) \in \mathbb{R}[\delta]^m$, is the system control considered known.
$y(t) \in \mathbb{R}[\delta]^p$, is the system output.

The initial condition $\varphi(t)$ is a piecewise continuous function $\varphi(t) : [-kh, 0] \to \mathbb{R}[\delta]^n$ for $k = \max k_a, k_b, k_c, k_d$; thereby, $x(t) = \varphi(t)$ over $[-kh, 0]$. A_i, B_i, C_i D_i are matrices of appropriate dimension with entries in $\mathbb{R}$. In order to facilitate the mathematical analysis, the delay operator is introduced$\delta^i : x(t) \to x(t-ih)$ as $\delta^k x(t) = x(t-kh), k \in \mathbb{N}_0$. Thus, the system in 2 can be represented with matricial form:

$$\begin{aligned} \dot{x}(t) &= A(\delta)x(t) + B(\delta)u(t) \\ y(t) &= C(\delta)x(t) + D(\delta)u(t) \end{aligned} \tag{3}$$

Where the fourfold of matrices $A(\delta)$, $B(\delta)$, $C(\delta)$, $D(\delta)$ are defined as

$$\begin{aligned} A(\delta) := \sum_{i=0}^{k_a} A_i \delta^i \ ; \ B(\delta) := \sum_{i=0}^{k_b} B_i \delta^i \\ C(\delta) := \sum_{i=0}^{k_c} C_i \delta^i \ ; \ D(\delta) := \sum_{i=0}^{k_d} D_i \delta^i \end{aligned}$$

A. Preliminaries

Following [2] we define a preliminary results analogous to the ones without delays. Let $\dot{x}_c(t) = A(\delta)x_c(t) + B(\delta)u(t)$ we have that the dynamic equation for $x_e(t) := x(t) - x_c(t)$ is given by $\dot{x}_e(t) = A(\delta)x_e(t)$ with the output $y_e(t) := y(t) - C(\delta)x_c(t) = Cx_e(t)$. Thus, the estimation of $x(t)$ is equivalent to the estimation of $x_e(t)$ since $x(t) = x_e(t) + x_c(t)$. This means that without the loss of generality it can be assumed that $u \equiv 0$.

Taking into account the results of [5], let us define the following generalized change of coordinates by $\xi(t) = T(\delta)x(t)$, where the matrix in $T(\delta)$ is developed with the form $T(\delta) = \begin{bmatrix} T_1(\delta) \\ \vdots \\ T_k(\delta) \end{bmatrix}$. Following the next construction:

$$\begin{aligned} T_1(\delta) &= C(\delta) \\ T_{i+1}(\delta) &= T_i(\delta)A(\delta) - \bar{H}_i(\delta)C(\delta), \quad \forall i = 1, \ \ldots, \ k-1 \end{aligned} \tag{4}$$

where $\bar{H}_i(\delta)'s$ are determined through $\begin{bmatrix} \bar{H}_k(\delta) & \ldots & \bar{H}_1(\delta) \end{bmatrix} = C(\delta)A^k(\delta)\mathcal{O}_k^L(\delta)$.

Therefore, the system in (2) can be transformed into the form:

$$\begin{aligned} \dot{\xi}(t) &= \bar{A}\xi(t) + \bar{H}(\delta)y(t) \\ y(t) &= \bar{C}\xi(t) \end{aligned} \tag{5}$$

Where $\bar{H}(\delta)y(t)$ is the term of output injection with $\bar{H}(\delta) \in \mathbb{R}[\delta]^{n \times p}$, and matrix$\bar{A}$ and $\bar{C}$ will be constant matrices and the pair $(\bar{A}, \bar{C})$ is observable. There, we can see that all the delay effects are presented as output injection, thereby, an observer for $\xi(t)$ can be designed as

$$\dot{\hat{\xi}}(t) = \bar{A}\hat{\xi}(t) + \bar{H}(\delta)y(t) + \bar{L}\left(y(t) - \bar{C}\hat{\xi}(t)\right) \tag{6}$$

a) Dynamic Error: The estimation error is defined as $e(t) = \xi(t) - \hat{\xi}(t)$. Therefore, taking into account (5) and (6), we obtain the dynamic error:

$$\dot{\mathbf{e}}(\mathbf{t}) = (\bar{\mathbf{A}} - \bar{\mathbf{L}}\bar{\mathbf{C}})\mathbf{e}(\mathbf{t}) \tag{7}$$

where, the matrix $\bar{L}$ is a constant matrix that turn the system into Hurwitz. The observer (6) gives the estimation of $\xi(t)$, so the last step to estimate $x(t)$ is to ensure that the matrix $T(\delta)$ has a left inverse so that $x(t) = T_L^{-1}(\delta)\xi(t)$. It has been proven in [5] that if the observability matrix $\mathcal{O}_x(\delta) \in \mathbb{R}[\delta]$ of the system (2):

$$\mathcal{O}_x(\delta) = \begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ \vdots \\ C(\delta)A^{n-1}(\delta) \end{bmatrix} \tag{8}$$

has a left inverse (or right) $T(\delta)$ has a left inverse, then we can define an observer for $x(t)$ by means of that of $\xi(t)$, that is with $\hat{x}(t) := T_L^{-1}(\delta)\hat{\xi}(t)$ we obtain that $\hat{x}(t) \to x(t)$ as $t \to \infty$.

