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Control via Finite-Time Terminal Sliding Mode

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Abstract: In this work, a trajectory tracking controller based on Slide-Mode Control (SMC), Terminal Sliding-Mode (TSM) and derived from finite time stability theory is presented. The main contributions from this algorithm are finite-time convergence of the states to zero and robustness against external perturbations and parametric uncertainties using only one tunable gain. Stability test are made to demonstrate the finite-time stability in closed loop and calculation of the system trajectories reaching time to the sliding surface.

Keywords: sliding mode, control, finite-time, trajectory, variable structure.

1. INTRODUCTION

When it comes to model a real plant system dynamics, usually, discrepancies and mismatches arise between the real system and its mathematical model. These differences usually come from parametric uncertainties and external perturbations. The most interesting and challenging duty of modern control theory is to design a valid control law to get the desired performance from the closed-loop system, overcoming the effects from these disturbances, this method is called robust control. The most common and probably one of the best methods that had arisen as a robust answer to disturbances and parametric uncertainties, is the Sliding-Mode Control (SMC), which has been studied in Utkin (1977) and Young et al. (1999). This method uses a variable structure system (VSS), which is made up with ordinary differential equations with discontinuous right hand side, see Filippov (1988). The discontinuous term drives the differential equations trajectories to the discontinuity surface (sliding surface) and keeps the trajectories "sliding" on the surface, once the trajectories are moving across the sliding surface, the system dynamics remain stable, even against perturbations affecting the system. Before the trajectories reach the sliding surface, there is a "reaching" time that starts from the initial conditions of the system and finishes when the trajectories reach the sliding surface. During this time, trajectories are vulnerable to perturbations and parametric variations. The ordinary SMC method or First Order Sliding-Mode (FOSM) drives the trajectories to the equilibrium point asymptotically, but using Terminal Sliding-Mode (TSM) the equilibrium point of the system is reached in finite-time, see Venkataraman and Gulati (1993). The reaching time can be adjusted by tuning the TSM parameters.

Control of mechanical systems has been an interesting duty due to its industrial applications. SMC can be designed for trajectory tracking purposes, for previous work about tracking and sliding-mode in mechanical systems see Slotine and Sastry (1983). For TSM related works see Yu et al. (2002, 2005).

This work discusses the design of a terminal sliding mode control law for a second order dynamical systems, besides this, the stability analysis is made to prove the finite-time convergence to the controller's tracking goal, for related works to finite-time stabilization see Sanyal and Bohn (2015); Zhu et al. (2011). The rest of the paper is organized as follows: In section 2 it is described the problem statement in second order dynamical systems, including mechanical systems of one degree of freedom on either rotational or translational links, also it is described the general structure of the controller and description of the discontinuous right hand side term of the equation. Section 3 explains the system in function of errors, control compensations, the variable sliding surface structure that splits in three different disjoint sets and the control law. Section 4 describes the stability analysis and it is established a equation for the reaching time of the trajectories to the sliding surface. The experiment results comparing the proposed controller with existing controllers is presented on Section 5 and final conclusions are on Section 6.

2. PROBLEM STATEMENT

The proposed problem is to design a discontinuous controller based on the FOSM control, see Shtessel et al. (2014), to solve the tracking problem for the system on equation (1).

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u + w(t) \quad (1)$$

Consider that $x = [x_1, x_2]^T$ is the state vector, $u \in \Re$ is the control input, the nominal dynamics of the system are given by the nonlinear function $f(x) \in \Re$, $g(x) \in \Re$ is a well known function, and $w(t)$ is a disturbance term for parametric uncertainties and external perturbations.

Assumption 1. The disturbance term $w(t)$ is unknown, although it is upper bounded by a known constant M that satisfies

$$\sup_{t\geq 0}|w(t)| \leq M \quad (2)$$

for a constant $M > 0$.

As equation (1) possess discontinuous right-hand side, the dynamic of such equation is defined throughout in the sense of Filippov, see Filippov (1988). Due to non-uniqueness of solutions, the system (1) may have some solutions that converge to the origin, while other solutions don't.

