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Adaptive Observer for a Class of Nonlinear Fractional-Order Systems ⋆

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Abstract: In this work we present an adaptive observer design for a class of Nonlinear-Fractional-Order Systems (NFOS) where, using an analysis based on quadratic Lyapunov functions and an extension of Barbalat's theorem to the fractional-order case, the asymptotic convergence of the observed states to the real ones is proven, as well as the boundedness of the parameter reconstruction. Numeric examples are presented to show the effectiveness of the proposed design.

Keywords: Fractional order systems, adaptive observers.

1. INTRODUCTION

Even though the concept of integro-differential fractional equations was first proposed at the end of the 17th century and taken into account as a research subject until 1884, their application to describe dynamic systems has been gaining attention in recent years due to the fact that several classes of physical systems, especially those including diffusion dynamics or friction, as well as memory and hereditary properties in materials and systems can be better and more succinctly described by fractional derivatives and integrals, rather than by their integer counterparts [Caponetto, 2010]. As usually the integer integral or derivative is represented by the operators J^n and D^m respectively, where $n \in \mathbb{N}$; so, fractional integral and derivative are typically described also as the operators J^β and D^α, where $\alpha, \beta \in \Re$, or even $\alpha, \beta \in \mathbb{C}$.

As the same tools used to analyse linear systems with integer differentials and integrals, such as the Laplace transform and the Fourier analysis can be extrapolated and used in linear fractional ones, some methods have been proposed to approximate the solution given by a fractional differential equation of fractional differential system (FOS), from a higher-order transfer function with integer derivatives [Mansouri et al., 2010, Oustaloup, 1991] to the analysis of the step response [Dorcák et al., 2002], similar to the case of first and second-order systems. However, in most cases the parameters of the FOS are assumed known or obtained from a physical analysis of the system, especially regarding the fractional values α and β.

Recently, some approaches have been proposed to identify the parameters of a FOS. One of them considers expanding the fractional differential equation to a larger integer order system [Sabatier et al., 2006], assuming that the fractional values α or β are known. When also this parameters are unknown, approximation methods have been proposed for the case of fractional-order chaotic systems [Yuan and Yang, 2012] by using a particle-swarm optimization and a numerical approximation of the solution of the FOS. In this sense, also Genetic Algorithms (GAs) have been proposed to tune the parameters of the $PI^\alpha D^\beta$ control [Cao et al., 2005], or to find the parameters of an integer-order system [Kristinsson and Dumont, 1992] or even for input-output linear systems [Zhou et al., 2013]. As it can be seen, the identification of a fractional order system is still an open and active research problem.

In this work we present a an adaptive observer design for a class of Nonlinear-Fractional-Order Systems (NFOS) which analysis is based on quadratic Lyapunov functions. The paper is organized as follows: In Section 2, a brief description of fractional calculus and systems are given and the problem statement is given. In Section 3, the algorithm for adaptive observer is presented, and in Section 4 results are shown in order to illustrate the effectiveness of the method. Finally, conclusions are discussed in Section 5.

2. ANTECEDENTS

From a mathematical point of view, a fractional order integral or derivative is defined as an extrapolation of the definition of the integer-order integral or derivative of a certain function $f(t)$, seen as a general fractional differential operator D^α. However, there exist different definitions of this operator, that in general do result in different solutions. Two of the main approaches and most generally used in control systems are the Riemann-Liouville and the Caputo fractional operator [Gorenflo and Mainardi, 1997].

Recall that, for $n \in \mathbb{N}$, given the Cauchy's formula for the repeated integration

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$$J^n f(t) \triangleq \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau) d\tau \cdots d\tau_2 d\tau_1$$
$$= \frac{1}{(n-1)!} \int_a^t f(\tau)(t-\tau)^{n-1} d\tau \quad (1)$$

if n is changed from an integer value to any real (and even complex) value $\alpha \in \Re$, then the definition is extrapolated in the so called *Riemman-Liouville fractional integral*, defined as

$$_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t f(\tau)(t-\tau)^{\alpha-1} d\tau. \quad (2)$$

where $\Gamma(w)$ is the Gamma function of $w \in \mathbb{C}$. From the previous definition, the Riemann-Liouville fractional differential operator D^α is then defined as

$$^{\mathcal{RL}}_{a}D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \\ \qquad \alpha \in (n-1, n),\ n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} f(t), \\ \qquad \alpha = n \in \mathbb{N}, \end{cases} \quad (3)$$

Note that this operator is a left-inverse for (2) [Caponetto, 2010], i.e., $D^\alpha(J^\alpha f(t)) = f(t)$.