Thus the problem here is to study the case when the observability matrix $\mathcal{O}_x(\delta)$ does not have a left inverse, that is to search for an observer for the observable part of the system.

B. Results Case I

a) Kalman-like decomposition of the system into its observable and unobservable part: If we consider $\mathscr{V}_{\bar{o}}$ as the largest submodule $\mathscr{V}$ of $\mathbb{R}[\delta]$ corresponding to the so-called unobservable part of the trajectories of the system (following the terminology of linear systems without delays) that satisfies the inclusion $\begin{pmatrix} A(\delta) \\ C(\delta) \end{pmatrix} \mathscr{V} \subset (\mathscr{V} \times 0)$. Let $V_{\bar{o}}(\delta)$ is a matrix whose columns form a basis of the the module $\mathscr{V}_{\bar{o}} = \ker\mathcal{O}(\delta)$ and $V_o(\delta)$ is a matrix such that $\begin{bmatrix} V_o(\delta) & \vdots & V_{\bar{o}}(\delta) \end{bmatrix}$ is unimodular (i..e. it is invertible). We define $V(\delta) \in \mathbb{R}[\delta]^{n \times n}$ as $V(\delta) = \begin{bmatrix} V_o(\delta) & \vdots & V_{\bar{o}}(\delta) \end{bmatrix}^{-1}$. Let us divide the matrix $V(\delta)$ as $V(\delta) = \begin{bmatrix} V_o^L(\delta) \\ --- \\ V_{\bar{o}}^R(\delta) \end{bmatrix}$ in such a way that

$$V(\delta)V^{-1}(\delta) = \begin{bmatrix} V_o^L(\delta)V_o(\delta) & V_o^L(\delta)V_{\bar{o}}(\delta) \\ V_{\bar{o}}^R(\delta)V_o(\delta) & V_{\bar{o}}^R(\delta)V_{\bar{o}}(\delta) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \tag{9}$$

And since we consider this submodule as $\mathscr{V}_{\bar{o}} = \ker\mathcal{O}(\delta)$ we can have the following algebraic equations:

$$\begin{aligned} A(\delta)V_{\bar{o}}(\delta) &= V_{\bar{o}}(\delta)Q(\delta) \\ C(\delta)V_{\bar{o}}(\delta) &= 0 \end{aligned} \tag{10}$$

Now, we can make a coordinate transformation $\bar{x}(t) = V(\delta)x(t)$, that transforms the system explicitly as

$$\dot{\bar{x}}(t) = \begin{bmatrix} V_o^L(\delta)A(\delta)V_o(\delta) & V_o^L(\delta)A(\delta)V_{\bar{o}}(\delta) \\ V_{\bar{o}}^R(\delta)A(\delta)V_o(\delta) & V_{\bar{o}}^R(\delta)A(\delta)V_{\bar{o}}(\delta) \end{bmatrix} \bar{x}(t) \tag{11}$$

$$y = \begin{bmatrix} C(\delta)V_o(\delta) & C(\delta)V_{\bar{o}}(\delta) \end{bmatrix} \tag{12}$$

Because of (10), it can be confirmed that $A_3 := V_o^L(\delta)A(\delta)V_{\bar{o}}(\delta) = V_o^L(\delta)V_{\bar{o}}(\delta)Q(\delta) = 0$. Moreover, since $C(\delta)V_{\bar{o}}(\delta) = 0$, we obtain the transformed system decomposed as follows

$$\begin{aligned} \dot{\bar{x}}(t) &= \begin{bmatrix} \bar{A}_1(\delta) & 0 \\ \bar{A}_2(\delta) & \bar{A}_4(\delta) \end{bmatrix} \begin{bmatrix} \bar{x}_o(t) \\ \bar{x}_{\bar{o}}(t) \end{bmatrix} \\ y(t) &= \bar{C}_1(\delta)\bar{x}_o(t) \end{aligned} \tag{13}$$

Thus, in order to design an observer for $\bar{x}_o(t)$, like (6), we only need to ensure that the observability matrix

$$\mathcal{O}_{\bar{x}_o}(\delta) = \begin{bmatrix} \bar{C}_1(\delta)\bar{A}_1(\delta) \\ \bar{C}_1(\delta)\bar{A}_1^2(\delta) \\ \vdots \\ \bar{C}_1(\delta)\bar{A}_1^{n-1}(\delta) \end{bmatrix} \tag{14}$$

is left invertible.