The discontinuous right-hand side is given by the *signum* function defined in equation (3).

$$\text{sign}(x_2) = \begin{cases} 1 & x_2 > 0 \\ 0 & x_2 = 0 \\ -1 & x_2 < 0. \end{cases} \quad (3)$$

3. CONTROL DESIGN

The control objective is to find a control u, depending on the desired position or trajectory x_d which is C^k, for a sufficiently large k, the generalized coordinates x_1 and x_2, such that the closed-loop response of system (1) satisfies the condition on equation (4).

$$\lim_{t\to\infty} |x(t) - x_d| = 0. \quad (4)$$

Let us propose a variable structure controller, based on the FOSM control, see Shtessel et al. (2014), to be applied on the system in equation (1). For this purpose, first shift the equilibrium point of (1) by defining the following transformation showed on equation (5).

$$e_1 = x_1 - x_d,$$
$$e_2 = \dot{x}_1 - \dot{x}_d = x_2 - \dot{x}_d. \quad (5)$$

Rewriting system (1) according to (5) and considering $e = [e_1, e_2]^T$

$$\dot{e}_1 = e_2$$
$$\dot{e}_2 = f(e) + g(e)u + w(t) - \ddot{x}_d. \quad (6)$$

For system (6) the following control design is proposed

$$u = -g(e)^{-1}\left(\widetilde{f}(e) - \tau - \ddot{x}_d\right) \quad (7)$$

Assumption 2. $\widetilde{f}(e) = f(e) + \Delta f(e)$ is an approximate compensation term for the nonlinear function $f(e)$, $\Delta f(e)$ represents the error between $f(e)$ and $\widetilde{f}(e)$, the term $\Delta f(e)$ is considered upper bounded by a constant N.

Substituting (7) in (6) renders to

$$\dot{e}_1 = e_2$$
$$\dot{e}_2 = -\Delta f(e) + \tau + w(t). \quad (8)$$

The proposed finite time sliding-mode controller is based on FOSM, see Rascón et al. (2016); Shtessel et al. (2014), the sliding surface is proposed as follows

$$s = \begin{cases} e_2 + e_1^{\frac{1}{2}} & e_1 > 0 \\ e_2 & e_1 = 0 \\ -e_2 + (-e_1)^{\frac{1}{2}} & e_1 < 0. \end{cases} \quad (9)$$

the expression (9) can be synthesized as

$$s = \begin{cases} \text{sign}(e_1)e_2 + |e_1|^{\frac{1}{2}} & e_1 \neq 0 \\ e_2 & e_1 = 0 \end{cases} \quad (10)$$

The dynamical behavior of the proposed finite time sliding surface (red) can be seen in Fig 1, notice the behavior of the sliding surface (blue) normally used in FOSM controllers that unlike the proposed finite time sliding surface, this converges asymptotically to the reference, see Shtessel et al. (2014).

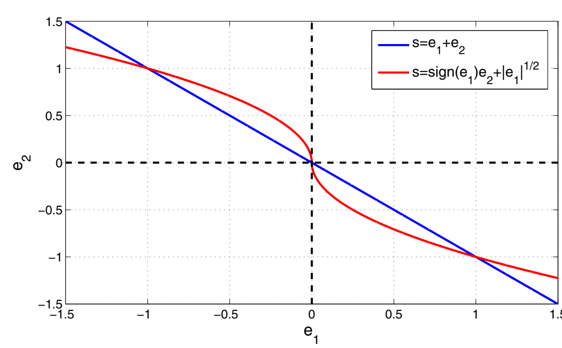

Fig. 1. Proposed finite time sliding surface and conventional sliding surface.

In order to design the control law τ from (9), the state space $\Re^2$ is splitted in three disjoint sets, Σ_-, Σ_0 and Σ_+, characterized by the sign of e_1, described by

$$\Sigma_- = \{e \in \Re^2 | e_1 < 0\},$$
$$\Sigma_0 = \{e \in \Re^2 | e_1 = 0\}, \quad (11)$$
$$\Sigma_+ = \{e \in \Re^2 | e_1 > 0\}.$$