A slightly different, but also valid definition of the differential operator, is given by [Caputo, 1967] and called the *Caputo Fractional Differential Operator*:

$$^{\mathcal{C}}_{a}D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{\frac{d^n f(\tau)}{dt^n}}{(t-\tau)^{\alpha+1-n}} d\tau, \\ \qquad \alpha \in (n-1, n),\ n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} f(t), \\ \qquad \alpha = n \in \mathbb{N}. \end{cases} \quad (4)$$

These two definitions are not always interchangeable. In the area of control systems, generally the Caputo's definition is preferred, rather than that of Riemann-Liouville, since in the first one the initial conditions typically associated with physical interpretation are involved, such as the integer derivative at $t = 0$. In the latter, the initial conditions involved do not have a clear physical interpretation [Podlubny, 1998]. In this work, the Caputo's definition is used for the fractional derivative and $a = t_0$, so in general, we use the simplified notation $^{\mathcal{C}}_{a}D_t^\alpha f(t) = D^{(\alpha)} = f^{(\alpha)}(t)$.

2.1 Problem Statement

Consider the class of single-input-single-output fractional-order nonlinear systems with commensurate order given by

$$\mathbf{z}^{(\alpha)}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{f}_0(y(t), u(t)) + \mathbf{b}\left(\sum_{i=1}^p \theta_i g_i(y(t), u(t))\right)$$
$$y = \mathbf{C}\mathbf{z} \quad (5)$$

where $\mathbf{z}(t) \in \Re^n$ is the pseudo-state vector, $0 < \alpha < 1$ the derivative order, $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{f}_0 : \Re \times \Re \to \Re^n$, $\mathbf{b} \in \Re^n$, $g_i : \Re \times \Re \to \Re$, $\mathbf{C} \in \Re^{1\times n}$ and $\theta = (\theta_1, \ldots, \theta_p)$ the p parameters of the system. The objective is to find an adaptive observer with the structure

$$\hat{\mathbf{z}}^{(\alpha)}(t) = h_z(\hat{\mathbf{z}}, u, y, \hat{\theta}) \quad (6)$$
$$\hat{\theta}^{(\alpha)} = h_\theta(u, y, \hat{\mathbf{z}}) \quad (7)$$

such that $\lim_{t\to\infty}(\mathbf{z} - \hat{\mathbf{z}}) = 0$ and $\lim_{t\to\infty}(\theta - \hat{\theta}) = 0$.

2.2 Analysis

Let $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}}$ be the observation error and $\tilde{\theta} = \theta - \hat{\theta}$ the parametric error. Calculating its dynamics we obtain:

$$\tilde{\mathbf{z}}^{(\alpha)}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{f}_0(y(t), u(t)) + \mathbf{b}\left(\sum_{i=1}^p \theta_i g_i(y(t), u(t))\right) - h_z(\hat{\mathbf{z}}, u, y, \hat{\theta}). \quad (8)$$

Choosing

$$h_z(\hat{\mathbf{z}}, u, y, \hat{\theta}) = \mathbf{A}\hat{\mathbf{z}}(t) + \mathbf{f}_0(y(t), u(t)) + \mathbf{b}\left(\sum_{i=1}^p \hat{\theta}_i g_i(y(t), u(t))\right) - \mathbf{K}\tilde{y}, \quad (9)$$

where $\mathbf{K} \in \Re^{n\times 1}$ is a design matrix, $\hat{y} = C\hat{\mathbf{z}}$, and $\tilde{y} = y - \hat{y}$, we obtain

$$\tilde{\mathbf{z}}^{(\alpha)}(t) = \mathbf{A}\tilde{\mathbf{z}}(t) + \mathbf{b}\left(\sum_{i=1}^p \tilde{\theta}_i g_i(y(t), u(t))\right) - \mathbf{K}\tilde{y} \quad (10)$$