Proposition 1. *Let* $r = \operatorname{rank}\mathcal{O}_x(\delta)$. *I f the observability matrix* $\mathcal{O}_x(\delta)$ *has is left invertible (i.e. if it has* n *constant invariant factors) then the observability matrix* $\mathcal{O}_{\bar{x}_o}(\delta)$*, of the observable part of the system, is also left invertible.*

Proof: We apply the matrix transformation

$$\mathcal{O}_x(\delta)V^{-1}(\delta) = \begin{bmatrix} C(\delta)V_o(\delta) & C(\delta)V_{\bar{o}}(\delta) \\ C(\delta)A(\delta)V_o(\delta) & C(\delta)A(\delta)V_{\bar{o}}(\delta) \\ \vdots & \vdots \\ C(\delta)A^{n-1}(\delta)V_o(\delta) & C(\delta)A^{n-1}(\delta)V_{\bar{o}}(\delta) \end{bmatrix} \tag{15}$$

But applying the relation in (10) we have $C(\delta)A(\delta)V_{\bar{o}}(\delta) = C(\delta)V_{\bar{o}}(\delta)Q(\delta) = 0$, which means:

$$\begin{aligned} \mathcal{O}_x(\delta)V^{-1}(\delta) &= \begin{bmatrix} C(\delta)V_o(\delta) & 0 \\ C(\delta)A(\delta)V_o(\delta) & 0 \\ \vdots & \vdots \\ C(\delta)A^{n-1}(\delta)V_o(\delta) & 0 \end{bmatrix} = \\ \begin{bmatrix} C_1(\delta) & 0 \\ C_1(\delta)A_1(\delta) & 0 \\ \vdots & \vdots \\ C_1(\delta)A_1^{n-1}(\delta) & 0 \end{bmatrix} &= \begin{bmatrix} \mathcal{O}_{\bar{x}_o}(\delta) & 0 \end{bmatrix} \end{aligned} \tag{16}$$

Thus we have that the invariant factors of $\mathcal{O}_x(\delta)$ and those of $\mathcal{O}_{\bar{x}_o}(\delta)$ are the same. Therefore according to the condition of the proposition, $\mathcal{O}_{\bar{x}_o}(\delta)$ is left invertible. ■

Thus, an observer for $\bar{x}_o(t)$ may be designed as

$$\begin{aligned} \dot{\hat{\xi}}_o(t) &= \bar{A}_1\hat{\xi}_o(t) + \bar{H}_1(\delta)y(t) + \bar{L}_1\left(y(t) - \bar{C}_1\hat{\xi}_o(t)\right) \\ \hat{x}_o(t) &= T_{oL}^{-1}(\delta)\hat{\xi}_o(t) \end{aligned}$$

III. Formulation of the problem. CASE II

The mathematical model is presented as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{k_a} A_i x(t-ih) + \sum_{i=0}^{k_f} F_i\omega(t-ih) \\ y(t) &= \sum_{i=0}^{k_c} C_i x(t-ih) + \sum_{i=0}^{k_e} E_i\omega(t-ih) \end{aligned} \tag{17}$$

Where:
$x(t) \in \mathbb{R}^n$, is the vector of trajectories of the system.
$y(t) \in \mathbb{R}^p$, is the system output.
$\omega(t) \in \mathbb{R}^m$, is the system input considered unknown.

The initial condition $\varphi(t)$ is a piecewise continuous function $\varphi(t) : [-kh, 0] \to \mathbb{R}^n$ for $k = \max k_a, k_f, k_c, k_e$; therefore, $x(t) = \varphi(t)$ $[-kh, 0]$. Now, A_i, F_i, C_i, E_i are matrices of appropriate dimension with entries in $\mathbb{R}$. Thus, the system in (17) can be represented such as:

$$\begin{aligned} \dot{x}(t) &= A(\delta)x(t) + F(\delta)\omega(t) \\ y(t) &= C(\delta)x(t) + E(\delta)\omega(t) \end{aligned} \tag{18}$$

Where each of the matrices are defined as $A(\delta) := \sum_{i=0}^{k_a} A_i\delta^i$, $F(\delta) := \sum_{i=0}^{k_f} F_i\delta^i$, $C(\delta) := \sum_{i=0}^{k_c} C_i\delta^i$, $E(\delta) := \sum_{i=0}^{k_e} E_i\delta^i$.