Consider the structure Σ_- of the system (the analysis of the structures Σ_0 and Σ_+ are similar). First let us get $\dot{s} = \frac{\partial s}{\partial e}\dot{e}$ given by

$$\dot{s} = \Delta f(e) - \tau - w(t) + \frac{1}{2}\frac{e_2}{\sqrt{e_1}} \tag{12}$$

now, let us consider (12) to design τ, which must fulfill the reachability law $s\dot{s} < 0$, defined in the literature Edwards and Spurgeon (1998). One control law τ which satisfies $s\dot{s} < 0$ is

$$\Sigma_- : -\tau = \frac{1}{2}\frac{e_2}{\sqrt{e_1}} + \beta \text{sign}(s) \tag{13}$$

where the reachability law is given by $s\dot{s} = -\beta|s| - (\Delta f(e) - w(t))s \leq -(\beta - (M+N))|s| < 0$, therefore all the trajectories converge to the sliding surface while $\beta > M + N$ is satisfied. The same procedure is applied to structures Σ_0 and Σ_+ to get τ, resulting as follows

$$\Sigma_0 : \tau = \beta \text{sign}(s) \tag{14}$$

$$\Sigma_+ : \tau = -\frac{1}{2}\frac{e_2}{\sqrt{-e_1}} - \beta \text{sign}(s) \tag{15}$$

where the same inequality $\beta > M + N$ still applies. The controllers designed for each structure can be unified in a single multivalued expression $\tau \in \Sigma_- \cup \Sigma_0 \cup \Sigma_+$ given by

$$\tau = \begin{cases} -\frac{1}{2}\frac{e_2}{\sqrt{|e_1|}} + \beta \text{sign}(s)\text{sign}(e_1) & e_1 \neq 0 \\ -\beta \text{sign}(s) & e_1 = 0 \end{cases} \tag{16}$$

the next section it is going to be proved the finite time convergence to the reference of the closed-loop system given above.

4. STABILITY ANALYSIS

In order to prove that the trajectories of the system can be brought to the sliding surface (9) at $s = 0$ in finite time, the system dynamics of the disjoint sets are defined by

$$\begin{aligned} \Sigma_- &\Rightarrow 0 = -e_2 + (-e)^{\frac{1}{2}} \rightarrow \dot{e}_1 = (-e_1)^{\frac{1}{2}}, \\ \Sigma_+ &\Rightarrow 0 = e_2 + e^{\frac{1}{2}} \qquad \rightarrow \dot{e}_1 = -e_1^{\frac{1}{2}}. \end{aligned} \tag{17}$$

Consider the structure Σ_- of the system (the analysis of the structure Σ_+ is similar) and integrating both sides of the equation (17)

$$\int_{e_1(t_0)}^{e_1(t_r)} de_1 = \int_{t_0}^{t_r} -\sqrt{e_1}dt \tag{18}$$

where reaching time (t_r) is the time when the trajectories reach the sliding surface. Considering $e_1(t_r) = 0$ and solving for the reaching time (t_r)

$$\Sigma_- : t_r = t_0 + 2\sqrt{-e_1(t_0)}. \tag{19}$$

The same procedure is applied to structure Σ_+, resulting as follows

$$\Sigma_+ : t_r = t_0 + 2\sqrt{e_1(t_0)}. \tag{20}$$

Unifying both expressions (19) and (20), gives the general time in which the errors will reach the sliding surface $s = 0$, no matter the sign of e_1.

$$t_r = t_0 + 2\sqrt{|e_1(t_0)|} \tag{21}$$

5. EXPERIMENT RESULTS

The real time experiments were developed in a simple pendulum system, showed on Fig. 2, with a data acquisition board DSPACE® and the acquisition sampling rate was set to 0.001 s.

Fig. 2. Single pendulum system used for test.