3. MAIN RESULT

Choose the quadratic Lyapunov candidate function:

$$V(\tilde{\mathbf{z}}, \tilde{\theta}) = \frac{1}{2}\tilde{\mathbf{z}}(t)^T \mathbf{P}\tilde{\mathbf{z}}(t) + \sum_i^p \tilde{\theta}_i^2(t)\gamma_i^{-1}, \quad (11)$$

where $\gamma_i > 0$ and $\mathbf{P} = \mathbf{P}^T \in \Re^{n\times n}$, $\mathbf{P} > 0$ a design matrix. Knowing from Duarte-Mermoud et al. that given a nonlinear system where $\mathbf{x} \in \Re^n$ defined by

$$\mathbf{x}^{(\alpha)}(t) = \mathbf{f}(\mathbf{x}(t), t),$$

and a quadratic function $V(\mathbf{x}(t)) = \frac{1}{2}\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$, where $\mathbf{P} = \mathbf{P}^T \in \Re^{n\times n}$, its fractional derivative complies with

$$V^{(\alpha)}(\mathbf{x}(t)) \le \mathbf{x}^T(t)\mathbf{P}\mathbf{x}^{(\alpha)}(t) = \mathbf{x}^T(t)\mathbf{P}\mathbf{f}(\mathbf{x}(t), t),$$

then, taking the fractional derivative of order α of $V(\tilde{\mathbf{z}}, \tilde{\theta})$, we get

$$V^{(\alpha)}(\tilde{\mathbf{z}}, \tilde{\theta}) \le \tilde{\mathbf{z}}^T\mathbf{P}\left((\mathbf{A} - \mathbf{K}\mathbf{C})\tilde{\mathbf{z}} + \mathbf{b}\sum_i^k \tilde{\theta}_i g_i(y(t), u(t))\right) + \gamma_i^{-1}\sum_i^p \tilde{\theta}_i\tilde{\theta}_i^{(\alpha)}$$
$$= \tilde{\mathbf{z}}^T(\mathbf{P}(\mathbf{A} - \mathbf{K}\mathbf{C}))\tilde{\mathbf{z}} + \sum_i^p \tilde{\theta}_i\left(\tilde{\mathbf{z}}^T\mathbf{P}\mathbf{b}g_i(y(t), u(t)) + \gamma_i^{-1}\tilde{\theta}_i^{(\alpha)}\right)$$

From the first term of $V^{(\alpha)}$, we get that

$$V_1 = \tilde{\mathbf{z}}^T(\mathbf{P}(\mathbf{A} - \mathbf{K}\mathbf{C}))\tilde{\mathbf{z}} = \tilde{\mathbf{z}}^T\mathbf{Q}_s\tilde{\mathbf{z}} + \tilde{\mathbf{z}}^T\mathbf{Q}_A\tilde{\mathbf{z}},$$

where $\mathbf{Q}_s = \frac{1}{2}\left(\mathbf{P}(\mathbf{A}-\mathbf{KC}) + (\mathbf{A}-\mathbf{KC})\mathbf{P}\right)^T$ is the symmetric part of $\mathbf{P}(\mathbf{A}-\mathbf{KC})$ and $\mathbf{Q}_A$ its antisymmetric part. Then,

$$V_1 = \tilde{\mathbf{z}}^T \mathbf{Q}_s \tilde{\mathbf{z}}.$$

and $V^{(\alpha)}(\tilde{\mathbf{z}}, \tilde{\theta})$, we can choose

$$\tilde{\theta}_i^{(\alpha)} = \hat{\theta}_i^{(\alpha)} = -\gamma_i \left(\tilde{\mathbf{z}}^T \mathbf{Pb} g_i(y(t), u(t))\right), \tag{12}$$

so

$$V^{(\alpha)}(\tilde{\mathbf{z}}, \tilde{\theta}) \leq \tilde{\mathbf{z}}^T \mathbf{Q}_s \tilde{\mathbf{z}}$$