A. Design conditions

In [10] are presented the next conditions:

Condition 1. For the quadruple $A(\delta), F(\delta), C(\delta), E(\delta)$ of the system in (18) exists a $k^* \in \mathbb{N}_0$, such as the $\operatorname{rank} M_{k^*}(\delta) = n_x$ being $M_{k^*}(\delta)$ unimodular over $\mathbb{R}[\delta]$

Condition 2. For the triple $F(\delta), C(\delta), E(\delta)$ of the system in (13), it is assumed that:

$$\operatorname{Inv}_s \begin{bmatrix} C(\delta)F(\delta) & E(\delta) \\ E(\delta) & 0 \\ F(\delta) & 0 \end{bmatrix} = \operatorname{Inv} \begin{bmatrix} C(\delta)F(\delta) & E(\delta) \\ E(\delta) & 0 \end{bmatrix} \tag{19}$$

Again from [10] it is proved that if Condition 2 is satisfied, then exists a matrix $W(\delta) \in \mathbb{R}^{(n_x+p)\times 2p}[\delta]$ such that fulfill:

$$W(\delta)\begin{bmatrix} C(\delta)F(\delta) & E(\delta) \\ E(\delta) & 0 \end{bmatrix} = \begin{bmatrix} F(\delta) \\ E(\delta) \end{bmatrix}$$

Going back to the system in (18):

$$\begin{aligned} \dot{x}(t) &= \tilde{A}(\delta)x(t) + K_1(\delta)\Lambda^{-1}(\delta)\dot{\bar{y}}(t) + K_1(\delta)\Lambda^{-1}(\delta)\bar{y}(t) \\ \bar{y} &= \begin{bmatrix} \tilde{C}(\delta) \\ 0 \end{bmatrix} x(t) + \bar{\Gamma}_1(\delta)\dot{\bar{y}}(t) + \bar{\Gamma}_1(\delta)\bar{y}(t) \end{aligned} \tag{20}$$

Where:
$\tilde{A}(\delta) \in \mathbb{R}^{n_x\times n_x}[\delta]$; $\tilde{C}(\delta)\mathbb{R}^{r\times n_x}[\delta]$; $\bar{\Gamma}_1(\delta), \bar{\Gamma}_2(\delta) \in \mathbb{R}^{p\times p}[\delta]$; $K_1(\delta), K_2(\delta) \in \mathbb{R}^{n_x\times p}[\delta]$

Analogous to the observer proposed in [10], we next transform (20) to its normal form with $z = T(\delta)x$:

$$\begin{aligned} \dot{z} &= A_0 z + [H(\delta), 0]\bar{y} + \bar{K}_1(\delta)\dot{\bar{y}} + \bar{K}_1(\delta)\bar{y} \\ \bar{y} &= \begin{bmatrix} C_0 \\ 0 \end{bmatrix} + \bar{\Gamma}_1(\delta)\dot{\bar{y}} + \bar{\Gamma}_2(\delta)\bar{y} \end{aligned} \tag{21}$$

Where

$$H(\delta) = \tilde{C}(\delta)\tilde{A}(\delta)[\tilde{\mathcal{O}}(\delta)]_L^{-1}$$
$$K_1 = T(\delta)K_1(\delta)\Lambda^{-1}(\delta)\mathbb{R}^{n_z\times p}[\delta]$$
$$K_2 = T(\delta)K_2(\delta)\Lambda^{-1}(\delta)\mathbb{R}^{n_z\times p}[\delta]$$

If Conditions 1 and 2 are satisfied for the system in (17), then the next dynamics is obtained:

$$\dot{\xi} = L_0\xi + J(\delta)\Lambda(\delta)$$
$$\hat{z} = \xi + P(\delta)\Lambda(\delta)y$$
$$\hat{x} = T_L^{-1}(\delta)\hat{z}$$

Where

$$L_0 = A_0 - G_0C_0$$
$$P(\delta) = \bar{K}_1(\delta) - [G_0, 0]\bar{\Lambda}_1(\delta)$$
$$J(\delta) = [H(\delta), 0] + K_2(\delta) + L_0P(\delta) - [G_0, 0]\Gamma_2(\delta) + [G_0, 0]$$

a) Dynamic Error.: The Estimation Error is thus$e_z = z - \hat{z}$. Differentiating from e_z:

$$\dot{e}_z = [A_0 - G_0C_0]e_z$$

B. Results CASE II.

a) Decomposition of the system into its observable and unobservable form: If the Condition 1 is not fulfilled, i.e., $M_{k^*}(\delta)$ has not n constant invariant factors over $\mathbb{R}[\delta]$ but it still has all constant invariant factors at its diagonal, then we still can transform the original system in (18).

Now, if we assume $\mathscr{V}_{\bar{o}}$ as the largest submodule $\mathscr{V}$ of $\mathbb{R}[\delta]$ corresponding to the unobservable part of the trajectories of the system that satisfies the inclusion:

$$\begin{pmatrix} A(\delta) \\ C(\delta) \end{pmatrix}\mathscr{V} \subset (A\times 0) + \mathrm{im}\begin{pmatrix} F(\delta) \\ E(\delta) \end{pmatrix} \quad (22)$$

Now, taking into account [3] we can find a relation between $\mathscr{V}_{\bar{o}}$ and $\mathscr{L}^* \triangleq \ker M_{k^*}(\delta)$ such as it allows the decomposition of the system. If we have the next algebraic relation:

$$\begin{aligned} A(\delta)V(\delta) + F(\delta)K(\delta) &= V(\delta)Q(\delta) \\ C(\delta)V(\delta) + E(\delta)K(\delta) &= 0 \end{aligned} \quad (23)$$