Let the system used for the experiment be a simple pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - bsin(x_1) + cu \end{aligned} \tag{22}$$

where $a = \frac{fv}{ml^2}$, $b = \frac{mgl}{ml^2}$, $c = \frac{1}{ml^2}$. Representing the system in function of errors and substituting (22) in (6)

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= -a(e_2 + \dot{x}_d) - bsin(x_1) + cu + w(t) - \ddot{x}_d \end{aligned} \tag{23}$$

and the designed control (7) is defined as

$$\begin{aligned} u &= -c^{-1}(-a(e_2 + \dot{x}_d) - bsin(x_1) - \tau - \ddot{x}_d) \\ \tau &= \begin{cases} -\frac{1}{2}\frac{e_2}{\sqrt{|e_1|}} + \beta \text{sign}(s)\text{sign}(e_1) & e_1 \neq 0 \\ -\beta \text{sign}(s) & e_1 = 0 \end{cases} \\ s &= \begin{cases} \text{sign}(e_1)e_2 + |e_1|^{\frac{1}{2}} & e_1 \neq 0 \\ e_2 & e_1 = 0 \end{cases} \end{aligned} \tag{24}$$

The FOSM algorithm used is shown in equation (25) and it's applied to the same system on equation (23).

$$u = -c^{-1}(-a(e_2 + \dot{x}_d) - bsin(x_1) - \tau - \ddot{x}_d)$$
$$\tau = -e_2 - \beta \text{sign}(s) \quad (25)$$
$$s = e_1 + e_2$$

where $e_2 = de_1/dt$, the FOSM controller can absorb perturbations and uncertainties $w(t)$ which are bounded by $|w(t)| \leq D$, and the condition $\beta > D$ must be hold for stability purposes, see Rascón et al. (2016).

Table 1. Experiment parameters

Plant parameters	
Notation	**Value**
$x_1(0)$	0
$x_2(0)$	0
x_d	$cos(t)$
a	17
b	93
c	103
Proposed controller	
β	11
FOSM	
β	10

The initial conditions and the plant parameters used for the experiment are shown on table 1.

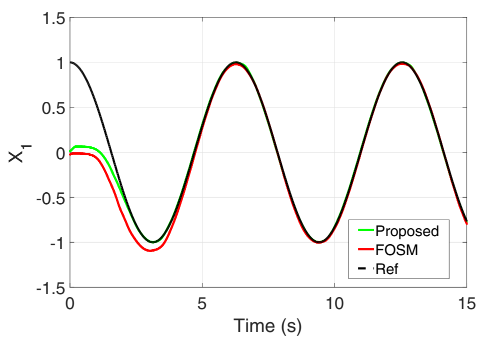

Fig. 3. Position measurements (experiment).

Fig. 3 shows the position measure (x_1), Fig. 4 represents the error between the position and the desired trajectory. It can be seen on these two first experiment figures that the proposed TSM algorithm converge faster to the desired trajectory than the FOSM controller. The control signal is shown on Fig. 5, showing that both signals are similar.

6. CONCLUSIONS AND FINAL COMMENTS

The designed controller uses the finite-time convergence theory from TSM to achieve the trajectory tracking goal, its implementation on mechanical systems become simple due to the only tunable gain the controller has. Robustness is an important feature for this controller, its algorithm allow to compensate bounded disturbances once the trajectories are inside the sliding surface. Performance compared with FOSM has been improved, achieving the desired trajectory in less and finite time, leading the position error to zero faster and directly, avoiding the

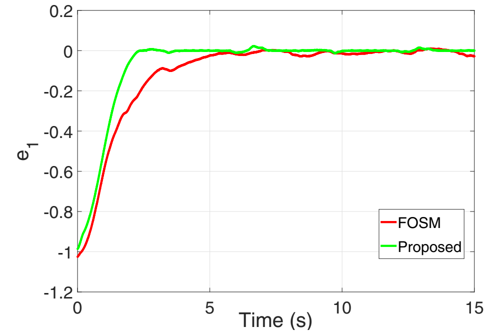

Fig. 4. Position error measurements (experiment).

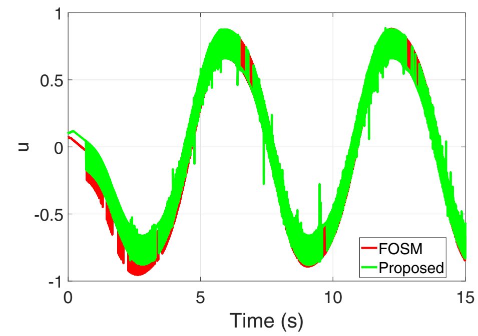

Fig. 5. Control signal comparison (experiment).

asymptotic behavior of the FOSM controller and without increasing the controller signal amplitude compared with FOSM control signal.

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