However, $\tilde{\mathbf{z}}$ is not available. But if there exists $\mathbf{P} > 0$ and $\mathbf{K}$ such that $\mathbf{Q}_s < 0$ under the restriction $\mathbf{Pb} = \mathbf{C}^T$, then

$$h_\theta(u, y, \hat{\mathbf{z}}) = -\gamma_i g_i\left(y(t), u(t)\right)\left(y - \mathbf{C}\hat{z}\right) \tag{13}$$

Given the previous facts, we can provide the following result

Theorem 1. Given the fractional-commensurate-order nonlinear fractional-order system (5) and the adaptive observer given by (9) and (13), if there exists a constant matrix $\mathbf{P} > 0$ and a gain $\mathbf{K} \in \Re^{n\times 1}$ such that $\mathbf{P}(\mathbf{A}-\mathbf{KC})^T + (\mathbf{A}-\mathbf{KC})\mathbf{P} < 0$ and $\mathbf{Pb} = \mathbf{C}^T$, then $\lim_{t\to\infty} \tilde{\mathbf{z}} = 0$ and the parameter error $\tilde{\theta}$ remains bounded.

Proof 1. Given that there exist $\mathbf{P} > 0$ and $\mathbf{K}$ such that $\mathbf{Q}_s = \frac{1}{2}\left(\mathbf{P}(\mathbf{A}-\mathbf{KC}) + (\mathbf{A}-\mathbf{KC})^T\mathbf{P}\right) < 0$ and $\mathbf{Pb} = \mathbf{C}^T$ [1], then from the quadratic Lyapunov function (11) we obtain, using (9) and (13), that

$$V^{(\alpha)}(\tilde{\mathbf{z}}, \tilde{\theta}) \leq \tilde{\mathbf{z}}^T \mathbf{Q}_s \tilde{\mathbf{z}}.$$

Then, there exist some $\lambda_P, \lambda_{Q_s} > 0$ such that $\tilde{\mathbf{z}}^T \mathbf{P} \mathbf{z} > \lambda_P ||\tilde{\mathbf{z}}||^2$ and $\tilde{\mathbf{z}}^T \mathbf{Q}_s \mathbf{z} < -\lambda_{Q_s} ||\tilde{\mathbf{z}}(t)||^2$. So,

$$V^{(\alpha)}(\tilde{\mathbf{z}}, \tilde{\theta}) \leq -\lambda_{Q_s} ||\tilde{\mathbf{z}}(t)||^2 \tag{14}$$

In consequence, from Gallegos et al. [2015] as $V(\tilde{\mathbf{z}}, \tilde{\theta})$ is nonnegative, therefore $\tilde{\mathbf{z}}$ and $\tilde{\theta}$ remain bounded, and necessarily $0 \leq V(\tilde{\mathbf{z}}, \tilde{\theta}) < \bar{V} < \infty$. Following Navarro-Guerrero and Tang [2017], applying the Riemann-Liouville fractional integral of order α to both sides if the previous equation, we get

$${}_{t_0}I_t^\alpha \left(V^{(\alpha)}\right) \leq -\lambda_{Q_s} {}_{t_0}I_t^\alpha ||\tilde{\mathbf{z}}(t)||^2. \tag{15}$$

By the Newton-Leibniz formula generalization we know that ${}_{t_0}I_t^\alpha \left(V^{(\alpha)}\right) = V(\tilde{\mathbf{z}}(t), \tilde{\theta}(t)) - V(\tilde{\mathbf{z}}(t_0), \tilde{\theta}(t_0))$, so

$${}_{t_0}I_t^\alpha ||\tilde{\mathbf{z}}(t)||^2 \leq -\frac{1}{\lambda_{Q_s}} \left(V(\tilde{\mathbf{z}}(t), \tilde{\theta}(t)) - V(\tilde{\mathbf{z}}(t_0), \tilde{\theta}(t_0)\right) \tag{16}$$

Then, ${}_{t_0}I_t^\alpha ||\tilde{\mathbf{z}}(t)||^2 < M < \infty$. In consequence, by the Barbalat's Lemma extension to the fractional case [Gallegos et al., 2015, Navarro-Guerrero and Tang, 2017], $\lim_{t\to\infty} \tilde{\mathbf{z}} = 0$, and the asymptotic convergence of $\hat{\mathbf{z}}(t)$ to $\mathbf{z}(t)$ is proven, and the parametric error $\tilde{\theta}$ remains bounded.