Also, let $S(\delta)$ be a polynomial matrix of $q\times s$ dimension with rank equal to r. There exists an invertible matrix $T(\delta)$ over $\mathbb{R}[\delta]$ (representing elementary row operations) such that $S(\delta)$ is put into (column) Hermite form. Thus, we have that $T(\delta)S(\delta) = \begin{bmatrix} S_1(\delta) \\ 0 \end{bmatrix}$, where $S_1(\delta)$ is of $r\times s$ dimensions, and S_1 has r invariant factors.

Proposition 2. *The number of invariant factors in the original matrix system affected by unknown inputs is the same as those corresponding to the observable part of the transformed system.*

Proof: We have to show that $\bar{\Delta}_1(\delta) = \Delta_1(\delta)P^{-1}(\delta)$. If we star with the algebraic relations:

$$T_k(\delta)\begin{bmatrix} \Delta_k(\delta)F(\delta) & \Delta_k(\delta)A(\delta) \\ \bar{F}_k(\delta) & G_k(\delta) \end{bmatrix} = \begin{bmatrix} \bar{F}_{k+1}(\delta) & G_{k+1}(\delta) \\ 0 & \Delta_{k+1}(\delta) \end{bmatrix} \quad (24)$$

$$M_{k+1}(\delta) \triangleq \begin{bmatrix} M_k(\delta) \\ \Delta_{k+1}(\delta) \end{bmatrix}, \qquad \forall\, k \geq 0 \quad (25)$$

to obtain $\Delta_1(\delta)$, we have that

$$\begin{bmatrix} G_1(\delta) \\ \Delta_1(\delta) \end{bmatrix} = T_0(\delta)\begin{bmatrix} \Delta_0(\delta)A(\delta) \\ G_0(\delta) \end{bmatrix}(3) \quad (26)$$

Where we can split the matrix $T_0(\delta) = \begin{bmatrix} T_0^{(a)} \\ T_0^{(b)} \end{bmatrix}$, taking into account $\Delta_0(\delta) \triangleq 0$ (dimension $1\times n$), $G_0(\delta) \triangleq C(\delta)$ and $\bar{F}_0(\delta) \triangleq E(\delta)$. If we know that $\Delta_1(\delta) = T_0^{(b)}(\delta)\begin{bmatrix} 0 \\ C(\delta) \end{bmatrix}$ and, applying the transformed $\bar{C}(\delta) = \left[C(\delta) + E(\delta)\overline{K}(\delta)\right]P^{-1}(\delta)$ and (24), we have $\bar{\Delta}_1(\delta)$ like:

$$\begin{aligned} \bar{\Delta}_1(\delta) &= T_0^{(b)}(\delta)\begin{bmatrix} 0 \\ C(\delta) + E(\delta)\overline{K}(\delta) \end{bmatrix}P^{-1}(\delta) \\ &= T_0^{(b)}(\delta)\begin{bmatrix} 0 \\ C(\delta) \end{bmatrix}P^{-1}(\delta) \\ &+ T_0^{(b)}(\delta)\begin{bmatrix} 0 \\ E(\delta) \end{bmatrix}\overline{K}(\delta)P^{-1}(\delta) \end{aligned} \quad (27)$$

Nevertheless, from (24) and because of the way it was defined $T_0(\delta)$ we have

$$T_0^{(b)}(\delta)\begin{bmatrix} \Delta_0(\delta)F(\delta) \\ \bar{F}_0(\delta) \end{bmatrix} = T_0^{(b)}(\delta)\begin{bmatrix} 0 \\ E(\delta) \end{bmatrix} = 0 \quad (28)$$

whereby, substituting (28) on (27) we have that $\bar{\Delta}_1(\delta) = T_0^{(b)}(\delta)\begin{bmatrix} 0 \\ C(\delta) \end{bmatrix}P^{-1}(\delta)$ implying:

$$\bar{\Delta}_1(\delta) = \Delta_1(\delta)P^{-1}(\delta) \quad (29)$$

Defined the previous one proceeds to replicate the test, but now for the step $j+1$. Based on the previous result, we have that $\bar{\Delta}_j(\delta) = \Delta_j(\delta)P^{-1}(\delta)$ and we look for the form $\bar{\Delta}_{j+1}(\delta) = \Delta_{j+1}(\delta)P^{-1}(\delta)$.

$$\Delta_{j+1}(\delta) = T_j^{(b)}(\delta)\begin{bmatrix} \Delta_j(\delta)A(\delta) \\ G_j(\delta) \end{bmatrix} \quad (30)$$