4. EXAMPLES

In this section we show some numeric examples to show the effectiveness of the proposed method. All simulations were run using Matlab & Simulink, using the numeric Grunwald-Letnikov approximation for Caputo fractional derivative, using a sampling time of 5 ms and a buffer of 500 samples.

[1] This is the same Kalman-Yakubovich condition required in the integer-order case

4.1 Example 1.

Consider the system (5) with $n = 2$ where $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{f}_0(y(t), u(t)) = \begin{pmatrix} -2y \\ -3y \end{pmatrix}$, $g_1(y(t), u(t)) = \sin(y(t))$, $g_2(y(t), u(t)) = \cos(u(t))$ with $y = z_2$, so $\mathbf{C} = (\,0\ 1\,)$, and assume that the unknown parameters with real values are $\theta(t) = \{\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}, t < 200; \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}, t \geq 200\}$, . Let $\mathbf{K}$ be the gain such that locates the poles of $(\mathbf{A}-\mathbf{KC})$ on the design location $\lambda\{(\mathbf{A}-\mathbf{KC})\} = \{\lambda_1, \lambda_2\}$. If $\lambda_1 = -2$, $\lambda_2 = -3$, then it is easy to find that $\mathbf{K} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Now, as the problem is to find a matrix $\mathbf{P} > 0$ such that $\mathbf{P}(\mathbf{A}-\mathbf{KC}) + (\mathbf{A}-\mathbf{KC})^T\mathbf{P}$ is negative definite and $\mathbf{Pb} = \mathbf{C}^T$, solving for $\mathbf{P} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$ we obtain the restrictions $p_2 = 0$ and $p_3 = 1$, $p_1 > 0$. Therefore, solving the matrix inequality $\mathbf{Q}_s = \frac{1}{2}\left(\mathbf{P}(\mathbf{A}-\mathbf{KC}) + (\mathbf{A}-\mathbf{KC})^T\mathbf{P}\right) < 0$, we find that the condition for a feasible solution is $0.0505 < p_1 < 4.9495$. In Fig. 2 to 5 it is shown the simulation results with $x(0) = [1, 1]^T$ and null initial conditions for the observer, with $p_1 = 2$ and $\gamma_1 = \gamma_2 = 10$, when the system is subject to the periodic input signal shown in Fig. 1. It can be seen how the observer state converges asymptotically to the actual state in Fig. 2 and 3, and how, although not proven yet, the parameters converge to their real values, as it can be seen in Fig. 4 and 5.

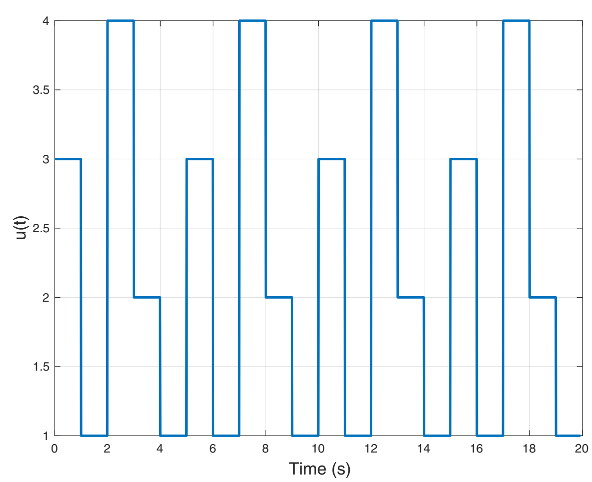

Fig. 1. Input to the system

4.2 Example 2. Neuro-fuzzy approximation

The adaptive observer algorithm can also be used, as in previous works [González-Olvera and Tang, 2010], as a method to train a neural network for identification purposes. In order to show that it can be also expanded to the fractional-order case, consider the same system as in the previous example, but now assuming that $g_i(\cdot,\cdot)$, $i =$