We also suppose $\bar{G}_j(\delta) = \left[G_j(\delta) + \bar{F}_j(\delta)\overline{K}(\delta)\right]P^{-1}(\delta)$. Thus,

$$\begin{aligned} &\bar{\Delta}_{j+1}(\delta) = \\ &T_j^{(b)}(\delta)\begin{bmatrix} \Delta_j(\delta)A(\delta) + \Delta_j(\delta)F(\delta)\overline{K}(\delta) \\ G_j(\delta) + \bar{F}_j(\delta)\overline{K}(\delta) \end{bmatrix}P^{-1}(\delta) \\ &= T_j^{(b)}(\delta)\begin{bmatrix} \Delta_j(\delta)A(\delta) \\ G_j(\delta) \end{bmatrix}P^{-1}(\delta) \\ &+ T_j^{(b)}(\delta)\begin{bmatrix} \Delta_j(\delta)F(\delta) \\ \bar{F}_j(\delta) \end{bmatrix}\overline{K}(\delta)P^{-1}(\delta) \end{aligned} \quad (31)$$

But, because of the property $T_j^{(b)}(\delta)\begin{bmatrix} \Delta_j(\delta)F(\delta) \\ \bar{F}_j(\delta) \end{bmatrix} = 0$. We have $\bar{\Delta}_{j+1}(\delta) = T_j^{(b)}(\delta)\begin{bmatrix} \Delta_j(\delta)A(\delta) \\ G_j(\delta) \end{bmatrix}P^{-1}(\delta)$. Having as a result that:

$$\bar{\Delta}_{j+1}(\delta) = \Delta_{j+1}(\delta)P^{-1}(\delta) \quad (32)$$

To finish the proof, we have to demonstrate that $\bar{G}_j(\delta) = \left[G_j(\delta) + \bar{F}_j(\delta)\overline{K}(\delta)\right] P^{-1}(\delta)$; in order to do that, we have to show that $\bar{G}_1(\delta) = \left[G_1(\delta) + \bar{F}_1(\delta)\overline{K}(\delta)\right] P^{-1}(\delta)$. Then, we have the next expression

$$\begin{aligned}\bar{G}_1(\delta) &= T_0^{(a)}(\delta)\begin{bmatrix}\Delta_0(\delta)A(\delta)+\Delta_0(\delta)F(\delta)\overline{K}(\delta)\\ G_0(\delta)+\bar{F}_0(\delta)\overline{K}(\delta)\end{bmatrix}P^{-1}(\delta)\\ &= T_0^{(a)}(\delta)\begin{bmatrix}\Delta_0(\delta)A(\delta)\\ G_0(\delta)\end{bmatrix}P^{-1}(\delta)\\ &+T_0^{(a)}(\delta)\begin{bmatrix}\Delta_0(\delta)F(\delta)\\ \bar{F}_0(\delta)\end{bmatrix}\overline{K}(\delta)P^{-1}(\delta)\end{aligned} \tag{33}$$

Of which, it is observed that effectively$\bar{G}_1(\delta) = \left[G_1(\delta) + \bar{F}_1(\delta)\overline{K}(\delta)\right] P^{-1}(\delta)$, and therefore, it can be assumed that $\bar{G}_j(\delta) = \left[G_j(\delta) + \bar{F}_j(\delta)\overline{K}(\delta)\right] P^{-1}(\delta)$, and in this way it has been proven by induction that $\bar{\Delta}_{j+1}(\delta) = \Delta_{j+1}(\delta)P^{-1}(\delta)$.

If in addition it is considered that the matrix of transformation has the form$P(\delta) = \begin{bmatrix} M_{k^*}(\delta)\\ V_L^{-1}(\delta)\end{bmatrix}$, we have the following matrices:

$$\begin{bmatrix}\bar{A}_1(\delta) & 0\\ \bar{A}_2(\delta) & \bar{A}_3(\delta)\end{bmatrix} = P(\delta)\left[A(\delta)+B(\delta)\overline{K}(\delta)\right]P^{-1}(\delta)$$
$$\begin{bmatrix}\bar{C}_1(\delta) & 0\end{bmatrix} = \left[C(\delta)+D(\delta)\overline{K}(\delta)\right]P^{-1}(\delta)$$
$$\bar{B}_1 = M_{k^*}(\delta)B(\delta)$$

And the matrix $M_{k^*}(\delta)$ is constructed like 25. In this way, it is possible to observe that the matrix product with respect to the transformation matrix $P(\delta)$ it is the part corresponding to the observable trajectories of the original system. Due to this, we can conclude that the invariant factors of the transformed part match with the invariant factors of the original system which in turn correspond to the factors of the observable part of the system. In addition, if it is taken into account that the second part of the system decomposition corresponds to the unobservable trajectories, it is appropriate to point out that the invariant factors of the original system will be the same as those of its observable part. ■

IV. Numeric Examples

A. Case I

Next, an academic example is presented to show the design of an observer in linear systems with commensurate delay of the transformed system.