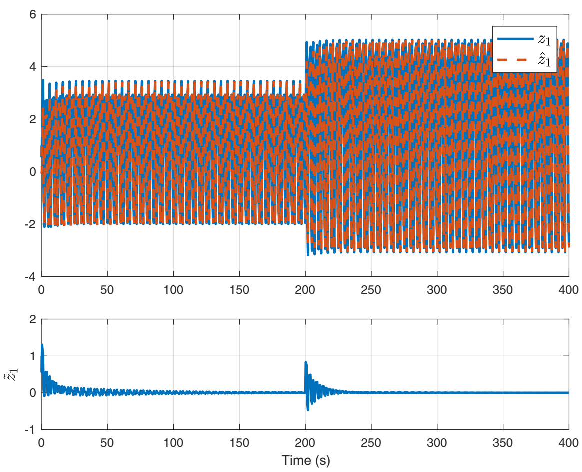

Fig. 2. Dynamics of the state z_1 and $\hat{z}_1$

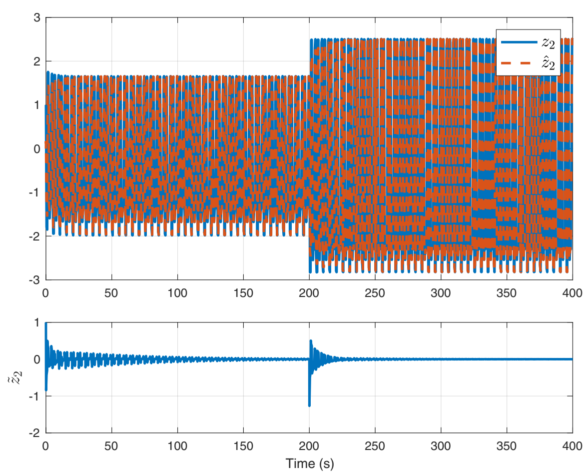

Fig. 3. Dynamics of the state z_2 and $\hat{z}_2$

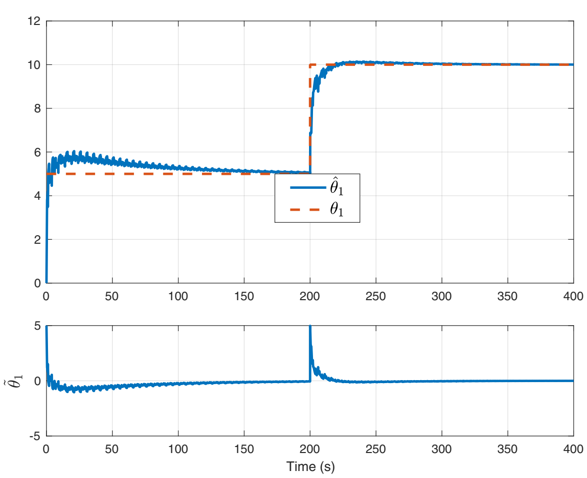

Fig. 4. Dynamics of the state θ_1 and $\hat{\theta}_1$

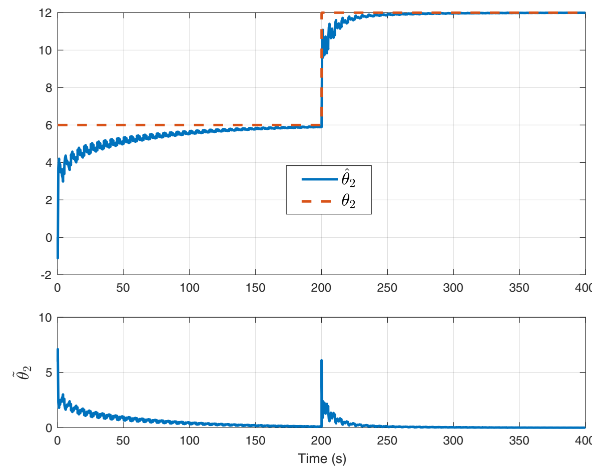

Fig. 5. Dynamics of the parameters θ_2 and $\hat{\theta}_2$

$1, 2$ functions in 5 are unknown, so its structure is to be approximated by a neurofuzzy network in the form

$$\sum_{i=1}^{p} \theta_i g_i(y(t), u(t)) \approx \Phi(\beta, y(t), u(t)) = \sum_{j=1}^{N} \beta_j \phi_i(y(t), u(t)), \tag{17}$$

where $\Phi(\beta, y(t), u(t))$ is a neurofuzzy function given by gaussian radial-basis functions for each rule in the form $r_j(y(t), u(t)) = e^{-\sigma_{u,j}(u(t)-\mu_{u-j})^2 - \sigma_{y,j}(u(t)-\mu_{y-j})^2}$, $\phi_j = \frac{r_j(y(t),u(t))}{\sum_j r_j}$ so β_j are the consequent parameters on each fuzzy rule. Assuming only the consequent parameters are to be found, using only the input and output signal information from the previous example, using $\sigma_{y-j} = \sigma_{u-j} = 1$, $\mu_{y-j} = \{1, 1, 1, 0, 0, 0, -1, -, 1, -1\}$, $\mu_{u,j} = \{1, 2, 3, 1, 2, 3, 1, 2, 3\}$, the parameter evolution is shown in Fig. 6, finding the set of consequent parameters

$$\begin{array}{c} \beta_j = \quad \{11.22, \quad 3.33, \quad -0.06139, \; 3.031, \\ -1.577, -1.627, \; -1.425, \; -11.27, \quad -10.14\} \end{array}.$$

The identification results are shown in Fig. 7, where it can be seen how the system with a neuro-fuzzy function does effectively approximate the actual one. It can be also observed that, although model uncertainties occur in the approximation process by the neurofuzzy network, the algorithm remains stable and convergent.

5. CONCLUSIONS

In this work we have presented a methodological design of an adaptive observer for a class of nonlinear fractional-order systems, where, assuming a linear parameterization, guarantees the asymptotic convergence of the observed states to the real ones, as well as the boundedness of the parametric error. Numeric examples are presented in order to show the effectiveness of the proposed scheme. It was shown how the algorithm remains stable and convergent even under parametric changes and model uncertainties. Further work includes the analysis of the convergence under bounded disturbances and model uncertainties.

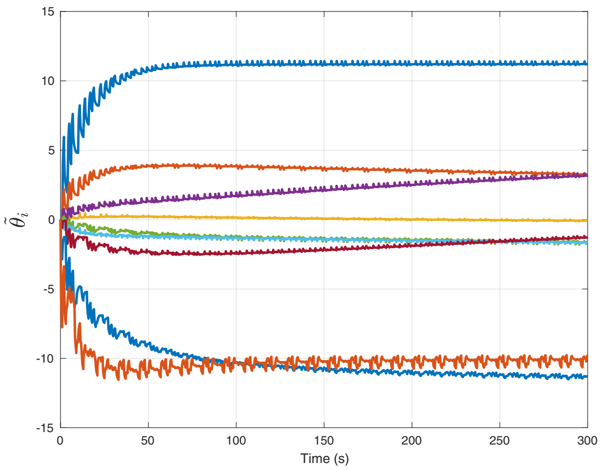

Fig. 6. Parameter evolution for Example 2.

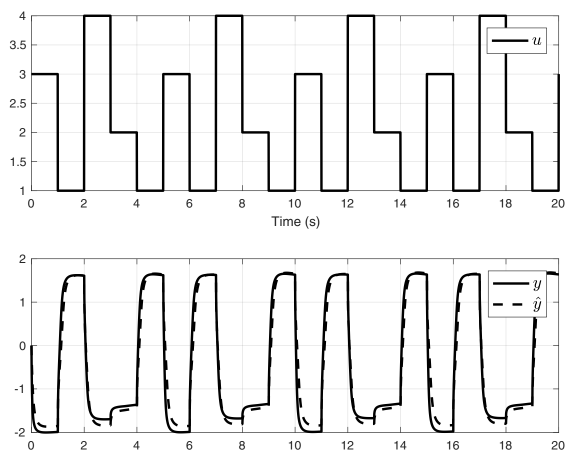

Fig. 7. Dynamics of the actual system with output y vs. the identified dynamics $\hat{y}$

6. ACKNOWLEDGEMENTS

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