You have a system like the one shown in (3) with the next matrices

$$A(\delta) = \begin{bmatrix}1 & -\delta(1+\delta)(1-\delta^2) & 1+\delta & -(1+\delta)(1-\delta^2) & 0\\ 0 & \delta & 0 & 1 & 0\\ 0 & \delta(1-\delta^2) & 0 & 1-\delta^2 & 0\\ 0 & 1+\delta-\delta^2 & -1 & -\delta^2 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$
$$C(\delta) = \begin{bmatrix}0 & 0 & 1+\delta & 1 & 0\end{bmatrix}$$

Making use of the transformation$\bar{x}(t) = V(\delta)x(t)$ in the system (3) you have the following matrices:

$$V(\delta)A(\delta)V^{-1}(\delta) = \begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 1+\delta & 0 & 1 & 0 & 0\\ 0 & -1+\delta^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$C(\delta)V^{-1}(\delta) = \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}$$

Where the following matrices are obtained:

$$\bar{A}_1(\delta) = \begin{bmatrix}0 & 1 & 0\\ 1+\delta & 0 & 1\\ 0 & -1+\delta^3 & 0\end{bmatrix};\ \bar{C}_1(\delta) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

Being the observability matrix$\mathcal{O}_{\bar{x}_o}(\delta)$ a left inverse matrix:

$$\mathcal{O}_{\bar{x}_o}(\delta) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1+\delta & 0 & 1\end{bmatrix}$$
$$\left[\mathcal{O}_{\bar{x}_o}(\delta)\right]_L^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1-\delta & 0 & 1\end{bmatrix}$$

Which indicates that a coordinate transformation can be found such that it can be possible to design the observer for the observable trajectories of the system:

$$\dot{\hat{\xi}}_o(t) = \bar{A}_1\hat{\xi}(t) + \bar{H}_1(\delta)y(t) + \bar{L}\left(y(t) - \bar{C}_1\hat{\xi}(t)\right)$$

Thus, the graph of the convergence of the estimation error $e(t) = \xi(t) - \hat{\xi}(t)$ for the case of $h = 1$s is presented in Figure 1:

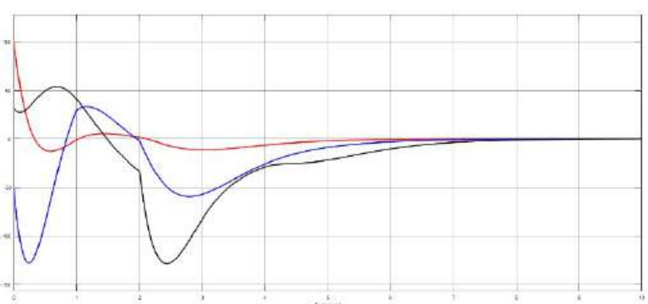

Fig. 1. $e(t) = \xi(t) - \hat{\xi}(t)$ considering the delay $h = 1$s

B. Case II

Now we continues to show an academic example for the design of an observer in linear systems transformed with system delays when there are unknown inputs. We have a system like the one shown in the following matrices:

$$A(\delta) = \begin{bmatrix}1 & a_{12} & a_{13} & 0 & 0 & \delta^2+\delta-1\\ 0 & \delta-1 & -\delta & 0 & 0 & -1\\ 0 & a_{32} & a_{33} & 0 & 0 & \delta-1\\ 1-4\delta+2\delta^2 & 4\delta-1-2\delta^2 & a_{43} & \delta-1 & 0 & a_{46}\\ a_{51} & a_{52} & \delta^3(1-\delta) & 1-2\delta+\delta^2 & 1-\delta & a_{56}\\ 0 & a_{62} & a_{63} & 0 & 0 & -\delta^2\end{bmatrix}$$

Where,

$a_{12} = \delta + (\delta-1)(\delta-2\delta^2-1)$
$a_{13} = \delta - 1 + \delta^2 + (1+\delta)(\delta^2-1)$
$a_{32} = 1 + (\delta-1)(1-2\delta)$
$a_{33} = 1 - 2\delta + (\delta-1)(1+\delta)$
$a_{43} = \delta^3 - \delta^2(1-\delta) + (1+\delta)(1+2\delta-\delta^3)$
$a_{46} = 1 + 2\delta + \delta(1-2\delta) + \delta(\delta-1) - \delta^3$
$a_{51} = -1 + 3\delta + 2\delta(\delta^2-1)^2$

$$a_{52} = (\delta - 1)^2 - \delta(2\delta^2 - \delta - \delta^3)$$
$$a_{56} = \delta(\delta - 1)^2 - 1 + \delta^2(2 + \delta - \delta^2) + \delta(3\delta + 5 - 2\delta^2)$$
$$a_{62} = (\delta - 1)(\delta^2 - 1 + \delta(\delta + 1)) - 2 - \delta$$
$$a_{63} = -1 + \delta(\delta + 1) - \delta^3$$

$$B(\delta) = \begin{bmatrix} \delta & 0 \\ 0 & 0 \\ 0 & 0 \\ 1+\delta-\delta^2 & \delta \\ \delta(1+2\delta-\delta^2) & 1-\delta+\delta^2 \\ -1 & 1 \end{bmatrix} ; \; D(\delta) = \begin{bmatrix} -1 & -\delta \\ 0 & 0 \\ -1 & -\delta \end{bmatrix}$$

$$C(\delta) = \begin{bmatrix} 0 & -\delta & 0 & 0 & 0 & 0 \\ 0 & 1-\delta & -1 & 0 & 0 & 0 \\ -1 & 0 & -(\delta^2+\delta+1) & 0 & 0 & -(1+\delta) \end{bmatrix}$$

Making use of the transformation $\bar{x}(t) = P(\delta)x(t)$ in the previously described system, the following matrices are obtained from the observable part

$$\bar{A}_1(\delta) = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -\delta & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -\delta & -1 & -1 & 1 \end{bmatrix}$$

$$\bar{C}_1(\delta) = \begin{bmatrix} -\delta & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} ; \; \bar{B}_1(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & -\delta \end{bmatrix}$$

Applying the conditions previously stated for the original system and the transformed in the observable part, it is confirmed that they are the same, i.e.:

Condition 1: It has to be: $(\bar{A}_1(\delta), \bar{B}_1(\delta), \bar{C}_1, D(\delta))$ with $k^* = 3$, such as $M_{k^*} = M_{k^*+1} = I_4$.

Condition 2: The invariant factors:

$$\text{Inv}_s \begin{bmatrix} \bar{C}_1(\delta)\bar{B}_1(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \text{Inv}_s \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \{1, 1, 1\}$$

Which indicates that a coordinate transformation can be found such that it can be possible to design the observer:

$$\dot{\xi}_o = L_0\xi + J(\delta)\Lambda(\delta)y$$
$$\hat{z}_o = \xi_o + H(\delta)\Lambda(\delta)y$$
$$\hat{x}_o = T_L^{-1}(\delta)\hat{z}_o$$

Thus, the graph of the convergence of the error $e_{\bar{x}_o} = T_L^{-1}(\delta)e_{z_o}$ is given in Figure 2 considering the delay $h = 0.01s$ and with unknown inputs $\omega_1(t) = 100\sin(100t)$ y $\omega_2(t) = 20\sin(20t)$.

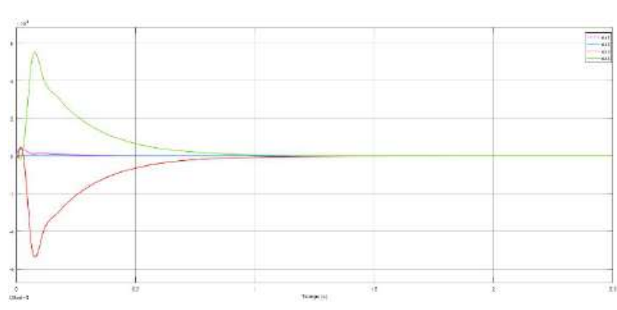

Fig. 2. Graph of convergence between the original system and the observer of the observable part of the transformed system considering unknown inputs and with $h = 0.01s$

Increasing the delay constant as $h = 1s$, we have obtained the estimation error presented in Figure 3.

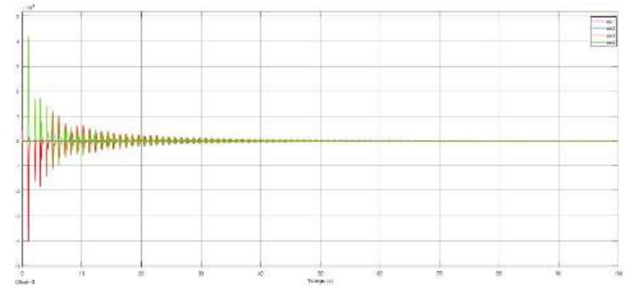

Fig. 3. Graph of convergence between the original system and the observer of the observable part of the transformed system considering unknown entries and with h = 1s.

V. Conclusions

The transformation of linear systems with commensurate delay into their observable and unobservable parts, both for systems without inputs and for systems with unknown inputs, can be carried out using the concepts of unimodularity of the observable matrix in [5], as well as the conditions obtained with respect to the invariant factors in [10]; this due to the existing equivalence with respect to the observable part of the systems transformed to the invariant factors of the original reference.

Acknowledgment

This work was sponsored by the Research and Postgraduate Secretariat of IPN, project SIP-20181391. Special thanks to Ecole Centrale de Lille and INRIA-Lille, France for allowing the use of the facilities during the development of the presented investigation. We also extend an acknowledgmet to the program of Mixed Scholarships of the National Council of Science and Technology for the financial support on the research stay in addition to the financing granted by the CCA-IPN through its program of International Mobility Scholarships.

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